

Amicable Numbers

Angeliki Koutsoukou-Argyaki

March 24, 2023

Abstract

This is a formalisation of Amicable Numbers, involving some relevant material including Euler's sigma function, some relevant definitions, results and examples as well as rules such as Thābit ibn Qurra's Rule, Euler's Rule, te Riele's Rule and Borho's Rule with breeders.

The main sources are [2] [3]. Some auxiliary material can be found in [1] [4]. If not otherwise stated, the source of definitions is [2]. In a few definitions where we refer to Wikipedia articles [5] [6] [7] this is explicitly mentioned.

Contents

1	Miscellaneous	3
2	Amicable Numbers	4
2.1	Preliminaries	4
2.2	Euler’s sigma function and properties	5
2.3	Amicable Numbers; definitions, some lemmas and examples	6
2.3.1	Regular Amicable Pairs	7
2.3.2	Twin Amicable Pairs	8
2.3.3	Isotopic Amicable Pairs	8
2.3.4	Betrothed (Quasi-Amicable) Pairs	8
2.3.5	Breeders	9
2.3.6	More examples	9
3	Euler’s Rule	9
4	Thābit ibn Qurra’s Rule and more examples	10
5	Te Riele’s Rule and Borho’s Rule with breeders	11
6	Acknowledgements	11

```

theory Amicable-Numbers
  imports HOL-Number-Theory.Number-Theory
            HOL-Computational-Algebra.Computational-Algebra
            Pratt-Certificate.Pratt-Certificate-Code
            Polynomial-Factorization.Prime-Factorization

```

```

begin

```

1 Miscellaneous

```

lemma mult-minus-eq-nat:
  fixes  $x::nat$  and  $y::nat$  and  $z::nat$ 
  assumes  $x+y = z$ 
  shows  $-x-y = -z$ 
  <proof>

```

```

lemma minus-eq-nat-subst: fixes  $A::nat$  and  $B::nat$  and  $C::nat$  and  $D::nat$  and
 $E::nat$ 
  assumes  $A = B-C-D$  and  $-E = -C-D$ 
  shows  $A = B-E$ 
  <proof>

```

```

lemma minus-eq-nat-subst-order: fixes  $A::nat$  and  $B::nat$  and  $C::nat$  and
 $D::nat$  and  $E::nat$ 
  assumes  $B-C-D > 0$  and  $A = B-C-D+B$  shows  $A = 2*B-C-D$ 
  <proof>

```

```

lemma auxiliary-ineq: fixes  $x::nat$  assumes  $x \geq (2::nat)$ 
  shows  $x+1 < (2::nat)*x$ 
  <proof>

```

```

lemma sum-strict-mono:
  fixes  $A :: nat$  set
  assumes finite  $B$   $A \subset B$   $0 \notin B$ 
  shows  $\sum A < \sum B$ 
  <proof>

```

```

lemma sum-image-eq:
  assumes inj-on  $f$   $A$ 
  shows  $\sum (f ` A) = (\sum i \in A. f i)$ 
  <proof>

```

```

lemma coprime-dvd-aux:
  assumes  $\text{gcd } m \ n = \text{Suc } 0$   $na \ \text{dvd } n$   $ma \ \text{dvd } m$   $mb \ \text{dvd } m$   $nb \ \text{dvd } n$  and  $eq: ma$ 
 $* na = mb * nb$ 

```

shows $ma = mb$
<proof>

2 Amicable Numbers

2.1 Preliminaries

definition $divisor :: nat \Rightarrow nat \Rightarrow bool$ (**infixr** $divisor$ 80)
where $n\ divisor\ m \equiv (n \geq 1 \wedge n \leq m \wedge n\ dvd\ m)$

definition $divisor\text{-}set$: $divisor\text{-}set\ m = \{n. n\ divisor\ m\}$

lemma $def\text{-}equiv\text{-}divisor\text{-}set$: $divisor\text{-}set\ (n::nat) = set(divisors\text{-}nat\ n)$
<proof>

definition $proper\text{-}divisor :: nat \Rightarrow nat \Rightarrow bool$ (**infixr** $properdiv$ 80)
where $n\ properdiv\ m \equiv (n \geq 1 \wedge n < m \wedge n\ dvd\ m)$

definition $properdiv\text{-}set$: $properdiv\text{-}set\ m = \{n. n\ properdiv\ m\}$

