Allen's Interval Calculus

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Contents

theory *xor-cal*

imports

Main

begin
definition $xor::bool \Rightarrow bool \Rightarrow bool$ (**infixl** ‹⊕› *60*)
where $xor\ A\ B \equiv (A \wedge \neg B) \vee (\neg A \wedge B)$

declare *xor-def* [*simp*]

interpretation *bool*:*semigroup* (⊕)
⟨*proof*⟩

lemma *xor-distr-L* [*simp*]:$A \oplus (B \oplus C) = (A\wedge\neg B\wedge\neg C)\vee(A\wedge B\wedge C)\vee(\neg A\wedge B\wedge\neg C)\vee(\neg A\wedge\neg B\wedge C)$
⟨*proof*⟩

lemma *xor-distr-R* [*simp*]:$(A \oplus B) \oplus C = A \oplus (B \oplus C)$
⟨*proof*⟩

end

theory *axioms*

imports
Main xor-cal

begin

1 Axioms

We formalize Allen's definition of theory of time in term of intervals (Allen, 1983). Two relations, namely meets and equality, are defined between intervals. Two interval meets if they are adjacent A set of 5 axioms ((M1) ∼ (M5)) are then defined based on relation meets.

We define a class interval whose assumptions are (i) properties of relations meets and, (ii) axioms (M1) ∼ (M5).

class *interval* =
fixes
$meets::'a \Rightarrow 'a \Rightarrow bool$ (**infixl** ‹‖› *60*) **and**
$\mathcal{I}::'a \Rightarrow bool$
assumes
meets-atrans:$[\![(p\|q);(q\|r)]\!] \Longrightarrow \neg(p\|r)$ **and**
meets-irrefl:$\mathcal{I}\ p \Longrightarrow \neg(p\|p)$ **and**
meets-asym:$(p\|q) \Longrightarrow \neg(q\|p)$ **and**

meets-wd: $p \| q \Longrightarrow \mathcal{I}\ p \wedge \mathcal{I}\ q$ **and**

M1: $[\![(p\|q);\ (p\|s);\ (r\|q)]\!] \Longrightarrow (r\|s)$ **and**
M2: $[\![(p\|q)\ ;\ (r\|s)]\!] \Longrightarrow p\|s \oplus ((\exists\, t.\ (p\|t)\wedge(t\|s)) \oplus (\exists\, t.\ (r\|t)\wedge(t\|q)))$ **and**
M3: $\mathcal{I}\ p \Longrightarrow (\exists\, q\ r.\ q\|p \wedge p\|r)$ **and**
M4: $[\![p\|q\ ;\ q\|s\ ;\ p\|r\ ;\ r\|s]\!] \Longrightarrow q = r$ **and**
M5exist: $p\|q \Longrightarrow (\exists\, r\ s\ t.\ r\|p \wedge p\|q \wedge q\|s \wedge r\|t \wedge t\|s)$

lemma (**in** *interval*) *trans2*: $[\![p\|t;\ t\|r;\ r\|q]\!] \Longrightarrow \neg p\|q$
⟨*proof*⟩

lemma (**in** *interval*) *nontrans1*: $u\|r \Longrightarrow \neg\ (\exists\, t.\ u\|t \wedge t\|r)$
⟨*proof*⟩

lemma (**in** *interval*) *nontrans2*: $u\|r \Longrightarrow \neg\ (\exists\, t.\ r\|t \wedge t\|u)$
⟨*proof*⟩

lemma (**in** *interval*) *nonmeets1*: $\neg\ (u\|r \wedge r\|u)$
⟨*proof*⟩

lemma (**in** *interval*) *nonmeets2*: $[\![\mathcal{I}\ u\ ;\ \mathcal{I}\ r\ ]\!] \Longrightarrow \neg\ (u\|r \wedge u = r)$
⟨*proof*⟩

lemma (**in** *interval*) *nonmeets3*: $\neg\ (u\|r \wedge (\exists\, p.\ u\|p \wedge p\|r))$
⟨*proof*⟩

lemma (**in** *interval*) *nonmeets4*: $\neg(u\|r \wedge (\exists\, p.\ r\|p \wedge p\|u))$
⟨*proof*⟩

lemma (**in** *interval*) *elimmeets*: $(p\ \|\ s \wedge (\exists\, t.\ p\ \|\ t \wedge t\ \|\ s) \wedge (\exists\, t.\ r\ \|\ t \wedge t\ \|\ q))$ $= \mathit{False}$
⟨*proof*⟩

lemma (**in** *interval*) *M5exist-var*:
assumes $x\|y$ $y\|z$ $z\|w$
shows $\exists\, t.\ x\|t \wedge t\|w$
⟨*proof*⟩

lemma (**in** *interval*) *M5exist-var2*:
assumes $p\|q$
shows $\exists\, r1\ r2\ r3\ s\ t.\ r1\|r2 \wedge r2\|r3 \wedge r3\|p \wedge p\|q \wedge q\|s \wedge r1\|t \wedge t\|s$
⟨*proof*⟩

lemma (**in** *interval*) *M5exist-var3*:
assumes $k\|l$ **and** $l\|q$ **and** $q\|t$ **and** $t\|r$
shows $\exists\, lqt.\ k\|lqt \wedge lqt\|r$
⟨*proof*⟩

end

2 Time interval relations

theory *allen*

imports

Main axioms
HOL−Eisbach.Eisbach-Tools

begin

3 Basic relations

We define 7 binary relations between time intervals. Relations e, m, b, ov, d, s and f stand for equal, meets, before, overlaps, during, starts and finishes, respectively.

class $arelations = interval +$
fixes
$e::('a \times 'a)\ set$ **and**
$m::('a \times 'a)\ set$ **and**
$b::('a \times 'a)\ set$ **and**
$ov::('a \times 'a)\ set$ **and**
$d::('a \times 'a)\ set$ **and**
$s::('a \times 'a)\ set$ **and**
$f::('a \times 'a)\ set$
assumes
$e:(p,q) \in e = (p = q)$ **and**
$m:(p,q) \in m = p \| q$ **and**
$b:(p,q) \in b = (\exists t::'a.\ p \| t \wedge t \| q)$ **and**
$ov:(p,q) \in ov = (\exists k\ l\ u\ v\ t::'a.$
$(k \| p \wedge p \| u \wedge u \| v) \wedge (k \| l \wedge l \| q \wedge q \| v) \wedge (l \| t \wedge t \| u))$ **and**
$s:(p,q) \in s = (\exists k\ u\ v::'a.\ k \| p \wedge p \| u \wedge u \| v \wedge k \| q \wedge q \| v)$ **and**
$f:(p,q) \in f = (\exists k\ l\ u::'a.\ k \| l \wedge l \| p \wedge p \| u \wedge k \| q \wedge q \| u)$ **and**
$d:(p,q) \in d = (\exists k\ l\ u\ v::'a.\ k \| l \wedge l \| p \wedge p \| u \wedge u \| v \wedge k \| q \wedge q \| v)$

3.1 e-composition

Relation e is the identity relation for composition.

lemma *cer*:
assumes $r \in \{e,m,b,ov,s,f,d,m\hat{}{-1},b\hat{}{-1},ov\hat{}{-1},s\hat{}{-1},f\hat{}{-1},d\hat{}{-1}\}$
shows $e\ O\ r = r$
⟨*proof*⟩

lemma *cre*:
assumes $r \in \{e,m,b,ov,s,f,d,m\hat{~}{-1},b\hat{~}{-1},ov\hat{~}{-1},s\hat{~}{-1},f\hat{~}{-1},d\hat{~}{-1}\}$
shows $r\ O\ e = r$
⟨*proof*⟩

lemmas *ceb* = *cer*[*of b*]
lemmas *cebi* = *cer*[*of b^−1*]
lemmas *cem* = *cer*[*of m*]
lemmas *cemi* = *cer*[*of m^−1*]
lemmas *cee* = *cer*[*of e*]
lemmas *ces* = *cer*[*of s*]
lemmas *cesi* = *cer*[*of s^−1*]
lemmas *cef* = *cer*[*of f*]
lemmas *cefi* = *cer*[*of f^−1*]
lemmas *ceov* = *cer*[*of ov*]
lemmas *ceovi* = *cer*[*of ov^−1*]
lemmas *ced* = *cer*[*of d*]
lemmas *cedi* = *cer*[*of d^−1*]
lemmas *cbe* = *cre*[*of b*]
lemmas *cbie* = *cre*[*of b^−1*]
lemmas *cme* = *cre*[*of m*]
lemmas *cmie* = *cre*[*of m^−1*]
lemmas *cse* = *cre*[*of s*]
lemmas *csie* = *cre*[*of s^−1*]
lemmas *cfe* = *cre*[*of f*]
lemmas *cfie* = *cre*[*of f^−1*]
lemmas *cove* = *cre*[*of ov*]
lemmas *covie* = *cre*[*of ov^−1*]
lemmas *cde* = *cre*[*of d*]
lemmas *cdie* = *cre*[*of d^−1*]

