

Allen's Interval Calculus

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theory *xor-cal*

imports

Main

begin

definition *xor*::*bool* \Rightarrow *bool* \Rightarrow *bool* (**infixl** \oplus 60)

where *xor* *A B* \equiv (*A* \wedge \neg *B*) \vee (\neg *A* \wedge *B*)

declare *xor-def* [*simp*]

interpretation *bool*:*semigroup* (\oplus)

\langle *proof* \rangle

lemma *xor-distr-L* [*simp*]: $A \oplus (B \oplus C) = (A \wedge \neg B \wedge \neg C) \vee (A \wedge B \wedge C) \vee (\neg A \wedge B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge C)$

\langle *proof* \rangle

lemma *xor-distr-R* [*simp*]: $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

\langle *proof* \rangle

end

theory *axioms*

imports

Main xor-cal

begin

1 Axioms

We formalize Allen's definition of theory of time in term of intervals (Allen, 1983). Two relations, namely meets and equality, are defined between intervals. Two interval meets if they are adjacent A set of 5 axioms ((M1) \sim (M5)) are then defined based on relation meets.

We define a class interval whose assumptions are (i) properties of relations meets and, (ii) axioms (M1) \sim (M5).

class *interval* =

fixes

meets::'*a* \Rightarrow '*a* \Rightarrow *bool* (**infixl** \parallel 60) **and**

I::'*a* \Rightarrow *bool*

assumes

meets-atrans:: $\llbracket (p \parallel q); (q \parallel r) \rrbracket \Longrightarrow \neg(p \parallel r)$ **and**

meets-irrefl:: $\mathcal{I} p \Longrightarrow \neg(p \parallel p)$ **and**

meets-asym:: $(p \parallel q) \Longrightarrow \neg(q \parallel p)$ **and**

meets-wd: $p \parallel q \implies \mathcal{I} p \wedge \mathcal{I} q$ **and**

M1: $\llbracket (p \parallel q); (p \parallel s); (r \parallel q) \rrbracket \implies (r \parallel s)$ **and**

M2: $\llbracket (p \parallel q); (r \parallel s) \rrbracket \implies p \parallel s \oplus ((\exists t. (p \parallel t) \wedge (t \parallel s)) \oplus (\exists t. (r \parallel t) \wedge (t \parallel q)))$ **and**

M3: $\mathcal{I} p \implies (\exists q r. q \parallel p \wedge p \parallel r)$ **and**

M4: $\llbracket p \parallel q; q \parallel s; p \parallel r; r \parallel s \rrbracket \implies q = r$ **and**

M5 *exist*: $p \parallel q \implies (\exists r s t. r \parallel p \wedge p \parallel q \wedge q \parallel s \wedge r \parallel t \wedge t \parallel s)$

lemma (*in interval*) *trans2*: $\llbracket p \parallel t; t \parallel r; r \parallel q \rrbracket \implies \neg p \parallel q$
<proof>

lemma (*in interval*) *nontrans1*: $u \parallel r \implies \neg (\exists t. u \parallel t \wedge t \parallel r)$
<proof>

lemma (*in interval*) *nontrans2*: $u \parallel r \implies \neg (\exists t. r \parallel t \wedge t \parallel u)$
<proof>

lemma (*in interval*) *nonmeets1*: $\neg (u \parallel r \wedge r \parallel u)$
<proof>

lemma (*in interval*) *nonmeets2*: $\llbracket \mathcal{I} u; \mathcal{I} r \rrbracket \implies \neg (u \parallel r \wedge u = r)$
<proof>

lemma (*in interval*) *nonmeets3*: $\neg (u \parallel r \wedge (\exists p. u \parallel p \wedge p \parallel r))$
<proof>

lemma (*in interval*) *nonmeets4*: $\neg (u \parallel r \wedge (\exists p. r \parallel p \wedge p \parallel u))$
<proof>

lemma (*in interval*) *elimmeets*: $(p \parallel s \wedge (\exists t. p \parallel t \wedge t \parallel s) \wedge (\exists t. r \parallel t \wedge t \parallel q))$
 $= \text{False}$
<proof>

lemma (*in interval*) *M5exist-var*:

assumes $x \parallel y \ y \parallel z \ z \parallel w$

shows $\exists t. x \parallel t \wedge t \parallel w$

<proof>

lemma (*in interval*) *M5exist-var2*:

assumes $p \parallel q$

shows $\exists r1 \ r2 \ r3 \ s \ t. r1 \parallel r2 \wedge r2 \parallel r3 \wedge r3 \parallel p \wedge p \parallel q \wedge q \parallel s \wedge r1 \parallel t \wedge t \parallel s$

<proof>

lemma (*in interval*) *M5exist-var3*:

assumes $k \parallel l$ **and** $l \parallel q$ **and** $q \parallel t$ **and** $t \parallel r$

shows $\exists lqt. k \parallel lqt \wedge lqt \parallel r$

<proof>

lemma *cre*:

assumes $r \in \{e, m, b, ov, s, f, d, m^{\wedge}-1, b^{\wedge}-1, ov^{\wedge}-1, s^{\wedge}-1, f^{\wedge}-1, d^{\wedge}-1\}$

shows $r \circ e = r$

<proof>

lemmas $ceb = cer[of\ b]$

lemmas $cebi = cer[of\ b^{\wedge}-1]$

lemmas $cem = cer[of\ m]$

lemmas $cemi = cer[of\ m^{\wedge}-1]$

lemmas $cee = cer[of\ e]$

lemmas $ces = cer[of\ s]$

lemmas $cesi = cer[of\ s^{\wedge}-1]$

lemmas $cef = cer[of\ f]$

lemmas $cefi = cer[of\ f^{\wedge}-1]$

lemmas $ceov = cer[of\ ov]$

lemmas $ceovi = cer[of\ ov^{\wedge}-1]$

lemmas $ced = cer[of\ d]$

lemmas $cedi = cer[of\ d^{\wedge}-1]$

lemmas $cbe = cre[of\ b]$

lemmas $cbie = cre[of\ b^{\wedge}-1]$

lemmas $cme = cre[of\ m]$

lemmas $cmie = cre[of\ m^{\wedge}-1]$

lemmas $cse = cre[of\ s]$

lemmas $csie = cre[of\ s^{\wedge}-1]$

lemmas $cfe = cre[of\ f]$

lemmas $cfie = cre[of\ f^{\wedge}-1]$

lemmas $cove = cre[of\ ov]$

lemmas $covie = cre[of\ ov^{\wedge}-1]$

lemmas $cde = cre[of\ d]$

lemmas $cdie = cre[of\ d^{\wedge}-1]$

3.2 r-composition

We prove compositions of the form $r_1 \circ r_2 \subseteq r$, where r is a basic relation.

method (**in** *arelations*) *r-compose* **uses** $r1\ r2\ r3 = ((auto, (subst\ (asm)\ r1\), (subst\ (asm)\ r2), (subst\ r3)) , (meson\ M5exist-var))$