lemma $example1\text{-}divisor$: **shows** $(2::nat) \in divisor\text{-}set\ (4::nat)$
<proof>

lemma $example2\text{-}properdiv\text{-}set$: $properdiv\text{-}set\ (Suc\ (Suc\ (Suc\ 0))) = \{(1::nat)\}$
<proof>

lemma $divisor\text{-}set\text{-}not\text{-}empty$: **fixes** $m::nat$ **assumes** $m \geq 1$
shows $m \in divisor\text{-}set\ m$
<proof>

lemma $finite\text{-}divisor\text{-}set$ [*simp*]: $finite(divisor\text{-}set\ n)$
<proof>

lemma $finite\text{-}properdiv\text{-}set$ [*simp*]: **shows** $finite(properdiv\text{-}set\ m)$
<proof>

lemma $divisor\text{-}set\text{-}mult$:
 $divisor\text{-}set\ (m*n) = \{i*j \mid i\ j. (i \in divisor\text{-}set\ m) \wedge (j \in divisor\text{-}set\ n)\}$
<proof>

lemma $divisor\text{-}set\text{-}1$ [*simp*]: $divisor\text{-}set\ (Suc\ 0) = \{Suc\ 0\}$
<proof>

lemma $divisor\text{-}set\text{-}one$: **shows** $divisor\text{-}set\ 1 = \{1\}$
<proof>

lemma $union\text{-}properdiv\text{-}set$: **assumes** $n \geq 1$ **shows** $divisor\text{-}set\ n = (properdiv\text{-}set\ n) \cup \{n\}$
<proof>

lemma *prime-div-set*: **assumes** *prime n* **shows** *divisor-set n = {n, 1}*
⟨*proof*⟩

lemma *div-set-prime*:
assumes *prime n*
shows *properdiv-set n = {1}*
⟨*proof*⟩

lemma *prime-gcd*: **fixes** *m::nat* **and** *n::nat* **assumes** *prime m* **and** *prime n*
and *m ≠ n* **shows** *gcd m n = 1* ⟨*proof*⟩

We refer to definitions from [5]:

definition *aliquot-sum* :: *nat* ⇒ *nat*
where *aliquot-sum n* ≡ \sum (*properdiv-set n*)

definition *deficient-number* :: *nat* ⇒ *bool*
where *deficient-number n* ≡ (*n* > *aliquot-sum n*)

definition *abundant-number* :: *nat* ⇒ *bool*
where *abundant-number n* ≡ (*n* < *aliquot-sum n*)

definition *perfect-number* :: *nat* ⇒ *bool*
where *perfect-number n* ≡ (*n* = *aliquot-sum n*)

lemma *example-perfect-6*: **shows** *perfect-number 6*
⟨*proof*⟩

2.2 Euler's sigma function and properties

The sources of the following useful material on Euler's sigma function are [2], [3], [4] and [1].

definition *Esigma* :: *nat* ⇒ *nat*
where *Esigma n* ≡ \sum (*divisor-set n*)

lemma *Esigma-properdiv-set*:
assumes *m* ≥ 1
shows *Esigma m* = (*aliquot-sum m*) + *m*
⟨*proof*⟩

lemma *Esigmanotzero*:
assumes *n* ≥ 1
shows *Esigma n* ≥ 1
⟨*proof*⟩

lemma *prime-sum-div*:
assumes *prime n*
shows *Esigma n* = *n* + (1::*nat*)

<proof>

lemma *sum-div-is-prime*:

assumes $Esigma\ n = n + (1::nat)$ **and** $n \geq 1$
shows *prime* n

<proof>

lemma *Esigma-prime-sum*:

fixes $k::nat$ **assumes** *prime* m $k \geq 1$
shows $Esigma\ (m^k) = (m^{k+(1::nat)} - (1::nat)) / (m-1)$