3.2 r-composition

We prove compositions of the form $r_1 \circ r_2 \subseteq r$, where r is a basic relation.

method (**in** *arelations*) *r-compose* **uses** *r1 r2 r3* = ((*auto*, (*subst* (*asm*) *r1*), (*subst* (*asm*) *r2*), (*subst r3*)) , (*meson M5exist-var*))

lemma (**in** *arelations*) *cbb*:$b\ O\ b \subseteq b$
⟨*proof*⟩

lemma (**in** *arelations*) *cbm*:$b\ O\ m \subseteq b$
⟨*proof*⟩

lemma *cbov*:$b\ O\ ov \subseteq b$
⟨*proof*⟩

lemma *cbfi*:$b\ O\ f\hat{~}{-1} \subseteq b$
⟨*proof*⟩

lemma *cbdi*:$b\ O\ d\hat{}{-1} \subseteq b$
⟨*proof*⟩

lemma *cbs*:$b\ O\ s \subseteq b$
⟨*proof*⟩

lemma *cbsi*:$b\ O\ s\hat{}{-1} \subseteq b$
⟨*proof*⟩

lemma (**in** *arelations*) *cmb*:$m\ O\ b \subseteq b$
⟨*proof*⟩

lemma *cmm*:$m\ O\ m \subseteq b$
⟨*proof*⟩

lemma *cmov*:$m\ O\ ov \subseteq b$
⟨*proof*⟩

lemma *cmfi*:$m\ O\ f\hat{}{-1} \subseteq b$
⟨*proof*⟩

lemma *cmdi*:$m\ O\ d\hat{}{-1} \subseteq b$
⟨*proof*⟩

lemma *cms*:$m\ O\ s \subseteq m$
⟨*proof*⟩

lemma *cmsi*:$m\ O\ s\hat{}{-1} \subseteq m$
⟨*proof*⟩

lemma *covb*:$ov\ O\ b \subseteq b$
⟨*proof*⟩

lemma *covm*:$ov\ O\ m \subseteq b$
⟨*proof*⟩

lemma *covs*:$ov\ O\ s \subseteq ov$
⟨*proof*⟩

lemma *cfib*:$f\hat{}{-1}\ O\ b \subseteq b$
⟨*proof*⟩

lemma *cfim*:$f\hat{}{-1}\ O\ m \subseteq m$
⟨*proof*⟩

lemma *cfiov*:$f\hat{}{-1}\ O\ ov \subseteq ov$
⟨*proof*⟩

lemma *cfifi*:f^−*1* O f^−*1* ⊆ f^−*1*
⟨*proof*⟩

lemma *cfidi*:f^−*1* O d^−*1* ⊆ d^−*1*
⟨*proof*⟩

lemma *cfis*:f^−*1* O s ⊆ *ov*
⟨*proof*⟩

lemma *cfisi*:f^−*1* O s^−*1* ⊆ d^−*1*
⟨*proof*⟩

lemma *cdifi*:d^−*1* O f^−*1* ⊆ d^−*1*
⟨*proof*⟩

lemma *cdidi*:d^−*1* O d^−*1* ⊆ d^−*1*
⟨*proof*⟩

lemma *cdisi*:d^−*1* O s^−*1* ⊆ d^−*1*
⟨*proof*⟩

lemma *csb*:s O b ⊆ b
⟨*proof*⟩

lemma *csm*:s O m ⊆ b
⟨*proof*⟩

lemma *css*:s O s ⊆ s
⟨*proof*⟩

lemma *csifi*:s^−*1* O f^−*1* ⊆ d^−*1*
⟨*proof*⟩

lemma *csidi*:s^−*1* O d^−*1* ⊆ d^−*1*
⟨*proof*⟩

lemma *cdb*:d O b ⊆ b
⟨*proof*⟩

lemma *cdm*:d O m ⊆ b
⟨*proof*⟩

lemma *cfb*:f O b ⊆ b
⟨*proof*⟩

lemma *cfm*:f O m ⊆ m
⟨*proof*⟩

3.3 α-composition

We prove compositions of the form $r_1 \circ r_2 \subseteq s \cup ov \cup d$.

lemma (**in** *arelations*) *cmd*:*m O d* ⊆ *s* ∪ *ov* ∪ *d*
⟨*proof*⟩

lemma (**in** *arelations*) *cmf*:*m O f* ⊆ *s* ∪ *ov* ∪ *d*
⟨*proof*⟩

lemma *cmovi*:*m O ov*^*−1* ⊆ *s* ∪ *ov* ∪ *d*
⟨*proof*⟩

lemma *covd*:*ov O d* ⊆ *s* ∪ *ov* ∪ *d*
⟨*proof*⟩

lemma *covf*:*ov O f* ⊆ *s* ∪ *ov* ∪ *d*
⟨*proof*⟩

lemma *cfid*:*f*^*−1 O d* ⊆ *s* ∪ *ov* ∪ *d*
⟨*proof*⟩

lemma *cfov*:*f O ov* ⊆ *ov* ∪ *s* ∪ *d*
⟨*proof*⟩

We prove compositions of the form $r_1 \circ r_2 \subseteq ov \cup f^{-1} \cup d^{-1}$.

lemma *covsi*:*ov O s*^*−1* ⊆ *ov* ∪ *f*^*−1* ∪ *d*^*−1*
⟨*proof*⟩

lemma *cdim*:*d*^*−1 O m* ⊆ *ov* ∪ *d*^*−1* ∪ *f*^*−1*
⟨*proof*⟩

lemma *cdiov*:*d*^*−1 O ov* ⊆ *ov* ∪ *f*^*−1* ∪ *d*^*−1*
⟨*proof*⟩

lemma *cdis*:*d*^*−1 O s* ⊆ *ov* ∪ *f*^*−1* ∪ *d*^*−1*
⟨*proof*⟩

lemma *csim*:*s*^*−1 O m* ⊆ *ov* ∪ *f*^*−1* ∪ *d*^*−1*
⟨*proof*⟩

lemma *csiov*:*s*^*−1 O ov* ⊆ *ov* ∪ *f*^*−1* ∪ *d*^*−1*
⟨*proof*⟩

lemma *covim*:*ov*^*−1 O m* ⊆ *ov* ∪ *f*^*−1* ∪ *d*^*−1*
⟨*proof*⟩

We prove compositions of the form $r_1 \circ r_2 \subseteq b \cup m \cup ov$.

lemma *covov*:*ov O ov* ⊆ *b* ∪ *m* ∪ *ov*
⟨*proof*⟩

lemma *covfi*:*ov O f*^−*1* ⊆ *b* ∪ *m* ∪ *ov*
⟨*proof*⟩

lemma *csov*:*s O ov* ⊆ *b* ∪ *m* ∪ *ov*
⟨*proof*⟩

lemma *csfi*:*s O f*^−*1* ⊆ *b* ∪ *m* ∪ *ov*
⟨*proof*⟩

We prove compositions of the form $r_1 \circ r_2 \subseteq f \cup f^{-1} \cup e$.

lemma *cmmi*:*m O m*^−*1* ⊆ *f* ∪ *f*^−*1* ∪ *e*
⟨*proof*⟩

lemma *cfif*:*f*^−*1 O f* ⊆ *e* ∪ *f*^−*1* ∪ *f*
⟨*proof*⟩

lemma *cffi*:*f O f*^−*1* ⊆ *e* ∪ *f* ∪ *f*^−*1*
⟨*proof*⟩

We prove compositions of the form $r_1 \circ r_2 \subseteq e \cup s \cup s^{-1}$.

lemma *cssi*:*s O s*^−*1* ⊆ *e* ∪ *s* ∪ *s*^−*1*
⟨*proof*⟩

lemma *csis*:*s*^−*1 O s* ⊆ *e* ∪ *s* ∪ *s*^−*1*
⟨*proof*⟩

lemma *cmim*:*m*^−*1 O m* ⊆ *s* ∪ *s*^−*1* ∪ *e*
⟨*proof*⟩

3.4 β-composition

We prove compositions of the form $r_1 \circ r_2 \subseteq b \cup m \cup ov \cup s \cup d$.