lemma (**in** *arelations*) $cbb:b\ O\ b \subseteq b$

<proof>

lemma (**in** *arelations*) $cbm:b\ O\ m \subseteq b$

<proof>

lemma $cbov:b\ O\ ov \subseteq b$

<proof>

lemma $cbfi:b\ O\ f^{\wedge}-1 \subseteq b$

<proof>

lemma *cdbi*: $b \ O \ d^{-1} \subseteq b$
<proof>

lemma *cbs*: $b \ O \ s \subseteq b$
<proof>

lemma *cbsi*: $b \ O \ s^{-1} \subseteq b$
<proof>

lemma (*in arelations*) *cmb*: $m \ O \ b \subseteq b$
<proof>

lemma *cmm*: $m \ O \ m \subseteq b$
<proof>

lemma *cmov*: $m \ O \ ov \subseteq b$
<proof>

lemma *cmfi*: $m \ O \ f^{-1} \subseteq b$
<proof>

lemma *cmdi*: $m \ O \ d^{-1} \subseteq b$
<proof>

lemma *cms*: $m \ O \ s \subseteq m$
<proof>

lemma *cmsi*: $m \ O \ s^{-1} \subseteq m$
<proof>

lemma *covb*: $ov \ O \ b \subseteq b$
<proof>

lemma *covm*: $ov \ O \ m \subseteq b$
<proof>

lemma *covs*: $ov \ O \ s \subseteq ov$
<proof>

lemma *cfib*: $f^{-1} \ O \ b \subseteq b$
<proof>

lemma *cfim*: $f^{-1} \ O \ m \subseteq m$
<proof>

lemma *cfiov*: $f^{-1} \ O \ ov \subseteq ov$
<proof>

lemma *cfif*: $f^{-1} O f^{-1} \subseteq f^{-1}$
<proof>

lemma *cfid*: $f^{-1} O d^{-1} \subseteq d^{-1}$
<proof>

lemma *cfis*: $f^{-1} O s \subseteq ov$
<proof>

lemma *cfisi*: $f^{-1} O s^{-1} \subseteq d^{-1}$
<proof>

lemma *cdifi*: $d^{-1} O f^{-1} \subseteq d^{-1}$
<proof>

lemma *cdidi*: $d^{-1} O d^{-1} \subseteq d^{-1}$
<proof>

lemma *cdisi*: $d^{-1} O s^{-1} \subseteq d^{-1}$
<proof>

lemma *csb*: $s O b \subseteq b$
<proof>

lemma *esm*: $s O m \subseteq b$
<proof>

lemma *css*: $s O s \subseteq s$
<proof>

lemma *csifi*: $s^{-1} O f^{-1} \subseteq d^{-1}$
<proof>

lemma *csidi*: $s^{-1} O d^{-1} \subseteq d^{-1}$
<proof>

lemma *cdb*: $d O b \subseteq b$
<proof>

lemma *cdm*: $d O m \subseteq b$
<proof>

lemma *cfb*: $f O b \subseteq b$
<proof>

lemma *cfm*: $f O m \subseteq m$
<proof>

3.3 α -composition

We prove compositions of the form $r_1 \circ r_2 \subseteq s \cup ov \cup d$.

lemma (in *arelations*) $cmd:m \ O \ d \subseteq s \cup ov \cup d$
<proof>

lemma (in *arelations*) $cmf:m \ O \ f \subseteq s \cup ov \cup d$
<proof>

lemma $cmovi:m \ O \ ov^{\wedge-1} \subseteq s \cup ov \cup d$
<proof>

lemma $covd:ov \ O \ d \subseteq s \cup ov \cup d$
<proof>

lemma $covf:ov \ O \ f \subseteq s \cup ov \cup d$
<proof>

lemma $cfid:f^{\wedge-1} \ O \ d \subseteq s \cup ov \cup d$
<proof>

lemma $cfov:f \ O \ ov \subseteq ov \cup s \cup d$
<proof>

We prove compositions of the form $r_1 \circ r_2 \subseteq ov \cup f^{-1} \cup d^{-1}$.

lemma $covsi:ov \ O \ s^{\wedge-1} \subseteq ov \cup f^{\wedge-1} \cup d^{\wedge-1}$
<proof>

lemma $cdim:d^{\wedge-1} \ O \ m \subseteq ov \cup d^{\wedge-1} \cup f^{\wedge-1}$
<proof>

lemma $cdiov:d^{\wedge-1} \ O \ ov \subseteq ov \cup f^{\wedge-1} \cup d^{\wedge-1}$
<proof>

lemma $cdis:d^{\wedge-1} \ O \ s \subseteq ov \cup f^{\wedge-1} \cup d^{\wedge-1}$
<proof>

lemma $csim:s^{\wedge-1} \ O \ m \subseteq ov \cup f^{\wedge-1} \cup d^{\wedge-1}$
<proof>

lemma $csiiov:s^{\wedge-1} \ O \ ov \subseteq ov \cup f^{\wedge-1} \cup d^{\wedge-1}$
<proof>

lemma $covim:ov^{\wedge-1} \ O \ m \subseteq ov \cup f^{\wedge-1} \cup d^{\wedge-1}$
<proof>

We prove compositions of the form $r_1 \circ r_2 \subseteq b \cup m \cup ov$.

lemma *covov:ov* $O\ ov \subseteq b \cup m \cup ov$
(proof)

lemma *covfi:ov* $O\ f^{-1} \subseteq b \cup m \cup ov$
(proof)

lemma *csov:s* $O\ ov \subseteq b \cup m \cup ov$
(proof)

lemma *csfi:s* $O\ f^{-1} \subseteq b \cup m \cup ov$
(proof)

We prove compositions of the form $r_1 \circ r_2 \subseteq f \cup f^{-1} \cup e$.

lemma *cmmi:m* $O\ m^{-1} \subseteq f \cup f^{-1} \cup e$
(proof)

lemma *cfif:f^{-1}* $O\ f \subseteq e \cup f^{-1} \cup f$
(proof)

lemma *cffif:f* $O\ f^{-1} \subseteq e \cup f \cup f^{-1}$
(proof)

We prove compositions of the form $r_1 \circ r_2 \subseteq e \cup s \cup s^{-1}$.

lemma *cssi:s* $O\ s^{-1} \subseteq e \cup s \cup s^{-1}$
(proof)

lemma *csis:s^{-1}* $O\ s \subseteq e \cup s \cup s^{-1}$
(proof)

lemma *cmim:m^{-1}* $O\ m \subseteq s \cup s^{-1} \cup e$
(proof)

3.4 β -composition

We prove compositions of the form $r_1 \circ r_2 \subseteq b \cup m \cup ov \cup s \cup d$.

lemma *cbd:b* $O\ d \subseteq b \cup m \cup ov \cup s \cup d$
(proof)

lemma *cbf:b* $O\ f \subseteq b \cup m \cup ov \cup s \cup d$
(proof)

lemma *cbovi:b* $O\ ov^{-1} \subseteq b \cup m \cup ov \cup s \cup d$
(proof)

lemma *cbmi:b* $O\ m^{-1} \subseteq b \cup m \cup ov \cup s \cup d$

<proof>

lemma *cdov*: $d \circ ov \subseteq b \cup m \cup ov \cup s \cup d$
<proof>

lemma *cdfi*: $d \circ f^{-1} \subseteq b \cup m \cup ov \cup s \cup d$
<proof>

We prove compositions of the form $r_1 \circ r_2 \subseteq b \cup m \cup ov \cup f^{-1} \cup d^{-1}$.

lemma *covdi*: $ov \circ d^{-1} \subseteq b \cup m \cup ov \cup f^{-1} \cup d^{-1}$
<proof>

lemma *cdib*: $d^{-1} \circ b \subseteq b \cup m \cup ov \cup f^{-1} \cup d^{-1}$
<proof>

lemma *csdi*: $s \circ d^{-1} \subseteq b \cup m \cup ov \cup f^{-1} \cup d^{-1}$
<proof>

lemma *csib*: $s^{-1} \circ b \subseteq b \cup m \cup ov \cup f^{-1} \cup d^{-1}$
<proof>

lemma *covib*: $ov^{-1} \circ b \subseteq b \cup m \cup ov \cup f^{-1} \cup d^{-1}$
<proof>

lemma *cmib*: $m^{-1} \circ b \subseteq b \cup m \cup ov \cup f^{-1} \cup d^{-1}$
<proof>

3.5 γ -composition

We prove compositions of the form $r_1 \circ r_2 \subseteq ov \cup s \cup d \cup f \cup e \cup f^{-1} \cup d^{-1} \cup s^{-1} \cup ov^{-1}$.

lemma *covovi*: $ov \circ ov^{-1} \subseteq e \cup ov \cup ov^{-1} \cup d \cup d^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1}$
<proof>

lemma *cdid*: $d^{-1} \circ d \subseteq e \cup ov \cup ov^{-1} \cup d \cup d^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1}$
<proof>

lemma *coviov*: $ov^{-1} \circ ov \subseteq e \cup ov \cup ov^{-1} \cup d \cup d^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1}$
<proof>

3.6 γ -composition

We prove compositions of the form $r_1 \circ r_2 \subseteq b \cup m \cup ov \cup s \cup d \cup f \cup e \cup f^{-1} \cup d^{-1} \cup s^{-1} \cup ov^{-1} \cup b^{-1} \cup m^{-1}$.