<proof>

lemma *prime-Esigma-mult*: **assumes** *prime* m **and** *prime* n **and** $m \neq n$

shows $Esigma\ (m*n) = (Esigma\ n)*(Esigma\ m)$

<proof>

lemma *gcd-Esigma-mult*:

assumes $gcd\ m\ n = 1$
shows $Esigma\ (m*n) = (Esigma\ m)*(Esigma\ n)$

<proof>

lemma *deficient-Esigma*:

assumes $Esigma\ m < 2*m$ **and** $m \geq 1$
shows *deficient-number* m
<proof>

lemma *abundant-Esigma*:

assumes $Esigma\ m > 2*m$ **and** $m \geq 1$
shows *abundant-number* m
<proof>

lemma *perfect-Esigma*:

assumes $Esigma\ m = 2*m$ **and** $m \geq 1$
shows *perfect-number* m
<proof>

2.3 Amicable Numbers; definitions, some lemmas and examples

definition *Amicable-pair* :: $nat \Rightarrow nat \Rightarrow bool$ (**infixr** *Amic* 80)

where $m\ Amic\ n \equiv ((m = aliquot-sum\ n) \wedge (n = aliquot-sum\ m))$

lemma *Amicable-pair-sym*: **fixes** $m::nat$ **and** $n::nat$

assumes $m\ Amic\ n$ **shows** $n\ Amic\ m$
<proof>

lemma *Amicable-pair-equiv-def*:

assumes $(m \text{ Amic } n)$ **and** $m \geq 1$ **and** $n \geq 1$
shows $(E\text{sigma } m = E\text{sigma } n) \wedge (E\text{sigma } m = m+n)$
<proof>

lemma *Amicable-pair-equiv-def-conv*:

assumes $m \geq 1$ **and** $n \geq 1$ **and** $(E\text{sigma } m = E\text{sigma } n) \wedge (E\text{sigma } m = m+n)$
shows $(m \text{ Amic } n)$
<proof>

definition *typeAmic* :: $\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat list}$

where *typeAmic* $n m =$
[[*card* { $i. \exists N. n = N * (\text{gcd } n m) \wedge \text{prime } i \wedge i \text{ dvd } N \wedge \neg i \text{ dvd } (\text{gcd } n m)$ }},
(*card* { $j. \exists M. m = M * (\text{gcd } n m) \wedge \text{prime } j \wedge j \text{ dvd } M \wedge \neg j \text{ dvd } (\text{gcd } n m)$ })]]

lemma *Amicable-pair-deficient*: **assumes** $m > n$ **and** $m \text{ Amic } n$

shows *deficient-number* m
<proof>

lemma *Amicable-pair-abundant*: **assumes** $m > n$ **and** $m \text{ Amic } n$

shows *abundant-number* n
<proof>

lemma *even-even-amicable*: **assumes** $m \text{ Amic } n$ **and** $m \geq 1$ **and** $n \geq 1$ **and** *even* m **and** *even* n

shows $(2 * m \neq n)$

<proof>

2.3.1 Regular Amicable Pairs

definition *regularAmicPair* :: $\text{nat} \Rightarrow \text{nat} \Rightarrow \text{bool}$ **where**

regularAmicPair $n m \longleftrightarrow (n \text{ Amic } m \wedge$
 $(\exists M N g. g = \text{gcd } m n \wedge m = M * g \wedge n = N * g \wedge \text{squarefree } M \wedge$
 $\text{squarefree } N \wedge \text{gcd } g M = 1 \wedge \text{gcd } g N = 1))$

lemma *regularAmicPair-sym*:

assumes *regularAmicPair* $n m$ **shows** *regularAmicPair* $m n$

<proof>

definition *irregularAmicPair* :: $\text{nat} \Rightarrow \text{nat} \Rightarrow \text{bool}$ **where**

irregularAmicPair $n m \longleftrightarrow ((n \text{ Amic } m) \wedge \neg \text{regularAmicPair } n m)$

lemma *irregularAmicPair-sym*:

assumes *irregularAmicPair* $n m$

shows *irregularAmicPair* $m n$

<proof>

2.3.2 Twin Amicable Pairs

We refer to the definition in [6]:

definition *twinAmicPair* :: nat ⇒ nat ⇒ bool **where**
twinAmicPair n m ↔
 (n Amic m) ∧ (¬(∃ k l. k > Min {n, m} ∧ k < Max {n, m} ∧ k Amic l))

lemma *twinAmicPair-sym*:
assumes *twinAmicPair* n m
shows *twinAmicPair* m n
 ⟨proof⟩

2.3.3 Isotopic Amicable Pairs

A way of generating an amicable pair from a given amicable pair under certain conditions is given below. Such amicable pairs are called Isotopic [2].