lemma *cbd*:*b O d* ⊆ *b* ∪ *m* ∪ *ov* ∪ *s* ∪ *d*
⟨*proof*⟩

lemma *cbf*:*b O f* ⊆ *b* ∪ *m* ∪ *ov* ∪ *s* ∪ *d*
⟨*proof*⟩

lemma *cbovi*:*b O ov*^−*1* ⊆ *b* ∪ *m* ∪ *ov* ∪ *s* ∪ *d*
⟨*proof*⟩

lemma *cbmi*:*b O m*^−*1* ⊆ *b* ∪ *m* ∪ *ov* ∪ *s* ∪ *d*

⟨*proof*⟩

lemma *cdov*:$d\ O\ ov \subseteq b \cup m \cup ov \cup s \cup d$
⟨*proof*⟩

lemma *cdfi*:$d\ O\ f\hat{}{-1} \subseteq b \cup m \cup ov \cup s \cup d$
⟨*proof*⟩

We prove compositions of the form $r_1 \circ r_2 \subseteq b \cup m \cup ov \cup f^{-1} \cup d^{-1}$.

lemma *covdi*:$ov\ O\ d\hat{}{-1} \subseteq b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *cdib*:$d\hat{}{-1}\ O\ b \subseteq b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *csdi*:$s\ O\ d\hat{}{-1} \subseteq b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *csib*:$s\hat{}{-1}\ O\ b \subseteq b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *covib*:$ov\hat{}{-1}\ O\ b \subseteq b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *cmib*:$m\hat{}{-1}\ O\ b \subseteq b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

3.5 γ-composition

We prove compositions of the form $r_1 \circ r_2 \subseteq ov \cup s \cup d \cup f \cup e \cup f^{-1} \cup d^{-1} \cup s^{-1} \cup ov^{-1}$.

lemma *covovi*:$ov\ O\ ov\hat{}{-1} \subseteq e \cup ov \cup ov\hat{}{-1} \cup d \cup d\hat{}{-1} \cup s \cup s\hat{}{-1} \cup f \cup f\hat{}{-1}$
⟨*proof*⟩

lemma *cdid*:$d\hat{}{-1}\ O\ d \subseteq e \cup ov \cup ov\hat{}{-1} \cup d \cup d\hat{}{-1} \cup s \cup s\hat{}{-1} \cup f \cup f\hat{}{-1}$
⟨*proof*⟩

lemma *coviov*:$ov\hat{}{-1}\ O\ ov \subseteq e \cup ov \cup ov\hat{}{-1} \cup d \cup d\hat{}{-1} \cup s \cup s\hat{}{-1} \cup f \cup f\hat{}{-1}$
⟨*proof*⟩

3.6 γ-composition

We prove compositions of the form $r_1 \circ r_2 \subseteq b \cup m \cup ov \cup s \cup d \cup f \cup e \cup f^{-1} \cup d^{-1} \cup s^{-1} \cup ov^{-1} \cup b^{-1} \cup m^{-1}$.

lemma *cbbi*:$b\ O\ b^{\frown}{-1} \subseteq b \cup b^{\frown}{-1} \cup m \cup m^{\frown}{-1} \cup e \cup ov \cup ov^{\frown}{-1} \cup s \cup s^{\frown}{-1} \cup d \cup d^{\frown}{-1} \cup f \cup f^{\frown}{-1}$ (**is** $b\ O\ b^{\frown}{-1} \subseteq ?R$)
⟨*proof*⟩

lemma *cbib*:$b^{\frown}{-1}\ O\ b \subseteq b \cup b^{\frown}{-1} \cup m \cup m^{\frown}{-1} \cup e \cup ov \cup ov^{\frown}{-1} \cup s \cup s^{\frown}{-1} \cup d \cup d^{\frown}{-1} \cup f \cup f^{\frown}{-1}$ (**is** $b^{\frown}{-1}\ O\ b \subseteq ?R$)
⟨*proof*⟩

lemma *cddi*:$d\ O\ d^{\frown}{-1} \subseteq b \cup b^{\frown}{-1} \cup m \cup m^{\frown}{-1} \cup e \cup ov \cup ov^{\frown}{-1} \cup s \cup s^{\frown}{-1} \cup d \cup d^{\frown}{-1} \cup f \cup f^{\frown}{-1}$ (**is** $d\ O\ d^{\frown}{-1} \subseteq ?R$)
⟨*proof*⟩

3.7 The rest of the composition table

Because of the symmetry $(r_1 \circ r_2)^{-1} = r_2^{-1} \circ r_1^{-1}$, the rest of the compositions is easily deduced.

lemma *cmbi*:$m\ O\ b^{\frown}{-1} \subseteq b^{\frown}{-1} \cup m^{\frown}{-1} \cup s^{\frown}{-1} \cup ov^{\frown}{-1} \cup d^{\frown}{-1}$
⟨*proof*⟩

lemma *covmi*:$ov\ O\ m^{\frown}{-1} \subseteq ov^{\frown}{-1} \cup d^{\frown}{-1} \cup s^{\frown}{-1}$
⟨*proof*⟩

lemma *covbi*:$ov\ O\ b^{\frown}{-1} \subseteq b^{\frown}{-1} \cup m^{\frown}{-1} \cup s^{\frown}{-1} \cup ov^{\frown}{-1} \cup d^{\frown}{-1}$
⟨*proof*⟩

lemma *cfiovi*:$f^{\frown}{-1}\ O\ ov^{\frown}{-1} \subseteq ov^{\frown}{-1} \cup s^{\frown}{-1} \cup d^{\frown}{-1}$
⟨*proof*⟩

lemma *cfimi*:$(f^{\frown}{-1}\ O\ m^{\frown}{-1}) \subseteq s^{\frown}{-1} \cup ov^{\frown}{-1} \cup d^{\frown}{-1}$
⟨*proof*⟩

lemma *cfibi*:$f^{\frown}{-1}\ O\ b^{\frown}{-1} \subseteq b^{\frown}{-1} \cup m^{\frown}{-1} \cup ov^{\frown}{-1} \cup s^{\frown}{-1} \cup d^{\frown}{-1}$
⟨*proof*⟩

lemma *cdif*:$d^{\frown}{-1}\ O\ f \subseteq ov^{\frown}{-1} \cup s^{\frown}{-1} \cup d^{\frown}{-1}$
⟨*proof*⟩

lemma *cdiovi*:$d^{\frown}{-1}\ O\ ov\ ^{\frown}{-1} \subseteq ov^{\frown}{-1} \cup s^{\frown}{-1} \cup d^{\frown}{-1}$
⟨*proof*⟩

lemma *cdimi*:$d^{\frown}{-1}\ O\ m^{\frown}{-1} \subseteq s^{\frown}{-1} \cup ov^{\frown}{-1} \cup d^{\frown}{-1}$
⟨*proof*⟩

lemma *cdibi*:$d^{\frown}{-1}\ O\ b^{\frown}{-1} \subseteq b^{\frown}{-1} \cup m^{\frown}{-1} \cup ov^{\frown}{-1} \cup s^{\frown}{-1} \cup d^{\frown}{-1}$
⟨*proof*⟩

lemma *csd*:$s\ O\ d \subseteq d$
⟨*proof*⟩

lemma *csf*:$s\ O\ f \subseteq d$
⟨*proof*⟩

lemma *csovi*:$s\ O\ ov\hat{}{-1} \subseteq ov\hat{}{-1} \cup f \cup d$
⟨*proof*⟩

lemma *csmi*:$s\ O\ m\hat{}{-1} \subseteq m\hat{}{-1}$
⟨*proof*⟩

lemma *csbi*:$s\ O\ b\hat{}{-1} \subseteq b\hat{}{-1}$
⟨*proof*⟩

lemma *csisi*:$s\hat{}{-1}\ O\ s\hat{}{-1} \subseteq s\hat{}{-1}$
⟨*proof*⟩

lemma *csid*:$s\hat{}{-1}\ O\ d \subseteq ov\hat{}{-1} \cup f \cup d$
⟨*proof*⟩

lemma *csif*:$s\hat{}{-1}\ O\ f \subseteq ov\hat{}{-1}$
⟨*proof*⟩

lemma *csiovi*:$s\hat{}{-1}\ O\ ov\hat{}{-1} \subseteq ov\hat{}{-1}$
⟨*proof*⟩

lemma *csimi*:$s\hat{}{-1}\ O\ m\hat{}{-1} \subseteq m\hat{}{-1}$
⟨*proof*⟩

lemma *csibi*:$s\hat{}{-1}\ O\ b\hat{}{-1} \subseteq b\hat{}{-1}$
⟨*proof*⟩

lemma *cds*:$d\ O\ s \subseteq d$
⟨*proof*⟩

lemma *cdsi*:$d\ O\ s\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1} \cup f \cup d$
⟨*proof*⟩