lemma *cbbi*: $b \circ b^{-1} \subseteq b \cup b^{-1} \cup m \cup m^{-1} \cup e \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup d \cup d^{-1} \cup f \cup f^{-1}$ (is $b \circ b^{-1} \subseteq ?R$)

<proof>

lemma *cbib*: $b^{-1} \circ b \subseteq b \cup b^{-1} \cup m \cup m^{-1} \cup e \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup d \cup d^{-1} \cup f \cup f^{-1}$ (is $b^{-1} \circ b \subseteq ?R$)

<proof>

lemma *cddi*: $d \circ d^{-1} \subseteq b \cup b^{-1} \cup m \cup m^{-1} \cup e \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup d \cup d^{-1} \cup f \cup f^{-1}$ (is $d \circ d^{-1} \subseteq ?R$)

<proof>

3.7 The rest of the composition table

Because of the symmetry $(r_1 \circ r_2)^{-1} = r_2^{-1} \circ r_1^{-1}$, the rest of the compositions is easily deduced.

lemma *cmbi*: $m \circ b^{-1} \subseteq b^{-1} \cup m^{-1} \cup s^{-1} \cup ov^{-1} \cup d^{-1}$

<proof>

lemma *covmi*: $ov \circ m^{-1} \subseteq ov^{-1} \cup d^{-1} \cup s^{-1}$

<proof>

lemma *covbi*: $ov \circ b^{-1} \subseteq b^{-1} \cup m^{-1} \cup s^{-1} \cup ov^{-1} \cup d^{-1}$

<proof>

lemma *cfiovi*: $f^{-1} \circ ov^{-1} \subseteq ov^{-1} \cup s^{-1} \cup d^{-1}$

<proof>

lemma *cfimi*: $(f^{-1} \circ m^{-1}) \subseteq s^{-1} \cup ov^{-1} \cup d^{-1}$

<proof>

lemma *cfibi*: $f^{-1} \circ b^{-1} \subseteq b^{-1} \cup m^{-1} \cup ov^{-1} \cup s^{-1} \cup d^{-1}$

<proof>

lemma *cdif*: $d^{-1} \circ f \subseteq ov^{-1} \cup s^{-1} \cup d^{-1}$

<proof>

lemma *cdiovi*: $d^{-1} \circ ov^{-1} \subseteq ov^{-1} \cup s^{-1} \cup d^{-1}$

<proof>

lemma *cdimi*: $d^{-1} \circ m^{-1} \subseteq s^{-1} \cup ov^{-1} \cup d^{-1}$

<proof>

lemma *cdibi*: $d^{-1} \circ b^{-1} \subseteq b^{-1} \cup m^{-1} \cup ov^{-1} \cup s^{-1} \cup d^{-1}$

<proof>

lemma *csd*: $s \ O \ d \subseteq d$
<proof>

lemma *csf*: $s \ O \ f \subseteq d$
<proof>

lemma *csovi*: $s \ O \ ov^{\wedge-1} \subseteq ov^{\wedge-1} \cup f \cup d$
<proof>

lemma *csmi*: $s \ O \ m^{\wedge-1} \subseteq m^{\wedge-1}$
<proof>

lemma *csbi*: $s \ O \ b^{\wedge-1} \subseteq b^{\wedge-1}$
<proof>

lemma *csisi*: $s^{\wedge-1} \ O \ s^{\wedge-1} \subseteq s^{\wedge-1}$
<proof>

lemma *csid*: $s^{\wedge-1} \ O \ d \subseteq ov^{\wedge-1} \cup f \cup d$
<proof>

lemma *csif*: $s^{\wedge-1} \ O \ f \subseteq ov^{\wedge-1}$
<proof>

lemma *csiovi*: $s^{\wedge-1} \ O \ ov^{\wedge-1} \subseteq ov^{\wedge-1}$
<proof>

lemma *csimi*: $s^{\wedge-1} \ O \ m^{\wedge-1} \subseteq m^{\wedge-1}$
<proof>

lemma *csibi*: $s^{\wedge-1} \ O \ b^{\wedge-1} \subseteq b^{\wedge-1}$
<proof>

lemma *cds*: $d \ O \ s \subseteq d$
<proof>

lemma *cdsi*: $d \ O \ s^{\wedge-1} \subseteq b^{\wedge-1} \cup m^{\wedge-1} \cup ov^{\wedge-1} \cup f \cup d$
<proof>

lemma *cdd*: $d \ O \ d \subseteq d$
<proof>

lemma *cdf*: $d \ O \ f \subseteq d$
<proof>

lemma *cdovi*: $d \ O \ ov^{\wedge-1} \subseteq b^{\wedge-1} \cup m^{\wedge-1} \cup ov^{\wedge-1} \cup f \cup d$
<proof>

lemma *cdmi*: $d \ O \ m^{\wedge-1} \subseteq b^{\wedge-1}$

<proof>

lemma *cdbi*: $d \cap b^{-1} \subseteq b^{-1}$
<proof>

lemma *cfdi*: $f \cap d^{-1} \subseteq b^{-1} \cup m^{-1} \cup ov^{-1} \cup s^{-1} \cup d^{-1}$
<proof>

lemma *cfs*: $f \cap s \subseteq d$
<proof>

lemma *cfsi*: $f \cap s^{-1} \subseteq b^{-1} \cup m^{-1} \cup ov^{-1}$
<proof>

lemma *cfid*: $f \cap d \subseteq d$
<proof>

lemma *cff*: $f \cap f \subseteq f$
<proof>

lemma *cfovi*: $f \cap ov^{-1} \subseteq b^{-1} \cup m^{-1} \cup ov^{-1}$
<proof>

lemma *cfmi*: $f \cap m^{-1} \subseteq b^{-1}$
<proof>

lemma *cfbi*: $f \cap b^{-1} \subseteq b^{-1}$
<proof>

lemma *covifi*: $ov^{-1} \cap f^{-1} \subseteq ov^{-1} \cup s^{-1} \cup d^{-1}$
<proof>

lemma *covidi*: $ov^{-1} \cap d^{-1} \subseteq b^{-1} \cup m^{-1} \cup s^{-1} \cup ov^{-1} \cup d^{-1}$
<proof>

lemma *covis*: $ov^{-1} \cap s \subseteq ov^{-1} \cup f \cup d$
<proof>

lemma *covisi*: $ov^{-1} \cap s^{-1} \subseteq b^{-1} \cup m^{-1} \cup ov^{-1}$
<proof>

lemma *covid*: $ov^{-1} \cap d \subseteq ov^{-1} \cup f \cup d$
<proof>

lemma *covif*: $ov^{-1} \cap f \subseteq ov^{-1}$
<proof>

lemma *coviovi*: $ov^{-1} \cap ov^{-1} \subseteq b^{-1} \cup m^{-1} \cup ov^{-1}$

<proof>

lemma *covimi:ov⁻¹ O m⁻¹ ⊆ b⁻¹*
<proof>

lemma *covibi:ov⁻¹ O b⁻¹ ⊆ b⁻¹*
<proof>

lemma *cmiov:m⁻¹ O ov ⊆ ov⁻¹ ∪ d ∪ f*
<proof>

lemma *cmift:m⁻¹ O f⁻¹ ⊆ m⁻¹*
<proof>

lemma *cmidi:m⁻¹ O d⁻¹ ⊆ b⁻¹*
<proof>

lemma *cmis:m⁻¹ O s ⊆ ov⁻¹ ∪ d ∪ f*
<proof>

lemma *cmisi:m⁻¹ O s⁻¹ ⊆ b⁻¹*
<proof>

lemma *cmid:m⁻¹ O d ⊆ ov⁻¹ ∪ d ∪ f*
<proof>

lemma *cmif:m⁻¹ O f ⊆ m⁻¹*
<proof>

lemma *cmiovi:m⁻¹ O ov⁻¹ ⊆ b⁻¹*
<proof>

lemma *cmimi:m⁻¹ O m⁻¹ ⊆ b⁻¹*
<proof>

lemma *cmibi:m⁻¹ O b⁻¹ ⊆ b⁻¹*
<proof>

lemma *cbim:b⁻¹ O m ⊆ b⁻¹ ∪ m⁻¹ ∪ ov⁻¹ ∪ f ∪ d*
<proof>

lemma *cbiov:b⁻¹ O ov ⊆ b⁻¹ ∪ m⁻¹ ∪ ov⁻¹ ∪ f ∪ d*
<proof>

lemma *cbifi:b⁻¹ O f⁻¹ ⊆ b⁻¹*
<proof>

lemma *cbidi:b⁻¹ O d⁻¹ ⊆ b⁻¹*
<proof>

lemma $cbis:b^{-1} O s \subseteq b^{-1} \cup m^{-1} \cup ov^{-1} \cup f \cup d$
<proof>