lemma *isotopic-amicable-pair*:
fixes m n g h M N :: nat
assumes m Amic n **and** m ≥ 1 **and** n ≥ 1 **and** m = g * M **and** n = g * N
and *Esigma* h = (h/g) * *Esigma* g **and** h ≠ g **and** h > 1 **and** g > 1
and gcd g M = 1 **and** gcd g N = 1 **and** gcd h M = 1 **and** gcd h N = 1
shows (h * M) Amic (h * N)

⟨proof⟩

lemma *isotopic-pair-example1*:
assumes (3³*5*11*17*227) Amic (3³*5*23*37*53)
shows (3²*7*13*11*17*227) Amic (3²*7*13*23*37*53)

⟨proof⟩

2.3.4 Betrothed (Quasi-Amicable) Pairs

We refer to the definition in [7]:

definition *QuasiAmicable-pair* :: nat ⇒ nat ⇒ bool (**infixr** *QAmic* 80)
where m *QAmic* n ↔ (m + 1 = aliquot-sum n) ∧ (n + 1 = aliquot-sum m)

lemma *QuasiAmicable-pair-sym* :
assumes m *QAmic* n **shows** n *QAmic* m
 ⟨proof⟩

lemma *QuasiAmicable-example*:
shows 48 *QAmic* 75

⟨proof⟩

2.3.5 Breeders

definition *breeder-pair* :: nat ⇒ nat ⇒ bool (infixr breeder 80)

where $m \text{ breeder } n \equiv (\exists x \in \mathbf{N}. x > 0 \wedge \text{Esigma } m = m + n * x \wedge \text{Esigma } m = (\text{Esigma } n) * (x + 1))$

lemma *breederAmic*:

fixes $x :: \text{nat}$

assumes $x > 0$ and $\text{Esigma } n = n + m * x$ and $\text{Esigma } n = \text{Esigma } m * (x + 1)$

and *prime* x and $\neg(x \text{ dvd } m)$

shows $n \text{ Amic } (m * x)$

⟨proof⟩

2.3.6 More examples

The first odd-odd amicable pair was discovered by Euler [2]. In the following proof, amicability is shown using the properties of Euler's sigma function.

lemma *odd-odd-amicable-Euler: 69615 Amic 87633*

⟨proof⟩

The following is the smallest odd-odd amicable pair [2]. In the following proof, amicability is shown directly by evaluating the sets of divisors.

lemma *Amicable-pair-example-smallest-odd-odd: 12285 Amic 14595*

⟨proof⟩

3 Euler's Rule

We present Euler's Rule as in [2]. The proof has been reconstructed.

theorem *Euler-Rule-Amicable*:

fixes $k \ l \ f \ p \ q \ r \ m \ n :: \text{nat}$

assumes $k > l$ and $l \geq 1$ and $f = 2^{l+1}$

and *prime* p and *prime* q and *prime* r

and $p = 2^{k-l} * f - 1$ and $q = 2^k * f - 1$ and $r = 2^{2*k-l} * f^2 - 1$

and $m = 2^k * p * q$ and $n = 2^k * r$

shows $m \text{ Amic } n$

⟨proof⟩

Another approach by Euler [2]:

theorem *Euler-Rule-Amicable-1*:

fixes $m \ n \ a :: \text{nat}$

assumes $m \geq 1$ and $n \geq 1$ and $a \geq 1$

and $\text{Esigma } m = \text{Esigma } n$ and $\text{Esigma } a * \text{Esigma } m = a * (m + n)$

and $\text{gcd } a \ m = 1$ and $\text{gcd } a \ n = 1$

shows $(a * m) \text{ Amic } (a * n)$

<proof>

4 Thābit ibn Qurra's Rule and more examples

Euler's Rule (theorem Euler_Rule_Amicable) is actually a generalisation of the following rule by Thābit ibn Qurra from the 9th century [2]. Thābit ibn Qurra's Rule is the special case for $l = 1$ thus $f = 3$.

corollary *Thabit-ibn-Qurra-Rule-Amicable:*

fixes $k\ l\ f\ p\ q\ r :: \text{nat}$

assumes $k > 1$ **and** *prime* p **and** *prime* q **and** *prime* r

and $p = 2^{k-1} * 3 - 1$ **and** $q = 2^k * 3 - 1$ **and** $r = 2^{2k-1} * 9 - 1$

shows $((2^k)*p*q)$ *Amic* $((2^k)*r)$

<proof>

In the following three example of amicable pairs, instead of evaluating the sum of the divisors or using the properties of Euler's sigma function as it was done in the previous examples, we prove amicability more directly as we can apply Thābit ibn Qurra's Rule.