lemma *cdd*:$d\ O\ d \subseteq d$
⟨*proof*⟩

lemma *cdf*:$d\ O\ f \subseteq d$
⟨*proof*⟩

lemma *cdovi*:$d\ O\ ov\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1} \cup f \cup d$
⟨*proof*⟩

lemma *cdmi*:$d\ O\ m\hat{}{-1} \subseteq b\hat{}{-1}$

⟨*proof*⟩

lemma *cdbi*:$d\ O\ b\widehat{\ }{-1} \subseteq b\widehat{\ }{-1}$
⟨*proof*⟩

lemma *cfdi*:$f\ O\ d\widehat{\ }{-1} \subseteq\ b\widehat{\ }{-1} \cup m\widehat{\ }{-1} \cup ov\widehat{\ }{-1} \cup s\widehat{\ }{-1} \cup d\widehat{\ }{-1}$
⟨*proof*⟩

lemma *cfs*:$f\ O\ s \subseteq d$
⟨*proof*⟩

lemma *cfsi*:$f\ O\ s\widehat{\ }{-1} \subseteq b\ \widehat{\ }{-1} \cup m\widehat{\ }{-1} \cup ov\ \widehat{\ }{-1}$
⟨*proof*⟩

lemma *cfd*:$f\ O\ d \subseteq d$
⟨*proof*⟩

lemma *cff*:$f\ O\ f \subseteq f$
⟨*proof*⟩

lemma *cfovi*:$f\ O\ ov\widehat{\ }{-1} \subseteq b\widehat{\ }{-1} \cup m\widehat{\ }{-1} \cup ov\widehat{\ }{-1}$
⟨*proof*⟩

lemma *cfmi*:$f\ O\ m\widehat{\ }{-1} \subseteq b\widehat{\ }{-1}$
⟨*proof*⟩

lemma *cfbi*:$f\ O\ b\widehat{\ }{-1} \subseteq b\widehat{\ }{-1}$
⟨*proof*⟩

lemma *covifi*:$ov\widehat{\ }{-1}\ O\ f\widehat{\ }{-1} \subseteq ov\widehat{\ }{-1} \cup s\widehat{\ }{-1} \cup d\widehat{\ }{-1}$
⟨*proof*⟩

lemma *covidi*:$ov\widehat{\ }{-1}\ O\ d\widehat{\ }{-1} \subseteq b\widehat{\ }{-1} \cup m\widehat{\ }{-1} \cup s\widehat{\ }{-1} \cup ov\widehat{\ }{-1} \cup d\widehat{\ }{-1}$
⟨*proof*⟩

lemma *covis*:$ov\widehat{\ }{-1}\ O\ s \subseteq ov\widehat{\ }{-1} \cup f \cup d$
⟨*proof*⟩

lemma *covisi*:$ov\widehat{\ }{-1}\ O\ s\widehat{\ }{-1} \subseteq b\widehat{\ }{-1} \cup m\widehat{\ }{-1} \cup ov\widehat{\ }{-1}$
⟨*proof*⟩

lemma *covid*:$ov\widehat{\ }{-1}\ O\ d \subseteq ov\widehat{\ }{-1} \cup f \cup d$
⟨*proof*⟩

lemma *covif*:$ov\widehat{\ }{-1}\ O\ f \subseteq ov\widehat{\ }{-1}$
⟨*proof*⟩

lemma *coviovi*:$ov\widehat{\ }{-1}\ O\ ov\widehat{\ }{-1} \subseteq b\widehat{\ }{-1} \cup m\widehat{\ }{-1} \cup ov\widehat{\ }{-1}$

⟨*proof*⟩

lemma *covimi*:ov^−1 O m^−1 ⊆ b^−1
⟨*proof*⟩

lemma *covibi*:ov^−1 O b^−1 ⊆ b^−1
⟨*proof*⟩

lemma *cmiov*:m^−1 O ov ⊆ ov^−1 ∪ d ∪ f
⟨*proof*⟩

lemma *cmifi*:m^−1 O f^−1 ⊆ m^−1
⟨*proof*⟩

lemma *cmidi*:m^−1 O d^−1 ⊆ b^−1
⟨*proof*⟩

lemma *cmis*:m^−1 O s ⊆ ov^−1 ∪ d ∪ f
⟨*proof*⟩

lemma *cmisi*:m^−1 O s^−1 ⊆ b^−1
⟨*proof*⟩

lemma *cmid*:m^−1 O d ⊆ ov^−1 ∪ d ∪ f
⟨*proof*⟩

lemma *cmif*:m^−1 O f ⊆ m^−1
⟨*proof*⟩

lemma *cmiovi*:m^−1 O ov^−1 ⊆ b^−1
⟨*proof*⟩

lemma *cmimi*:m^−1 O m^−1 ⊆ b^−1
⟨*proof*⟩

lemma *cmibi*:m^−1 O b^−1 ⊆ b^−1
⟨*proof*⟩

lemma *cbim*:b^−1 O m ⊆ b^−1 ∪ m^−1 ∪ ov^−1 ∪ f ∪ d
⟨*proof*⟩

lemma *cbiov*:b^−1 O ov ⊆ b^−1 ∪ m^−1 ∪ ov^−1 ∪ f ∪ d
⟨*proof*⟩

lemma *cbifi*:b^−1 O f^−1 ⊆ b^−1
⟨*proof*⟩

lemma *cbidi*:b^−1 O d^−1 ⊆ b^−1
⟨*proof*⟩

lemma *cbis*:$b^{\frown}{-1}$ O $s \subseteq b^{\frown}{-1} \cup m^{\frown}{-1} \cup ov^{\frown}{-1} \cup f \cup d$
⟨*proof*⟩

lemma *cbisi*:$b^{\frown}{-1}$ O $s^{\frown}{-1} \subseteq b^{\frown}{-1}$
⟨*proof*⟩

lemma *cbid*:$b^{\frown}{-1}$ O $d \subseteq b^{\frown}{-1} \cup m^{\frown}{-1} \cup ov^{\frown}{-1} \cup f \cup d$
⟨*proof*⟩

lemma *cbif*:$b^{\frown}{-1}$ O $f \subseteq b^{\frown}{-1}$
⟨*proof*⟩

lemma *cbiovi*:$b^{\frown}{-1}$ O $ov^{\frown}{-1} \subseteq b^{\frown}{-1}$
⟨*proof*⟩

lemma *cbimi*:$b^{\frown}{-1}$ O $m^{\frown}{-1} \subseteq b^{\frown}{-1}$
⟨*proof*⟩

lemma *cbibi*:$b^{\frown}{-1}$ O $b^{\frown}{-1} \subseteq b^{\frown}{-1}$
⟨*proof*⟩

3.8 Composition rules

named-theorems *ce-rules* **declare** *cem*[*ce-rules*] **and** *ceb*[*ce-rules*] **and** *ceov*[*ce-rules*]
and *ces*[*ce-rules*] **and** *cef*[*ce-rules*] **and** *ced*[*ce-rules*] **and**
cemi[*ce-rules*] **and** *cebi*[*ce-rules*] **and** *ceovi*[*ce-rules*] **and** *cesi*[*ce-rules*] **and** *cefi*[*ce-rules*]
and *cedi*[*ce-rules*]

named-theorems *cm-rules* **declare** *cme*[*cm-rules*] **and** *cmb*[*cm-rules*] **and** *cmm*[*cm-rules*]
and *cmov*[*cm-rules*] **and** *cms* [*cm-rules*] **and** *cmd*[*cm-rules*] **and** *cmf*[*cm-rules*]
and
cmbi[*cm-rules*] **and** *cmmi*[*cm-rules*] **and** *cmovi*[*cm-rules*] **and** *cmsi*[*cm-rules*] **and**
cmdi[*cm-rules*] **and** *cmfi*[*cm-rules*]

named-theorems *cb-rules* **declare** *cbe*[*cb-rules*] **and** *cbm*[*cb-rules*] **and** *cbb*[*cb-rules*]
and *cbov*[*cb-rules*] **and** *cbs* [*cb-rules*] **and** *cbd*[*cb-rules*] **and** *cbf*[*cb-rules*] **and**
cbbi[*cb-rules*] **and** *cbbi*[*cb-rules*] **and** *cbovi*[*cb-rules*] **and** *cbsi*[*cb-rules*] **and** *cbdi*[*cb-rules*]
and *cbfi*[*cb-rules*]