lemma $cbisi:b^{-1} O s^{-1} \subseteq b^{-1}$
<proof>

lemma $cbid:b^{-1} O d \subseteq b^{-1} \cup m^{-1} \cup ov^{-1} \cup f \cup d$
<proof>

lemma $cbif:b^{-1} O f \subseteq b^{-1}$
<proof>

lemma $cbiovi:b^{-1} O ov^{-1} \subseteq b^{-1}$
<proof>

lemma $cbimi:b^{-1} O m^{-1} \subseteq b^{-1}$
<proof>

lemma $cbibi:b^{-1} O b^{-1} \subseteq b^{-1}$
<proof>

3.8 Composition rules

named-theorems *ce-rules* **declare** $cem[ce-rules]$ **and** $ceb[ce-rules]$ **and** $ceov[ce-rules]$
and $ces[ce-rules]$ **and** $cef[ce-rules]$ **and** $ced[ce-rules]$ **and**
 $cemi[ce-rules]$ **and** $cebi[ce-rules]$ **and** $ceovi[ce-rules]$ **and** $cesi[ce-rules]$ **and** $cefi[ce-rules]$
and $cedi[ce-rules]$

named-theorems *cm-rules* **declare** $cme[cm-rules]$ **and** $cmb[cm-rules]$ **and** $cmm[cm-rules]$
and $cmov[cm-rules]$ **and** $cms[cm-rules]$ **and** $cmd[cm-rules]$ **and** $cmf[cm-rules]$
and
 $cmbi[cm-rules]$ **and** $cmmi[cm-rules]$ **and** $cmovi[cm-rules]$ **and** $cmsi[cm-rules]$ **and**
 $cmdi[cm-rules]$ **and** $cmfi[cm-rules]$

named-theorems *cb-rules* **declare** $cbe[cb-rules]$ **and** $cbm[cb-rules]$ **and** $cbb[cb-rules]$
and $cbov[cb-rules]$ **and** $cbs[cb-rules]$ **and** $cbd[cb-rules]$ **and** $cbf[cb-rules]$ **and**
 $cbbi[cb-rules]$ **and** $cbbi[cb-rules]$ **and** $cbovi[cb-rules]$ **and** $cbsi[cb-rules]$ **and** $cbdi[cb-rules]$
and $cbfi[cb-rules]$

named-theorems *cov-rules* **declare** $cove[cov-rules]$ **and** $covb[cov-rules]$ **and** $covb[cov-rules]$
and $covov[cov-rules]$ **and** $covs[cov-rules]$ **and** $covd[cov-rules]$ **and** $covf[cov-rules]$
and
 $covbi[cov-rules]$ **and** $covbi[cov-rules]$ **and** $covovi[cov-rules]$ **and** $covsi[cov-rules]$ **and**
 $covdi[cov-rules]$ **and** $covfi[cov-rules]$

named-theorems *cs-rules* **declare** $cse[cs-rules]$ **and** $csb[cs-rules]$ **and** $csb[cs-rules]$
and $csov[cs-rules]$ **and** $css[cs-rules]$ **and** $csd[cs-rules]$ **and** $csf[cs-rules]$ **and**
 $csbi[cs-rules]$ **and** $csbi[cs-rules]$ **and** $csovi[cs-rules]$ **and** $cssi[cs-rules]$ **and** $csdi[cs-rules]$

and *csfi*[*cs-rules*]

named-theorems *cf-rules* **declare** *cfe*[*cf-rules*] **and** *cfb*[*cf-rules*] **and** *cfb*[*cf-rules*]
and *cfov*[*cf-rules*] **and** *cfs* [*cf-rules*] **and** *cfid*[*cf-rules*] **and** *cff*[*cf-rules*] **and**
cfbi[*cf-rules*] **and** *cfbi*[*cf-rules*] **and** *cfovi*[*cf-rules*] **and** *cfsi*[*cf-rules*] **and** *cfdi*[*cf-rules*]
and *cff*[*cf-rules*]

named-theorems *cd-rules* **declare** *cde*[*cd-rules*] **and** *cdb*[*cd-rules*] **and** *cdb*[*cd-rules*]
and *cdov*[*cd-rules*] **and** *cds* [*cd-rules*] **and** *cdd*[*cd-rules*] **and** *cdf*[*cd-rules*] **and**
cdbi[*cd-rules*] **and** *cdbi*[*cd-rules*] **and** *cdovi*[*cd-rules*] **and** *cdsi*[*cd-rules*] **and** *cddi*[*cd-rules*]
and *cdfi*[*cd-rules*]

named-theorems *cmi-rules* **declare** *cmie*[*cmi-rules*] **and** *cmib*[*cmi-rules*] **and**
cmib[*cmi-rules*] **and** *cmiov*[*cmi-rules*] **and** *cmis* [*cmi-rules*] **and** *cmid*[*cmi-rules*]
and *cmif*[*cmi-rules*] **and**
cmibi[*cmi-rules*] **and** *cmibi*[*cmi-rules*] **and** *cmiovi*[*cmi-rules*] **and** *cmisi*[*cmi-rules*]
and *cmidi*[*cmi-rules*] **and** *cmifi*[*cmi-rules*]

named-theorems *cbi-rules* **declare** *cbie*[*cbi-rules*] **and** *cbim*[*cbi-rules*] **and** *cbib*[*cbi-rules*]
and *cbiov*[*cbi-rules*] **and** *cbis* [*cbi-rules*] **and** *cbid*[*cbi-rules*] **and** *cbif*[*cbi-rules*] **and**
cbimi[*cbi-rules*] **and** *cbibi*[*cbi-rules*] **and** *cbiovi*[*cbi-rules*] **and** *cbisi*[*cbi-rules*] **and**
cbidi[*cbi-rules*] **and** *cbifi*[*cbi-rules*]

named-theorems *covi-rules* **declare** *covie*[*covi-rules*] **and** *covib*[*covi-rules*] **and**
covib[*covi-rules*] **and** *coviov*[*covi-rules*] **and** *covis* [*covi-rules*] **and** *covid*[*covi-rules*]
and *covif*[*covi-rules*] **and**
covibi[*covi-rules*] **and** *covibi*[*covi-rules*] **and** *coviovi*[*covi-rules*] **and** *covisi*[*covi-rules*]
and *covidi*[*covi-rules*] **and** *covifi*[*covi-rules*]

named-theorems *csi-rules* **declare** *csie*[*csi-rules*] **and** *csib*[*csi-rules*] **and** *csib*[*csi-rules*]
and *csiov*[*csi-rules*] **and** *csis* [*csi-rules*] **and** *csid*[*csi-rules*] **and** *csif*[*csi-rules*] **and**
csibi[*csi-rules*] **and** *csibi*[*csi-rules*] **and** *csiovi*[*csi-rules*] **and** *csisi*[*csi-rules*] **and**
csidi[*csi-rules*] **and** *csifi*[*csi-rules*]

named-theorems *cfi-rules* **declare** *cfie*[*cfi-rules*] **and** *cfib*[*cfi-rules*] **and** *cfib*[*cfi-rules*]
and *cfiov*[*cfi-rules*] **and** *cfis* [*cfi-rules*] **and** *cfid*[*cfi-rules*] **and** *cfif*[*cfi-rules*] **and**
cfibi[*cfi-rules*] **and** *cfibi*[*cfi-rules*] **and** *cfiovi*[*cfi-rules*] **and** *cfisi*[*cfi-rules*] **and** *cfidi*[*cfi-rules*]
and *cfifi*[*cfi-rules*]