The following is the first example of an amicable pair known to the Pythagoreans and can be derived from Thābit ibn Qurra's Rule with $k = 2$ [2].

lemma *Amicable-Example-Pythagoras:*

shows 220 *Amic* 284

<proof>

The following example of an amicable pair was (re)discovered by Fermat and can be derived from Thābit ibn Qurra's Rule with $k = 4$ [2].

lemma *Amicable-Example-Fermat:*

shows 17296 *Amic* 18416

<proof>

The following example of an amicable pair was (re)discovered by Descartes and can be derived from Thābit ibn Qurra's Rule with $k = 7$ [2].

lemma *Amicable-Example-Descartes:*

shows 9363584 *Amic* 9437056

<proof>

In fact, the Amicable Pair (220, 284) is Regular and of type (2,1):

lemma *regularAmicPairExample: regularAmicPair 220 284 \wedge typeAmic 220 284 = [2, 1]*

<proof>

lemma *abundant220ex: abundant-number 220*
 ⟨*proof*⟩

lemma *deficient284ex: deficient-number 284*
 ⟨*proof*⟩

5 Te Riele's Rule and Borho's Rule with breeders

With the following rule [2] we can get an amicable pair from a known amicable pair under certain conditions.

theorem *teRiele-Rule-Amicable:*

fixes $a\ u\ p\ r\ c\ q :: \text{nat}$
assumes $a \geq 1$ **and** $u \geq 1$
and *prime* p **and** *prime* r **and** *prime* c **and** *prime* q **and** $r \neq c$
and $\neg(p \text{ dvd } a)$ **and** $(a*u) \text{ Amic } (a*p)$ **and** $\text{gcd } a\ (r*c)=1$
and $q = r+c+u$ **and** $\text{gcd } (a*u)\ q=1$ **and** $r*c = p*(r + c + u) + p + u$
shows $(a*u*q) \text{ Amic } (a*r*c)$

⟨*proof*⟩

By replacing the assumption that $(a*u) \text{ Amic } (a*p)$ in the above rule by te Riele with the assumption that $(a*u) \text{ breeder } u$, we obtain Borho's Rule with breeders [2].

theorem *Borho-Rule-breeders-Amicable:*

fixes $a\ u\ r\ c\ q\ x :: \text{nat}$
assumes $x \geq 1$ **and** $a \geq 1$ **and** $u \geq 1$
and *prime* r **and** *prime* c **and** *prime* q **and** $r \neq c$
and $\text{Esigma } (a*u) = a*u + a*x$ $\text{Esigma } (a*u) = (\text{Esigma } a)*(x+1)$ **and** $\text{gcd } a\ (r * c) = 1$
and $\text{gcd } (a*u)\ q = 1$ **and** $r * c = x+u + x*u + r*x + x*c$ **and** $q = r+c+u$
shows $(a*u*q) \text{ Amic } (a*r*c)$

⟨*proof*⟩

no-notation *divisor (infixr divisor 80)*

6 Acknowledgements

The author was supported by the ERC Advanced Grant ALEXANDRIA (Project 742178) funded by the European Research Council and led by Professor Lawrence Paulson at the University of Cambridge, UK. Many thanks to Lawrence Paulson for his help and suggestions. Number divisors were initially looked up on <https://onlinemathtools.com/find-all-divisors>.

end

References

- [1] E. Escott. Amicable numbers. *Scripta Mathematica*, 12:61–72, 1946.
- [2] M. García, J. Pedersen, and H. te Riele. Amicable pairs, a survey. *REPORT MAS-R0307*, 2003.
- [3] M. García, J. Pedersen, and H. te Riele. Amicable pairs, a survey. *Fields Institute Communications*, 41:1–19, 2004.
- [4] E. Sandifer. Amicable pairs. 2005. How Euler Did It, The Euler Archive.
- [5] Wikipedia. Aliquot sum. https://en.wikipedia.org/wiki/Aliquot_sum, 2020.
- [6] Wikipedia. Amicable numbers. https://en.wikipedia.org/wiki/Amicable_numbers, 2020.
- [7] Wikipedia. Betrothed numbers. https://en.wikipedia.org/wiki/Betrothed_numbers, 2020.