named-theorems *cov-rules* **declare** *cove*[*cov-rules*] **and** *covb*[*cov-rules*] **and** *covb*[*cov-rules*]
and *covov*[*cov-rules*] **and** *covs* [*cov-rules*] **and** *covd*[*cov-rules*] **and** *covf*[*cov-rules*]
and
covbi[*cov-rules*] **and** *covbi*[*cov-rules*] **and** *covovi*[*cov-rules*] **and** *covsi*[*cov-rules*] **and**
covdi[*cov-rules*] **and** *covfi*[*cov-rules*]

named-theorems *cs-rules* **declare** *cse*[*cs-rules*] **and** *csb*[*cs-rules*] **and** *csb*[*cs-rules*]
and *csov*[*cs-rules*] **and** *css* [*cs-rules*] **and** *csd*[*cs-rules*] **and** *csf*[*cs-rules*] **and**
csbi[*cs-rules*] **and** *csbi*[*cs-rules*] **and** *csovi*[*cs-rules*] **and** *cssi*[*cs-rules*] **and** *csdi*[*cs-rules*]

and *csfi*[*cs-rules*]

named-theorems *cf-rules* **declare** *cfe*[*cf-rules*] **and** *cfb*[*cf-rules*] **and** *cfb*[*cf-rules*] **and** *cfov*[*cf-rules*] **and** *cfs* [*cf-rules*] **and** *cfd*[*cf-rules*] **and** *cff* [*cf-rules*] **and** *cfbi*[*cf-rules*] **and** *cfbi*[*cf-rules*] **and** *cfovi*[*cf-rules*] **and** *cfsi*[*cf-rules*] **and** *cfdi*[*cf-rules*] **and** *cffi*[*cf-rules*]

named-theorems *cd-rules* **declare** *cde*[*cd-rules*] **and** *cdb*[*cd-rules*] **and** *cdb*[*cd-rules*] **and** *cdov*[*cd-rules*] **and** *cds* [*cd-rules*] **and** *cdd*[*cd-rules*] **and** *cdf* [*cd-rules*] **and** *cdbi*[*cd-rules*] **and** *cdbi*[*cd-rules*] **and** *cdovi*[*cd-rules*] **and** *cdsi*[*cd-rules*] **and** *cddi*[*cd-rules*] **and** *cdfi*[*cd-rules*]

named-theorems *cmi-rules* **declare** *cmie*[*cmi-rules*] **and** *cmib*[*cmi-rules*] **and** *cmib*[*cmi-rules*] **and** *cmiov*[*cmi-rules*] **and** *cmis* [*cmi-rules*] **and** *cmid*[*cmi-rules*] **and** *cmif* [*cmi-rules*] **and**
cmibi[*cmi-rules*] **and** *cmibi*[*cmi-rules*] **and** *cmiovi*[*cmi-rules*] **and** *cmisi*[*cmi-rules*] **and** *cmidi*[*cmi-rules*] **and** *cmifi*[*cmi-rules*]

named-theorems *cbi-rules* **declare** *cbie*[*cbi-rules*] **and** *cbim*[*cbi-rules*] **and** *cbib*[*cbi-rules*] **and** *cbiov*[*cbi-rules*] **and** *cbis* [*cbi-rules*] **and** *cbid*[*cbi-rules*] **and** *cbif* [*cbi-rules*] **and** *cbimi*[*cbi-rules*] **and** *cbibi*[*cbi-rules*] **and** *cbiovi*[*cbi-rules*] **and** *cbisi*[*cbi-rules*] **and** *cbidi*[*cbi-rules*] **and** *cbifi*[*cbi-rules*]

named-theorems *covi-rules* **declare** *covie*[*covi-rules*] **and** *covib*[*covi-rules*] **and** *covib*[*covi-rules*] **and** *coviov*[*covi-rules*] **and** *covis* [*covi-rules*] **and** *covid*[*covi-rules*] **and** *covif* [*covi-rules*] **and**
covibi[*covi-rules*] **and** *covibi*[*covi-rules*] **and** *coviovi*[*covi-rules*] **and** *covisi*[*covi-rules*] **and** *covidi*[*covi-rules*] **and** *covifi*[*covi-rules*]

named-theorems *csi-rules* **declare** *csie*[*csi-rules*] **and** *csib*[*csi-rules*] **and** *csib*[*csi-rules*] **and** *csiov*[*csi-rules*] **and** *csis* [*csi-rules*] **and** *csid*[*csi-rules*] **and** *csif* [*csi-rules*] **and** *csibi*[*csi-rules*] **and** *csibi*[*csi-rules*] **and** *csiovi*[*csi-rules*] **and** *csisi*[*csi-rules*] **and** *csidi*[*csi-rules*] **and** *csifi*[*csi-rules*]

named-theorems *cfi-rules* **declare** *cfie*[*cfi-rules*] **and** *cfib*[*cfi-rules*] **and** *cfib*[*cfi-rules*] **and** *cfiov*[*cfi-rules*] **and** *cfis* [*cfi-rules*] **and** *cfid*[*cfi-rules*] **and** *cfif* [*cfi-rules*] **and** *cfibi*[*cfi-rules*] **and** *cfibi*[*cfi-rules*] **and** *cfiovi*[*cfi-rules*] **and** *cfisi*[*cfi-rules*] **and** *cfidi*[*cfi-rules*] **and** *cfifi*[*cfi-rules*]

named-theorems *cdi-rules* **declare** *cdie*[*cdi-rules*] **and** *cdib*[*cdi-rules*] **and** *cdib*[*cdi-rules*] **and** *cdiov*[*cdi-rules*] **and** *cdis* [*cdi-rules*] **and** *cdid*[*cdi-rules*] **and** *cdif* [*cdi-rules*] **and** *cdibi*[*cdi-rules*] **and** *cdibi*[*cdi-rules*] **and** *cdiovi*[*cdi-rules*] **and** *cdisi*[*cdi-rules*] **and** *cdidi*[*cdi-rules*] **and** *cdifi*[*cdi-rules*]

named-theorems *cre-rules* **declare** *cee*[*cre-rules*] **and** *cme*[*cre-rules*] **and** *cbe*[*cre-rules*] **and** *cove*[*cre-rules*] **and** *cse*[*cre-rules*] **and** *cfe*[*cre-rules*] **and** *cde*[*cre-rules*] **and** *cmie*[*cre-rules*] **and** *cbie*[*cre-rules*] **and** *covie*[*cre-rules*] **and** *csie*[*cre-rules*] **and** *cfie*[*cre-rules*] **and** *cdie*[*cre-rules*]

named-theorems *crm-rules* **declare** *cem*[*crm-rules*] **and** *cbm*[*crm-rules*] **and** *cmm*[*crm-rules*] **and** *covm*[*crm-rules*] **and** *csm*[*crm-rules*] **and** *cfm*[*crm-rules*] **and** *cdm*[*crm-rules*] **and**
cmim[*crm-rules*] **and** *cbim*[*crm-rules*] **and** *covim*[*crm-rules*] **and** *csim*[*crm-rules*] **and** *cfim*[*crm-rules*] **and** *cdim*[*crm-rules*]

named-theorems *crmi-rules* **declare** *cemi*[*crmi-rules*] **and** *cbmi*[*crmi-rules*] **and** *cmmi*[*crmi-rules*] **and** *covmi*[*crmi-rules*] **and** *csmi*[*crmi-rules*] **and** *cfmi*[*crmi-rules*] **and** *cdmi*[*crmi-rules*] **and**
cmimi[*crmi-rules*] **and** *cbimi*[*crmi-rules*] **and** *covimi*[*crmi-rules*] **and** *csimi*[*crmi-rules*] **and** *cfimi*[*crmi-rules*] **and** *cdimi*[*crmi-rules*]

named-theorems *crs-rules* **declare** *ces*[*crs-rules*] **and** *cbs*[*crs-rules*] **and** *cms*[*crs-rules*] **and** *covs*[*crs-rules*] **and** *css*[*crs-rules*] **and** *cfs*[*crs-rules*] **and** *cds*[*crs-rules*] **and**
cmis[*crs-rules*] **and** *cbis*[*crs-rules*] **and** *covis*[*crs-rules*] **and** *csis*[*crs-rules*] **and** *cfis*[*crs-rules*] **and** *cdis*[*crs-rules*]