named-theorems *cdi-rules* **declare** *cdie*[*cdi-rules*] **and** *cdib*[*cdi-rules*] **and** *cdib*[*cdi-rules*]
and *cdiov*[*cdi-rules*] **and** *cdis* [*cdi-rules*] **and** *cdid*[*cdi-rules*] **and** *cdif*[*cdi-rules*] **and**
cdibi[*cdi-rules*] **and** *cdibi*[*cdi-rules*] **and** *cdiovi*[*cdi-rules*] **and** *cdisi*[*cdi-rules*] **and**
cdidi[*cdi-rules*] **and** *cdifi*[*cdi-rules*]

named-theorems *cre-rules* **declare** *cee*[*cre-rules*] **and** *cme*[*cre-rules*] **and** *cbe*[*cre-rules*]
and *cove*[*cre-rules*] **and** *cse*[*cre-rules*] **and** *cfe*[*cre-rules*] **and** *cde*[*cre-rules*] **and**
cmie[*cre-rules*] **and** *cbie*[*cre-rules*] **and** *covie*[*cre-rules*] **and** *csie*[*cre-rules*] **and**
cfie[*cre-rules*] **and** *cdie*[*cre-rules*]

named-theorems *crm-rules* **declare** *cem*[*crm-rules*] **and** *cbm*[*crm-rules*] **and** *cmm*[*crm-rules*] **and** *covm*[*crm-rules*] **and** *csm*[*crm-rules*] **and** *cfm*[*crm-rules*] **and** *cdm*[*crm-rules*] **and** *cmim*[*crm-rules*] **and** *cbim*[*crm-rules*] **and** *covim*[*crm-rules*] **and** *csim*[*crm-rules*] **and** *cfim*[*crm-rules*] **and** *cdim*[*crm-rules*]

named-theorems *crmi-rules* **declare** *cemi*[*crmi-rules*] **and** *cbmi*[*crmi-rules*] **and** *cmmi*[*crmi-rules*] **and** *covmi*[*crmi-rules*] **and** *csmi*[*crmi-rules*] **and** *cfmi*[*crmi-rules*] **and** *cdmi*[*crmi-rules*] **and** *cmimi*[*crmi-rules*] **and** *cbimi*[*crmi-rules*] **and** *covimi*[*crmi-rules*] **and** *csimi*[*crmi-rules*] **and** *cfmi*[*crmi-rules*] **and** *cdimi*[*crmi-rules*]

named-theorems *crs-rules* **declare** *ces*[*crs-rules*] **and** *cbs*[*crs-rules*] **and** *cms*[*crs-rules*] **and** *covs*[*crs-rules*] **and** *css*[*crs-rules*] **and** *cfs*[*crs-rules*] **and** *cds*[*crs-rules*] **and** *cmis*[*crs-rules*] **and** *cbis*[*crs-rules*] **and** *covis*[*crs-rules*] **and** *csis*[*crs-rules*] **and** *cfis*[*crs-rules*] **and** *cdis*[*crs-rules*]

named-theorems *crsi-rules* **declare** *cesi*[*crsi-rules*] **and** *cbsi*[*crsi-rules*] **and** *cmsi*[*crsi-rules*] **and** *covsi*[*crsi-rules*] **and** *cssi*[*crsi-rules*] **and** *cfsi*[*crsi-rules*] **and** *cdsi*[*crsi-rules*] **and** *cmisi*[*crsi-rules*] **and** *cbisi*[*crsi-rules*] **and** *covisi*[*crsi-rules*] **and** *csisi*[*crsi-rules*] **and** *cfisi*[*crsi-rules*] **and** *cdisi*[*crsi-rules*]

named-theorems *crb-rules* **declare** *ceb*[*crb-rules*] **and** *cbb*[*crb-rules*] **and** *cmb*[*crb-rules*] **and** *covb*[*crb-rules*] **and** *csb*[*crb-rules*] **and** *cfb*[*crb-rules*] **and** *cdb*[*crb-rules*] **and** *cmib*[*crb-rules*] **and** *cbib*[*crb-rules*] **and** *covib*[*crb-rules*] **and** *csib*[*crb-rules*] **and** *cfib*[*crb-rules*] **and** *cdib*[*crb-rules*]

named-theorems *crbi-rules* **declare** *cebi*[*crbi-rules*] **and** *cbbi*[*crbi-rules*] **and** *cmbi*[*crbi-rules*] **and** *covbi*[*crbi-rules*] **and** *csbi*[*crbi-rules*] **and** *cfbi*[*crbi-rules*] **and** *cdbi*[*crbi-rules*] **and** *cmibi*[*crbi-rules*] **and** *cbibi*[*crbi-rules*] **and** *covibi*[*crbi-rules*] **and** *csibi*[*crbi-rules*] **and** *cfibi*[*crbi-rules*] **and** *cdibi*[*crbi-rules*]

named-theorems *crov-rules* **declare** *ceov*[*crov-rules*] **and** *cbov*[*crov-rules*] **and** *cmov*[*crov-rules*] **and** *covov*[*crov-rules*] **and** *csov*[*crov-rules*] **and** *cfov*[*crov-rules*] **and** *cdov*[*crov-rules*] **and** *cmiov*[*crov-rules*] **and** *cbiov*[*crov-rules*] **and** *coviov*[*crov-rules*] **and** *csiouv*[*crov-rules*] **and** *cfiov*[*crov-rules*] **and** *cdiov*[*crov-rules*]

named-theorems *crovi-rules* **declare** *ceovi*[*crovi-rules*] **and** *cbovi*[*crovi-rules*] **and** *cmovi*[*crovi-rules*] **and** *covovi*[*crovi-rules*] **and** *csovi*[*crovi-rules*] **and** *cfovi*[*crovi-rules*] **and** *cdovi*[*crovi-rules*] **and** *cmiovi*[*crovi-rules*] **and** *cbiovi*[*crovi-rules*] **and** *coviovi*[*crovi-rules*] **and** *csiovi*[*crovi-rules*] **and** *cfiovi*[*crovi-rules*] **and** *cdiovi*[*crovi-rules*]

named-theorems *crf-rules* **declare** *cef*[*crf-rules*] **and** *cbf*[*crf-rules*] **and** *cmf*[*crf-rules*] **and** *covf*[*crf-rules*] **and** *csf*[*crf-rules*] **and** *cff*[*crf-rules*] **and** *cdf*[*crf-rules*] **and** *cmif*[*crf-rules*] **and** *cbif*[*crf-rules*] **and** *covif*[*crf-rules*] **and** *csif*[*crf-rules*] **and** *cfif*[*crf-rules*]

and *cdif*[*crf-rules*]

named-theorems *crfi-rules* **declare** *cefi*[*crfi-rules*] **and** *cbfi*[*crfi-rules*] **and** *cmfi*[*crfi-rules*]
and *covfi*[*crfi-rules*] **and** *csfi*[*crfi-rules*] **and** *cfffi*[*crfi-rules*] **and** *cdfi*[*crfi-rules*] **and**

cmifi[*crfi-rules*] **and** *cbifi*[*crfi-rules*] **and** *covifi*[*crfi-rules*] **and** *csifi*[*crfi-rules*] **and**
cfifi[*crfi-rules*] **and** *cdifi*[*crfi-rules*]

named-theorems *crd-rules* **declare** *ced*[*crd-rules*] **and** *cbd*[*crd-rules*] **and** *cmd*[*crd-rules*]
and *covd*[*crd-rules*] **and** *csd*[*crd-rules*] **and** *cfid*[*crd-rules*] **and** *cdd*[*crd-rules*] **and**
cmid[*crd-rules*] **and** *cbid*[*crd-rules*] **and** *covid*[*crd-rules*] **and** *csid*[*crd-rules*] **and**
cfid[*crd-rules*] **and** *cdid*[*crd-rules*]

named-theorems *crdi-rules* **declare** *cedi*[*crdi-rules*] **and** *cbdi*[*crdi-rules*] **and** *cmdi*[*crdi-rules*]
and *covdi*[*crdi-rules*] **and** *csdi*[*crdi-rules*] **and** *cfdi*[*crdi-rules*] **and** *cddi*[*crdi-rules*]
and
cmidi[*crdi-rules*] **and** *cbidi*[*crdi-rules*] **and** *covidi*[*crdi-rules*] **and** *csidi*[*crdi-rules*]
and *cfidi*[*crdi-rules*] **and** *cdidi*[*crdi-rules*]

end

theory *disjoint-relations*

imports

allen

begin

4 PD property

The 13 time interval relations (i.e. *e*, *b*, *m*, *s*, *f*, *d*, *ov* and their inverse relations) are pairwise disjoint.