named-theorems *crsi-rules* **declare** *cesi*[*crsi-rules*] **and** *cbsi*[*crsi-rules*] **and** *cmsi*[*crsi-rules*] **and** *covsi*[*crsi-rules*] **and** *cssi*[*crsi-rules*] **and** *cfsi*[*crsi-rules*] **and** *cdsi*[*crsi-rules*] **and**
cmisi[*crsi-rules*] **and** *cbisi*[*crsi-rules*] **and** *covisi*[*crsi-rules*] **and** *csisi*[*crsi-rules*] **and** *cfisi*[*crsi-rules*] **and** *cdisi*[*crsi-rules*]

named-theorems *crb-rules* **declare** *ceb*[*crb-rules*] **and** *cbb*[*crb-rules*] **and** *cmb*[*crb-rules*] **and** *covb*[*crb-rules*] **and** *csb*[*crb-rules*] **and** *cfb*[*crb-rules*] **and** *cdb*[*crb-rules*] **and**
cmib[*crb-rules*] **and** *cbib*[*crb-rules*] **and** *covib*[*crb-rules*] **and** *csib*[*crb-rules*] **and** *cfib*[*crb-rules*] **and** *cdib*[*crb-rules*]

named-theorems *crbi-rules* **declare** *cebi*[*crbi-rules*] **and** *cbbi*[*crbi-rules*] **and** *cmbi*[*crbi-rules*] **and** *covbi*[*crbi-rules*] **and** *csbi*[*crbi-rules*] **and** *cfbi*[*crbi-rules*] **and** *cdbi*[*crbi-rules*] **and**
cmibi[*crbi-rules*] **and** *cbibi*[*crbi-rules*] **and** *covibi*[*crbi-rules*] **and** *csibi*[*crbi-rules*] **and** *cfibi*[*crbi-rules*] **and** *cdibi*[*crbi-rules*]

named-theorems *crov-rules* **declare** *ceov*[*crov-rules*] **and** *cbov*[*crov-rules*] **and** *cmov*[*crov-rules*] **and** *covov*[*crov-rules*] **and** *csov*[*crov-rules*] **and** *cfov*[*crov-rules*] **and** *cdov*[*crov-rules*] **and**
cmiov[*crov-rules*] **and** *cbiov*[*crov-rules*] **and** *coviov*[*crov-rules*] **and** *csiov*[*crov-rules*] **and** *cfiov*[*crov-rules*] **and** *cdiov*[*crov-rules*]

named-theorems *crovi-rules* **declare** *ceovi*[*crovi-rules*] **and** *cbovi*[*crovi-rules*] **and** *cmovi*[*crovi-rules*] **and** *covovi*[*crovi-rules*] **and** *csovi*[*crovi-rules*] **and** *cfovi*[*crovi-rules*] **and** *cdovi*[*crovi-rules*] **and**
cmiovi[*crovi-rules*] **and** *cbiovi*[*crovi-rules*] **and** *coviovi*[*crovi-rules*] **and** *csiovi*[*crovi-rules*] **and** *cfiovi*[*crovi-rules*] **and** *cdiovi*[*crovi-rules*]

named-theorems *crf-rules* **declare** *cef*[*crf-rules*] **and** *cbf*[*crf-rules*] **and** *cmf*[*crf-rules*] **and** *covf*[*crf-rules*] **and** *csf*[*crf-rules*] **and** *cff*[*crf-rules*] **and** *cdf*[*crf-rules*] **and**
cmif[*crf-rules*] **and** *cbif*[*crf-rules*] **and** *covif*[*crf-rules*] **and** *csif*[*crf-rules*] **and** *cfif*[*crf-rules*]

and *cdif*[*crf-rules*]

named-theorems *crfi-rules* **declare** *cefi*[*crfi-rules*] **and** *cbfi*[*crfi-rules*] **and** *cmfi*[*crfi-rules*] **and** *covfi*[*crfi-rules*] **and** *csfi*[*crfi-rules*] **and** *cffi*[*crfi-rules*] **and** *cdfi*[*crfi-rules*] **and**

cmifi[*crfi-rules*] **and** *cbifi*[*crfi-rules*] **and** *covifi*[*crfi-rules*] **and** *csifi*[*crfi-rules*] **and** *cfifi*[*crfi-rules*] **and** *cdifi*[*crfi-rules*]

named-theorems *crd-rules* **declare** *ced*[*crd-rules*] **and** *cbd*[*crd-rules*] **and** *cmd*[*crd-rules*] **and** *covd*[*crd-rules*] **and** *csd*[*crd-rules*] **and** *cfd*[*crd-rules*] **and** *cdd*[*crd-rules*] **and** *cmid*[*crd-rules*] **and** *cbid*[*crd-rules*] **and** *covid*[*crd-rules*] **and** *csid*[*crd-rules*] **and** *cfid*[*crd-rules*] **and** *cdid*[*crd-rules*]

named-theorems *crdi-rules* **declare** *cedi*[*crdi-rules*] **and** *cbdi*[*crdi-rules*] **and** *cmdi*[*crdi-rules*] **and** *covdi*[*crdi-rules*] **and** *csdi*[*crdi-rules*] **and** *cfdi*[*crdi-rules*] **and** *cddi*[*crdi-rules*] **and**
cmidi[*crdi-rules*] **and** *cbidi*[*crdi-rules*] **and** *covidi*[*crdi-rules*] **and** *csidi*[*crdi-rules*] **and** *cfidi*[*crdi-rules*] **and** *cdidi*[*crdi-rules*]

end

theory *disjoint-relations*

imports
allen

begin

4 PD property

The 13 time interval relations (i.e. e, b, m, s, f, d, ov and their inverse relations) are pairwise disjoint.

lemma *em* : $e \cap m = \{\}$
⟨*proof*⟩

lemma *eb* : $e \cap b = \{\}$
⟨*proof*⟩

lemma *eov* : $e \cap ov = \{\}$
⟨*proof*⟩

lemma *es* : $e \cap s = \{\}$
⟨*proof*⟩

lemma *ef* :$e \cap f = \{\}$
⟨*proof*⟩

lemma *ed* :$e \cap d = \{\}$
⟨*proof*⟩

lemma *emi* :$e \cap m\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *ebi* :$e \cap b\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *eovi* :$e \cap ov\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *esi* :$e \cap s\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *efi* :$e \cap f\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *edi* :$e \cap d\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *mb* :$m \cap b = \{\}$
⟨*proof*⟩

lemma *mov* : $m \cap ov = \{\}$
⟨*proof*⟩

lemma *ms* :$m \cap s = \{\}$
⟨*proof*⟩

lemma *mf* :$m \cap f = \{\}$
⟨*proof*⟩

lemma *md* :$m \cap d = \{\}$
⟨*proof*⟩

lemma *mi* :$m \cap m\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *mbi* :$m \cap b\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *movi* :$m \cap ov\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma $msi : m \cap s\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma $mfi : m \cap f\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma $mdi : m \cap d\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma $bov : b \cap ov = \{\}$
⟨*proof*⟩

lemma $bs : b \cap s = \{\}$
⟨*proof*⟩

lemma $bf : b \cap f = \{\}$
⟨*proof*⟩

lemma $bd : b \cap d = \{\}$
⟨*proof*⟩

lemma $bmi : b \cap m\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma $bi : b \cap b\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma $bovi : b \cap ov\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma $bsi : b \cap s\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma $bfi : b \cap f\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma $bdi : b \cap d\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma $ovs : ov \cap s = \{\}$
⟨*proof*⟩

lemma $ovf : ov \cap f = \{\}$
⟨*proof*⟩

lemma *ovd* :$ov \cap\ d = \{\}$
⟨*proof*⟩

lemma *ovmi* :$ov \cap m\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *ovbi* :$ov \cap b\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *ovi* :$ov \cap ov\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *ovsi* :$ov \cap s\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *ovfi* :$ov \cap f\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *ovdi* :$ov \cap d\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *sf* :$s \cap f = \{\}$
⟨*proof*⟩

lemma *sd* :$s \cap d = \{\}$
⟨*proof*⟩

lemma *smi* :$s \cap m\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *sbi* :$s \cap b\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *sovi* :$s \cap ov\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *si* :$s \cap s\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *sfi* :$s \cap f\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *sdi* :$s\cap d\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *fd* :$f \cap d = \{\}$