lemma *em* : $e \cap m = \{\}$
<proof>

lemma *eb* : $e \cap b = \{\}$
<proof>

lemma *eov* : $e \cap ov = \{\}$
<proof>

lemma *es* : $e \cap s = \{\}$
<proof>

lemma $ef : e \cap f = \{\}$
<proof>

lemma $ed : e \cap d = \{\}$
<proof>

lemma $emi : e \cap m^{\wedge-1} = \{\}$
<proof>

lemma $ebi : e \cap b^{\wedge-1} = \{\}$
<proof>

lemma $eovi : e \cap ov^{\wedge-1} = \{\}$
<proof>

lemma $esi : e \cap s^{\wedge-1} = \{\}$
<proof>

lemma $efi : e \cap f^{\wedge-1} = \{\}$
<proof>

lemma $edi : e \cap d^{\wedge-1} = \{\}$
<proof>

lemma $mb : m \cap b = \{\}$
<proof>

lemma $mov : m \cap ov = \{\}$
<proof>

lemma $ms : m \cap s = \{\}$
<proof>

lemma $mf : m \cap f = \{\}$
<proof>

lemma $md : m \cap d = \{\}$
<proof>

lemma $mi : m \cap m^{\wedge-1} = \{\}$
<proof>

lemma $mbi : m \cap b^{\wedge-1} = \{\}$
<proof>

lemma $movi : m \cap ov^{\wedge-1} = \{\}$
<proof>

lemma $msi : m \cap s^{\wedge-1} = \{\}$
<proof>

lemma $mfi : m \cap f^{\wedge-1} = \{\}$
<proof>

lemma $mdi : m \cap d^{\wedge-1} = \{\}$
<proof>

lemma $bov : b \cap ov = \{\}$
<proof>

lemma $bs : b \cap s = \{\}$
<proof>

lemma $bf : b \cap f = \{\}$
<proof>

lemma $bd : b \cap d = \{\}$
<proof>

lemma $bmi : b \cap m^{\wedge-1} = \{\}$
<proof>

lemma $bi : b \cap b^{\wedge-1} = \{\}$
<proof>

lemma $bovi : b \cap ov^{\wedge-1} = \{\}$
<proof>

lemma $bsi : b \cap s^{\wedge-1} = \{\}$
<proof>

lemma $bfi : b \cap f^{\wedge-1} = \{\}$
<proof>

lemma $bdi : b \cap d^{\wedge-1} = \{\}$
<proof>

lemma $ovs : ov \cap s = \{\}$
<proof>

lemma $ovf : ov \cap f = \{\}$
<proof>

lemma $ovd : ov \cap d = \{\}$
 $\langle proof \rangle$

lemma $ovmi : ov \cap m^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $ovbi : ov \cap b^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $ovi : ov \cap ov^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $ovsi : ov \cap s^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $ovfi : ov \cap f^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $ovdi : ov \cap d^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $sf : s \cap f = \{\}$
 $\langle proof \rangle$

lemma $sd : s \cap d = \{\}$
 $\langle proof \rangle$

lemma $smi : s \cap m^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $sbi : s \cap b^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $sovi : s \cap ov^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $si : s \cap s^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $sfi : s \cap f^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $sdi : s \cap d^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $fd : f \cap d = \{\}$

$\langle proof \rangle$

lemma $fmi : f \cap m^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $fbi : f \cap b^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $fovi : f \cap ov^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $fsi : f \cap s^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $fi : f \cap f^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $fdi : f \cap d^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $dmi : d \cap m^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $dbi : d \cap b^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $dovi : d \cap ov^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $dsi : d \cap s^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $dfi : d \cap f^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $di : d \cap d^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $mibi : m^{\wedge-1} \cap b^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $miovi : m^{\wedge-1} \cap ov^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma $misi : m^{\wedge-1} \cap s^{\wedge-1} = \{\}$
 $\langle proof \rangle$

lemma *mifi* : $m^{-1} \cap f^{-1} = \{\}$
<proof>

lemma *midi* : $m^{-1} \cap d^{-1} = \{\}$
<proof>

lemma *bid* : $b^{-1} \cap d = \{\}$
<proof>

lemma *bimi* : $b^{-1} \cap m^{-1} = \{\}$
<proof>

lemma *biovi* : $b^{-1} \cap ov^{-1} = \{\}$
<proof>

lemma *bisi* : $b^{-1} \cap s^{-1} = \{\}$
<proof>

lemma *bifi* : $b^{-1} \cap f^{-1} = \{\}$
<proof>

lemma *bidi* : $b^{-1} \cap d^{-1} = \{\}$
<proof>

lemma *ovisi* : $ov^{-1} \cap s^{-1} = \{\}$
<proof>

lemma *ovifi* : $ov^{-1} \cap f^{-1} = \{\}$
<proof>

lemma *ovidi* : $ov^{-1} \cap d^{-1} = \{\}$
<proof>

lemma *sifi* : $s^{-1} \cap f^{-1} = \{\}$
<proof>

lemma *sidi* : $s^{-1} \cap d^{-1} = \{\}$
<proof>

lemma *fidi* : $f^{-1} \cap d^{-1} = \{\}$
<proof>

lemma *eei[simp]:* $e^{\wedge-1} = e$
<proof>

lemma *rdisj-sym:* $A \cap B = \{\} \implies B \cap A = \{\}$
<proof>

4.1 Intersection rules

named-theorems *e-rules* **declare** *em[e-rules]* **and** *eb[e-rules]* **and** *eov[e-rules]*
and *es[e-rules]* **and** *ef[e-rules]* **and** *ed[e-rules]* **and** *emi[e-rules]* **and** *ebi[e-rules]*
and *eovi[e-rules]*
and *esi[e-rules]* **and** *efi[e-rules]* **and** *edi[e-rules]*

named-theorems *m-rules* **declare** *em[THEN rdisj-sym, m-rules]* **and** *mb [m-rules]*
and *ms [m-rules]* **and** *mov [m-rules]* **and** *mf[m-rules]* **and**
md[m-rules] **and** *mi [m-rules]* **and** *mbi [m-rules]* **and** *movi [m-rules]* **and** *msi*
[m-rules] **and** *mfi [m-rules]* **and** *mdi [m-rules]* **and** *emi[m-rules]*

named-theorems *b-rules* **declare** *eb[THEN rdisj-sym, b-rules]* **and** *mb [THEN*
rdisj-sym, b-rules] **and** *bs [b-rules]* **and** *bov [b-rules]* **and** *bf[b-rules]* **and**
bd[b-rules] **and** *bmi [b-rules]* **and** *bi [b-rules]* **and** *bovi [b-rules]* **and** *bsi [b-rules]*
and *bfi [b-rules]* **and** *bdi [b-rules]* **and** *ebi[b-rules]*

named-theorems *ov-rules* **declare** *eov[THEN rdisj-sym, ov-rules]* **and** *mov [THEN*
rdisj-sym, ov-rules] **and** *ovs [ov-rules]* **and** *bov [THEN rdisj-sym,ov-rules]* **and**
ovf[ov-rules] **and**
ovd[ov-rules] **and** *ovmi [ov-rules]* **and** *ovi [ov-rules]* **and** *ovsi [ov-rules]* **and** *ovfi*
[ov-rules] **and** *ovdi [ov-rules]* **and** *eovi[ov-rules]*