⟨*proof*⟩

lemma *fmi* :$f \cap m\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *fbi* :$f \cap b\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *fovi* :$f \cap ov\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *fsi* :$f \cap s\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *fi* :$f \cap f\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *fdi* :$f \cap d\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *dmi* :$d \cap m\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *dbi* :$d \cap b\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *dovi* :$d \cap ov\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *dsi* :$d \cap s\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *dfi* :$d \cap f\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *di* :$d \cap d\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *mibi* :$m\hat{}{-1} \cap b\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *miovi* :$m\hat{}{-1} \cap ov\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *misi* :$m\hat{}{-1} \cap s\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *mifi* :$m^\frown-1 \cap f^\frown-1 = \{\}$
⟨*proof*⟩

lemma *midi* :$m^\frown-1 \cap d^\frown-1 = \{\}$
⟨*proof*⟩

lemma *bid* :$b^\frown-1 \cap d = \{\}$
⟨*proof*⟩

lemma *bimi* :$b^\frown-1 \cap m^\frown-1 = \{\}$
⟨*proof*⟩

lemma *biovi* :$b^\frown-1 \cap ov^\frown-1 = \{\}$
⟨*proof*⟩

lemma *bisi* :$b^\frown-1 \cap s^\frown-1 = \{\}$
⟨*proof*⟩

lemma *bifi* :$b^\frown-1 \cap f^\frown-1 = \{\}$
⟨*proof*⟩

lemma *bidi* :$b^\frown-1 \cap d^\frown-1 = \{\}$
⟨*proof*⟩

lemma *ovisi* :$ov^\frown-1 \cap s^\frown-1 = \{\}$
⟨*proof*⟩

lemma *ovifi* :$ov^\frown-1 \cap f^\frown-1 = \{\}$
⟨*proof*⟩

lemma *ovidi* :$ov^\frown-1 \cap d^\frown-1 = \{\}$
⟨*proof*⟩

lemma *sifi* :$s^\frown-1 \cap f^\frown-1 = \{\}$
⟨*proof*⟩

lemma *sidi* :$s^\frown-1 \cap d^\frown-1 = \{\}$
⟨*proof*⟩

lemma *fidi* :$f^\frown-1 \cap d^\frown-1 = \{\}$
⟨*proof*⟩

lemma *eei*[*simp*]:$e^\frown\!-1 = e$
⟨*proof*⟩

lemma *rdisj-sym*:$A \cap B = \{\} \Longrightarrow B \cap A = \{\}$
⟨*proof*⟩

4.1 Intersection rules

named-theorems *e-rules* **declare** *em*[*e-rules*] **and** *eb*[*e-rules*] **and** *eov*[*e-rules*] **and** *es*[*e-rules*] **and** *ef*[*e-rules*] **and** *ed*[*e-rules*] **and** *emi*[*e-rules*] **and** *ebi*[*e-rules*] **and** *eovi*[*e-rules*]
and *esi*[*e-rules*] **and** *efi*[*e-rules*] **and** *edi*[*e-rules*]

named-theorems *m-rules* **declare** *em*[*THEN rdisj-sym*, *m-rules*] **and** *mb* [*m-rules*] **and** *ms* [*m-rules*] **and** *mov* [*m-rules*] **and** *mf*[*m-rules*] **and**
md[*m-rules*] **and** *mi* [*m-rules*] **and** *mbi* [*m-rules*] **and** *movi* [*m-rules*] **and** *msi* [*m-rules*] **and** *mfi* [*m-rules*] **and** *mdi* [*m-rules*] **and** *emi*[*m-rules*]

named-theorems *b-rules* **declare** *eb*[*THEN rdisj-sym*, *b-rules*] **and** *mb* [*THEN rdisj-sym*, *b-rules*] **and** *bs* [*b-rules*] **and** *bov* [*b-rules*] **and** *bf*[*b-rules*] **and**
bd[*b-rules*] **and** *bmi* [*b-rules*] **and** *bi* [*b-rules*] **and** *bovi* [*b-rules*] **and** *bsi* [*b-rules*] **and** *bfi* [*b-rules*] **and** *bdi* [*b-rules*] **and** *ebi*[*b-rules*]

named-theorems *ov-rules* **declare** *eov*[*THEN rdisj-sym*, *ov-rules*] **and** *mov* [*THEN rdisj-sym*, *ov-rules*] **and** *ovs* [*ov-rules*] **and** *bov* [*THEN rdisj-sym*,*ov-rules*] **and** *ovf*[*ov-rules*] **and**
ovd[*ov-rules*] **and** *ovmi* [*ov-rules*] **and** *ovi* [*ov-rules*] **and** *ovsi* [*ov-rules*] **and** *ovfi* [*ov-rules*] **and** *ovdi* [*ov-rules*] **and** *eovi*[*ov-rules*]

named-theorems *s-rules* **declare** *es*[*THEN rdisj-sym*, *s-rules*] **and** *ms* [*THEN rdisj-sym*, *s-rules*] **and** *ovs* [*THEN rdisj-sym*, *s-rules*] **and** *bs* [*THEN rdisj-sym*,*s-rules*] **and** *sf*[*s-rules*] **and**
sd[*s-rules*] **and** *smi* [*s-rules*] **and** *sovi* [*s-rules*] **and** *si* [*s-rules*] **and** *sfi* [*s-rules*] **and** *sdi* [*s-rules*]

named-theorems *d-rules* **declare** *ed*[*THEN rdisj-sym*, *d-rules*] **and** *md* [*THEN rdisj-sym*, *d-rules*] **and** *sd* [*THEN rdisj-sym*, *d-rules*] **and** *fd*[*THEN rdisj-sym*, *d-rules*] **and**
ovd[*THEN rdisj-sym*,*d-rules*] **and** *dmi* [*d-rules*] **and** *dovi* [*d-rules*] **and** *dsi* [*d-rules*] **and** *dfi* [*d-rules*] **and** *di* [*d-rules*]

named-theorems *f-rules* **declare** *ef*[*THEN rdisj-sym*, *f-rules*] **and** *mf* [*THEN rdisj-sym*, *f-rules*] **and** *sf* [*THEN rdisj-sym*, *f-rules*] **and** *ovf* [*THEN rdisj-sym*,*f-rules*] **and** *fd*[*f-rules*] **and**
fmi [*f-rules*] **and** *fovi* [*f-rules*] **and** *fsi* [*f-rules*] **and** *fi* [*f-rules*] **and** *fdi* [*f-rules*]

end

theory *jointly-exhaustive*

imports
allen

begin

5 JE property

The 13 time interval relations are jointly exhaustive. For any two intervals x and y, we can find a basic relation r such that $(x, y) \in r$.

lemma (**in** *arelations*) *jointly-exhaustive*:
assumes $\mathcal{I}\ p\ \mathcal{I}\ q$
shows $(p{::}'a,q{::}'a) \in b \lor (p,q) \in m \lor (p,q) \in ov \lor (p,q) \in s \lor (p,q) \in d \lor (p,q) \in f\hat{}{-1} \lor (p,q) \in e \lor$
$(p,q) \in f \lor (p,q) \in s\hat{}{-1} \lor (p,q) \in d\hat{}{-1} \lor (p,q) \in ov\hat{}{-1} \lor (p,q) \in m\hat{}{-1} \lor (p,q) \in b\hat{}{-1}$ (**is** *?R*)
⟨*proof*⟩

lemma (**in** *arelations*) *JE*:
assumes $\mathcal{I}\ p\ \mathcal{I}\ q$
shows $(p{::}'a,q{::}'a) \in b \cup m \cup ov \cup s \cup d \cup f\hat{}{-1} \cup e \cup f \cup s\hat{}{-1} \cup d\hat{}{-1} \cup ov\hat{}{-1} \cup m\hat{}{-1} \cup b\hat{}{-1}$
⟨*proof*⟩

end

theory *examples*

imports

disjoint-relations

begin

6 Examples

6.1 Compositions of non-basic relations

Basic relations are the 13 time interval relations. The unions of basic relations are also relations and their compositions is the union of compositions. We prove few of these compositions that are required in theory nest.thy.