named-theorems *s-rules* **declare** *es[THEN rdisj-sym, s-rules]* **and** *ms [THEN*
rdisj-sym, s-rules] **and** *ovs [THEN rdisj-sym, s-rules]* **and** *bs [THEN rdisj-sym,s-rules]*
and *sf[s-rules]* **and**
sd[s-rules] **and** *smi [s-rules]* **and** *sovi [s-rules]* **and** *si [s-rules]* **and** *sfi [s-rules]*
and *sdi [s-rules]*

named-theorems *d-rules* **declare** *ed[THEN rdisj-sym, d-rules]* **and** *md [THEN*
rdisj-sym, d-rules] **and** *sd [THEN rdisj-sym, d-rules]* **and** *fd[THEN rdisj-sym,*
d-rules] **and**
ovd[THEN rdisj-sym,d-rules] **and** *dmi [d-rules]* **and** *dovi [d-rules]* **and** *dsi [d-rules]*
and *dfi [d-rules]* **and** *di [d-rules]*

named-theorems *f-rules* **declare** *ef[THEN rdisj-sym, f-rules]* **and** *mf [THEN*
rdisj-sym, f-rules] **and** *sf [THEN rdisj-sym, f-rules]* **and** *ovf [THEN rdisj-sym,f-rules]*
and *fd[f-rules]* **and**
fmi [f-rules] **and** *fovi [f-rules]* **and** *fsi [f-rules]* **and** *fi [f-rules]* **and** *fdi [f-rules]*

end

theory *jointly-exhaustive*

imports

allen

begin

5 JE property

The 13 time interval relations are jointly exhaustive. For any two intervals x and y , we can find a basic relation r such that $(x, y) \in r$.

lemma (in *arelations*) *jointly-exhaustive*:

assumes $\mathcal{I} p \mathcal{I} q$

shows $(p::'a, q::'a) \in b \vee (p, q) \in m \vee (p, q) \in ov \vee (p, q) \in s \vee (p, q) \in d \vee (p, q) \in f^{-1} \vee (p, q) \in e \vee$

$(p, q) \in f \vee (p, q) \in s^{-1} \vee (p, q) \in d^{-1} \vee (p, q) \in ov^{-1} \vee (p, q) \in m^{-1} \vee (p, q) \in b^{-1}$ (is ?R)

<proof>

lemma (in *arelations*) *JE*:

assumes $\mathcal{I} p \mathcal{I} q$

shows $(p::'a, q::'a) \in b \cup m \cup ov \cup s \cup d \cup f^{-1} \cup e \cup f \cup s^{-1} \cup d^{-1} \cup ov^{-1} \cup m^{-1} \cup b^{-1}$

<proof>

end

theory *examples*

imports

disjoint-relations

begin

6 Examples

6.1 Compositions of non-basic relations

Basic relations are the 13 time interval relations. The unions of basic relations are also relations and their compositions is the union of compositions. We prove few of these compositions that are required in theory nest.thy.

method (in *arelations*) *e-compose* = (match conclusion in $e \ O \ b \subseteq - \Rightarrow \langle insert \ c \ e \ b, \ blast \rangle$

$| - \Rightarrow \langle match \ conclusion \ in \ e \ O \ m \subseteq - \Rightarrow \langle insert \ c \ e \ m, \ blast \rangle$ $| - \Rightarrow \langle fail \rangle$)

declare [[*simp-trace-depth-limit=4*]]

lemma *eovisidifmifiOm*: $(e \cup \text{ov}^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup m^{-1} \cup \widehat{f^{-1}}) \ O \ m \subseteq m \cup \text{ov} \cup \widehat{f^{-1}} \cup \widehat{d^{-1}} \cup s \cup s^{-1} \cup e$
 $\langle proof \rangle$

lemma *ovsmfidiesiOmi*: $(\text{ov} \cup s \cup m \cup \widehat{f^{-1}} \cup \widehat{d^{-1}} \cup e \cup \widehat{s^{-1}}) \ O \ m \widehat{-1} \subseteq \widehat{d^{-1}} \cup \widehat{s^{-1}} \cup \widehat{\text{ov}^{-1}} \cup \widehat{m^{-1}} \cup \widehat{f^{-1}} \cup f \cup e$
 $\langle proof \rangle$

lemma *ovsmfidiesiOm*: $(\text{ov} \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}) \ O \ m \subseteq b \cup \text{ov} \cup \widehat{f^{-1}} \cup \widehat{d^{-1}} \cup m$
 $\langle proof \rangle$

lemma *ovsmfidiesiOssie*: $(\text{ov} \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}) \ O \ (s \cup \widehat{s^{-1}} \cup e) \subseteq \text{ov} \cup \widehat{f^{-1}} \cup \widehat{d^{-1}} \cup s \cup e \cup \widehat{s^{-1}} \cup m$
 $\langle proof \rangle$

lemma $(b \cup m \cup \text{ov} \cup s \cup d) \ O \ (b \cup m \cup \text{ov} \cup s \cup d) \subseteq b \cup m \cup \text{ov} \cup s \cup d$
 $\langle proof \rangle$

lemma *ebmovovissifsiddib*: $(e \cup b \cup m \cup \text{ov} \cup \text{ov}^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1}) \ O \ b \subseteq b \cup m \cup \text{ov} \cup \widehat{f^{-1}} \cup \widehat{d^{-1}}$
 $\langle proof \rangle$

lemma *ebmovovissiffiddibmovsd*: $(e \cup b \cup m \cup \text{ov} \cup \text{ov}^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1}) \ O \ (b \cup m \cup \text{ov} \cup s \cup d) \subseteq (b \cup m \cup \text{ov} \cup s \cup d \cup \widehat{f^{-1}} \cup \widehat{d^{-1}} \cup \widehat{\text{ov}^{-1}} \cup s^{-1} \cup f \cup e)$
 $\langle proof \rangle$

lemma *difimov*: $(\widehat{d^{-1}} \cup \widehat{f^{-1}} \cup \text{ov} \cup e \cup f \cup m \cup b \cup \widehat{s^{-1}} \cup s) \ O \ (m \cup \text{ov} \cup s \cup d \cup b \cup \widehat{f^{-1}} \cup f \cup e) \subseteq (e \cup b \cup m \cup \text{ov} \cup \widehat{\text{ov}^{-1}} \cup s \cup \widehat{s^{-1}} \cup f \cup f^{-1} \cup d \cup d^{-1})$

$\langle proof \rangle$

lemma *difibs*: $(d^{-1} \cup f^{-1} \cup ov \cup e \cup f \cup m \cup b \cup s^{-1} \cup s) \ O \ (b \cup s \cup m) \subseteq (b \cup m \cup ov \cup f^{-1} \cup d^{-1} \cup d \cup e \cup s \cup s^{-1})$

$\langle proof \rangle$

lemma *bebmovovissiffiddi*: $b \ O \ (e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1}) \subseteq (b \cup m \cup ov \cup s \cup d)$

$\langle proof \rangle$

lemma *ovsmfidiesi*: $((ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}) \ O \ (ov^{-1} \cup s^{-1} \cup m^{-1} \cup f \cup d \cup e \cup s)) \subseteq (s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1} \cup e \cup ov \cup ov^{-1} \cup m \cup m^{-1})$

$\langle proof \rangle$

lemma *pii*: $(p, i) \in ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1} \implies (i, q) \in ov^{-1} \cup s^{-1} \cup m^{-1} \cup f \cup d \cup e \cup s \implies (p, q) \in s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1} \cup e \cup ov \cup ov^{-1} \cup m \cup m^{-1}$

$\langle proof \rangle$

lemma *ceovisidiffimi-ffie*: $(e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}) \ O \ (f \cup f^{-1} \cup e) \subseteq e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$

$\langle proof \rangle$

lemma *ceovisidiffimi-ffie-simp*: $(p, i) \in (e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}) \implies (i, q) \in (f \cup f^{-1} \cup e) \implies (p, q) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$

$\langle proof \rangle$

lemma *ceovisidiffimi-fife*: $(e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}) \ O \ (f^{-1} \cup f \cup e) \subseteq e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$

$\langle proof \rangle$

lemma $(x, j) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1} \implies (j, i) \in f^{-1} \cup f \cup e \implies (x, i) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$

$\langle proof \rangle$

lemma *m-ovsmfidiesi*: $m \ O \ (ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}) \subseteq b \cup s \cup m$

$\langle proof \rangle$

lemma *ovsmfidiesi-d*: $(ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}) \ O \ d \subseteq e \cup s \cup d \cup ov \cup ov^{-1} \cup s^{-1} \cup f \cup f^{-1} \cup d^{-1}$

$\langle proof \rangle$

lemma *cbi-esdovovisiffidi*: $b^{-1} \ O \ (e \cup s \cup d \cup ov \cup ov^{-1} \cup s^{-1} \cup f \cup f^{-1} \cup d^{-1}) \subseteq b^{-1} \cup m^{-1} \cup ov^{-1} \cup f \cup d$

$\langle proof \rangle$

lemma *cm-alpha1ialpha4mi:m* $O (e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}) \subseteq m \cup ov \cup s \cup d \cup b \cup \widehat{f^{-1}} \cup f \cup e$
 ⟨proof⟩

lemma *cbi-alpha1ialpha4mi:b^-1* $O (e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}) \subseteq \widehat{b^{-1}}$
 ⟨proof⟩

lemma *cbeta2-beta2:(b ∪ m ∪ ov ∪ f^-1 ∪ d^-1)* $O (b \cup m \cup ov \cup f^{-1} \cup d^{-1}) \subseteq b \cup m \cup ov \cup f^{-1} \cup d^{-1}$
 ⟨proof⟩

lemma *cbeta2-gammabm:* $(b \cup m \cup ov \cup f^{-1} \cup d^{-1}) O (e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1}) \subseteq (e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1})$
 ⟨proof⟩

lemma *calpha1-alpha1:(b ∪ m ∪ ov ∪ s ∪ d)* $O (b \cup m \cup ov \cup s \cup d) \subseteq (b \cup m \cup ov \cup s \cup d)$
 ⟨proof⟩

6.2 Intersection of non-basic relations

lemma *inter-ov:*

assumes $(i, j) \in (b \cup m \cup ov \cup f^{-1} \cup d^{-1}) \cap (e \cup \widehat{b^{-1}} \cup \widehat{m^{-1}} \cup \widehat{ov^{-1}} \cup ov \cup \widehat{s^{-1}} \cup s \cup \widehat{f^{-1}} \cup f \cup \widehat{d^{-1}} \cup d) \cap (b \cup m \cup ov \cup s \cup d)$

shows $(i, j) \in ov$

⟨proof⟩

lemma *neq-beta2i-alpha2alpha5m:*

assumes $(q, j) \in b^{-1} \cup d \cup f \cup ov^{-1} \cup m^{-1}$ **and** $(q, j) \in ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}$

shows *False*

⟨proof⟩

lemma *neq-bi-alpha1ialpha4mi:*

assumes $(q, i) \in \widehat{b^{-1}}$ **and** $(q, i) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$

shows *False*

⟨proof⟩

end

theory *nest*

imports

Main jointly-exhaustive examples
HOL-Eisbach.Eisbach-Tools

begin

7 Nests

Nests are sets of intervals that share a meeting point. We define relation before between nests that give the ordering properties of points.

7.1 Definitions

type-synonym *'a nest* = *'a set*

definition (in *arelations*) *BEGIN* :: *'a* ⇒ *'a nest*

where *BEGIN* *i* = {*j* | *j*. (*j*,*i*) ∈ *ov* ∪ *s* ∪ *m* ∪ *f*^{^-1} ∪ *d*^{^-1} ∪ *e* ∪ *s*^{^-1}}

definition (in *arelations*) *END* :: *'a* ⇒ *'a nest*

where *END* *i* = {*j* | *j*. (*j*,*i*) ∈ *e* ∪ *ov*^{^-1} ∪ *s*^{^-1} ∪ *d*^{^-1} ∪ *f* ∪ *f*^{^-1} ∪ *m*^{^-1}}

definition (in *arelations*) *NEST* :: *'a nest* ⇒ *bool*

where *NEST* *S* ≡ ∃ *i*. *I* *i* ∧ (*S* = *BEGIN* *i* ∨ *S* = *END* *i*)

definition (in *arelations*) *before* :: *'a nest* ⇒ *'a nest* ⇒ *bool* (**infix** <= 100)

where *before* *N M* ≡ *NEST* *N* ∧ *NEST* *M* ∧ (∃ *n m*. *I* *n* ∧ *I* *m* ∧ *n* ∈ *N* ∧ *m* ∈ *M* ∧ (*n*,*m*) ∈ *b*)

7.2 Properties of Nests

lemma *intv1*:

assumes *I* *i*

shows *i* ∈ *BEGIN* *i*

⟨*proof*⟩

lemma *intv2*:

assumes *I* *i*

shows *i* ∈ *END* *i*

⟨*proof*⟩

lemma *NEST-nonempty*:

assumes *NEST* *S*

shows *S* ≠ {}

⟨*proof*⟩

lemma *NEST-BEGIN*:

assumes *I* *i*

shows *NEST* (*BEGIN* *i*)

<proof>

lemma *NEST-END*:
assumes $\mathcal{I} i$
shows $NEST (END i)$
<proof>

lemma *before*:
assumes $a:\mathcal{I} i$
shows $BEGIN i \lll END i$
<proof>

lemma *meets*:
fixes $i j$
assumes $\mathcal{I} i$ **and** $\mathcal{I} j$
shows $(i,j) \in m = ((END i) = (BEGIN j))$
<proof>

lemma *starts*:
fixes $i j$
assumes $\mathcal{I} i$ **and** $\mathcal{I} j$
shows $((i,j) \in s \cup s^{-1} \cup e) = (BEGIN i = BEGIN j)$
<proof>

lemma *xj-set*: $x \in \{a \mid a. (a, j) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}\} =$
 $((x,j) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1})$
<proof>

lemma *ends*:
fixes $i j$
assumes $\mathcal{I} i$ **and** $\mathcal{I} j$
shows $((i,j) \in f \cup f^{-1} \cup e) = (END i = END j)$
<proof>

lemma *before-irrefl*:
fixes a
shows $\neg a \lll a$
<proof>

lemma *BEGIN-before*:
fixes $i j$
assumes $\mathcal{I} i$ **and** $\mathcal{I} j$
shows $BEGIN i \lll BEGIN j = ((i,j) \in b \cup m \cup ov \cup f^{-1} \cup d^{-1})$
<proof>

lemma *BEGIN-END-before*:
fixes $i j$
assumes $\mathcal{I} i$ **and** $\mathcal{I} j$

shows $BEGIN\ i \ll END\ j = ((i,j) \in e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1})$
 ⟨proof⟩

lemma *END-BEGIN-before*:

fixes $i\ j$

assumes $\mathcal{I}\ i$ and $\mathcal{I}\ j$

shows $END\ i \ll BEGIN\ j = ((i,j) \in b)$

⟨proof⟩

lemma *END-END-before*:

fixes $i\ j$

assumes $\mathcal{I}\ i$ and $\mathcal{I}\ j$

shows $END\ i \ll END\ j = ((i,j) \in b \cup m \cup ov \cup s \cup d)$

⟨proof⟩

lemma *overlaps*:

assumes $\mathcal{I}\ i$ and $\mathcal{I}\ j$

shows $(i,j) \in ov = ((BEGIN\ i \ll BEGIN\ j) \wedge (BEGIN\ j \ll END\ i) \wedge (END\ i \ll END\ j))$

⟨proof⟩

7.3 Ordering of nests

class *strict-order* =

fixes $ls::'a\ nest \Rightarrow 'a\ nest \Rightarrow bool$

assumes

irrefl: $\neg ls\ a\ a$ and

trans: $ls\ a\ c \Longrightarrow ls\ c\ g \Longrightarrow ls\ a\ g$ and

asym: $ls\ a\ c \Longrightarrow \neg ls\ c\ a$

class *total-strict-order* = *strict-order* +

assumes *trichotomy*: $a = c \Longrightarrow (\neg (ls\ a\ c) \wedge \neg (ls\ c\ a))$

interpretation *nest*: *total-strict-order* (\ll)

⟨proof⟩

end