method (**in** *arelations*) *e-compose* = (*match* **conclusion in** *e O b* ⊆ - ⇒ ‹*insert ceb*, *blast*›
| - ⇒ ‹*match conclusion in e O m* ⊆ - ⇒ ‹*insert cem*, *blast*› | - ⇒ ‹*fail*››)

declare [[*simp-trace-depth-limit=4*]]

lemma *eovisidifmifiOm*:$(e \cup\ ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup m^{-1} \cup f\hat{}{-1})\ O\ m \subseteq m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1} \cup s \cup s^{-1} \cup e$
⟨*proof*⟩

lemma *ovsmfidiesiOmi*:$(ov \cup s \cup m \cup f\hat{}{-1} \cup d\hat{}{-1} \cup e \cup s\hat{}{-1})\ O\ m\hat{}{-1} \subseteq d\hat{}{-1} \cup s\hat{}{-1} \cup\ ov\hat{}{-1} \cup m\hat{}{-1} \cup f\hat{}{-1} \cup f \cup e$
⟨*proof*⟩

lemma *ovsmfidiesiOm*:$(ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1})\ O\ m \subseteq b \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1} \cup m$
⟨*proof*⟩

lemma *ovsmfidiesiOssie*:$(ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1})\ O\ (s \cup s\hat{}{-1} \cup e) \subseteq ov \cup f\hat{}{-1} \cup d\hat{}{-1} \cup s \cup e \cup s\hat{}{-1} \cup m$
⟨*proof*⟩

lemma $(b \cup m \cup ov \cup s \cup d)\ O\ (b \cup m \cup ov \cup s \cup d) \subseteq\ b \cup m \cup ov \cup s \cup d$
⟨*proof*⟩

lemma *ebmovovissifsiddib*:$(e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1})\ O\ b \subseteq b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *ebmovovissiffiddibmovsd*:$(e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1})\ O\ (\ b \cup m \cup ov \cup s \cup d) \subseteq (b \cup m \cup ov \cup s \cup d \cup f\hat{}{-1} \cup d\hat{}{-1} \cup ov\hat{}{-1} \cup s^{-1} \cup f \cup e)$
⟨*proof*⟩

lemma *difimov*:$(d\hat{}{-1} \cup f\hat{}{-1} \cup ov \cup e \cup f \cup m \cup b \cup s\hat{}{-1} \cup s)\ O\ (\ m \cup ov \cup s \cup d \cup b \cup f\hat{}{-1} \cup f \cup e) \subseteq (\ e \cup b \cup m \cup ov \cup ov\hat{}{-1} \cup s \cup s\hat{}{-1} \cup f \cup f^{-1} \cup d \cup d^{-1})$

⟨*proof*⟩

lemma *difibs*:$(d^{-1} \cup f^{-1} \cup ov \cup e \cup f \cup m \cup b \cup s^{-1} \cup s)\ O\ (b \cup s \cup m) \subseteq (b \cup m \cup ov \cup f^{-1} \cup d^{-1} \cup d \cup e \cup s \cup s^{-1})$
⟨*proof*⟩

lemma *bebmovovissiffiddi*:$b\ O\ (e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1}) \subseteq (b \cup m \cup ov \cup s \cup d)$
⟨*proof*⟩

lemma *ovsmfidiesi*:$(((ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1})\ O\ (ov\hat{}{-1} \cup s\hat{}{-1} \cup m\hat{}{-1} \cup f \cup d \cup e \cup s)) \subseteq (s \cup s\hat{}{-1} \cup f \cup f\hat{}{-1} \cup d \cup d\hat{}{-1} \cup e \cup ov \cup ov\hat{}{-1} \cup m \cup m\hat{}{-1}))$
⟨*proof*⟩

lemma *piiq*:$(p,i) \in ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1} \Longrightarrow (i,q) \in ov\hat{}{-1} \cup s\hat{}{-1} \cup m\hat{}{-1} \cup f \cup d \cup e \cup s \Longrightarrow (p,q) \in s \cup s\hat{}{-1} \cup f \cup f\hat{}{-1} \cup d \cup d\hat{}{-1} \cup e \cup ov \cup ov\hat{}{-1} \cup m \cup m\hat{}{-1}$
⟨*proof*⟩

lemma *ceovisidiffimi-ffie*:$(e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1})\ O\ (f \cup f^{-1} \cup e) \subseteq\ e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$
⟨*proof*⟩

lemma *ceovisidiffimi-ffie-simp*:$(p,i) \in (e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}) \Longrightarrow (i,\ q) \in (f \cup f^{-1} \cup e) \Longrightarrow (p,q) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$

⟨*proof*⟩

lemma *ceovisidiffimi-fife*: $(e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1})\ O\ (f^{-1} \cup f \cup e)\ \subseteq\ e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$
⟨*proof*⟩

lemma $(x,\ j) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1} \Longrightarrow (j,\ i) \in f^{-1} \cup f \cup e \Longrightarrow (x,\ i) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$
⟨*proof*⟩

lemma *m-ovsmfidiesi*:$m\ O\ (ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}) \subseteq b \cup s \cup m$
⟨*proof*⟩

lemma *ovsmfidiesi-d*:$(ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1})\ O\ d \subseteq\ e \cup\ s \cup d \cup ov \cup ov\hat{}{-1} \cup s\hat{}{-1} \cup f \cup f\hat{}{-1}\ \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *cbi-esdovovisiffidi*:$b\hat{}{-1}\ O\ (e \cup s \cup d \cup ov \cup ov^{-1} \cup s^{-1} \cup f \cup f^{-1} \cup d^{-1}) \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1} \cup f \cup d$
⟨*proof*⟩

lemma *cm-alpha1ialpha4mi*:$m\ O\ \ (e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}) \subseteq m \cup ov \cup s \cup d \cup b \cup f\hat{}{-1} \cup f \cup e$
⟨*proof*⟩

lemma *cbi-alpha1ialpha4mi*:$b\hat{}{-1}\ O\ (\ e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}) \subseteq\ \ b\hat{}{-1}$
⟨*proof*⟩

lemma *cbeta2-beta2*:$(\ b \cup m \cup ov \cup f^{-1} \cup d^{-1})\ O\ (\ b \cup m \cup ov \cup f^{-1} \cup d^{-1}) \subseteq b \cup m \cup ov \cup f^{-1} \cup d^{-1}$
⟨*proof*⟩

lemma *cbeta2-gammabm*: $(b \cup m \cup ov \cup f^{-1} \cup d^{-1})\ O\ (\ e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1}) \subseteq (\ e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1})$
⟨*proof*⟩

lemma *calpha1-alpha1*:$(b \cup m \cup ov \cup s \cup d)\ O\ (\ b \cup m \cup ov \cup s \cup d) \subseteq (\ b \cup m \cup ov \cup s \cup d)$
⟨*proof*⟩

6.2 Intersection of non-basic relations

lemma *inter-ov*:
assumes $(i,j) \in\ (b \cup m \cup ov \cup f^{-1} \cup d^{-1}) \cap (e \cup b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1} \cup ov \cup s\hat{}{-1} \cup s \cup f\hat{}{-1} \cup f \cup d\hat{}{-1} \cup d) \cap (b \cup m \cup ov \cup s \cup d)$
shows $(i,j) \in ov$
⟨*proof*⟩

lemma *neq-beta2i-alpha2alpha5m*:
assumes $(q,\ j) \in b^{-1} \cup d \cup f \cup ov^{-1} \cup m^{-1}$ **and** $(q,\ j) \in ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}$
shows *False*
⟨*proof*⟩

lemma *neq-bi-alpha1ialpha4mi*:
assumes $(q,i) \in b\hat{}{-1}$ **and** $(q,\ i) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$
shows *False*
⟨*proof*⟩

end

theory *nest*

imports
Main *jointly-exhaustive* *examples*
HOL−Eisbach.Eisbach-Tools

begin

7 Nests

Nests are sets of intervals that share a meeting point. We define relation before between nests that give the ordering properties of points.

7.1 Definitions

type-synonym *'a nest* = *'a set*

definition (**in** *arelations*) *BEGIN* :: *'a* ⇒ *'a nest*
where *BEGIN i* = {*j* | *j*. (*j*,*i*) ∈ *ov* ∪ *s* ∪ *m* ∪ *f*^−*1* ∪ *d*^−*1* ∪ *e* ∪ *s*^−*1*}

definition (**in** *arelations*) *END* :: *'a* ⇒ *'a nest*
where *END i* = {*j* | *j*. (*j*,*i*) ∈ *e* ∪ *ov*^−*1* ∪ *s*^−*1* ∪ *d*^−*1* ∪ *f* ∪ *f*^−*1* ∪ *m*^−*1*}

definition (**in** *arelations*) *NEST* ::*'a nest* ⇒ *bool*
where *NEST S* ≡ ∃ *i*. $\mathcal{I}$ *i* ∧ (*S* = *BEGIN i* ∨ *S* = *END i*)

definition (**in** *arelations*) *before* :: *'a nest* ⇒ *'a nest* ⇒ *bool* (**infix** ‹≪› *100*)
where *before N M* ≡ *NEST N* ∧ *NEST M* ∧ (∃ *n m*. ~~ℐ m N ℐ n N~~ *n* ∈ *N* ∧ *m* ∈ *M* ∧ (*n*,*m*) ∈ *b*)

7.2 Properties of Nests

lemma *intv1*:
assumes $\mathcal{I}$ *i*
shows *i* ∈ *BEGIN i*
⟨*proof*⟩

lemma *intv2*:
assumes $\mathcal{I}$ *i*
shows *i* ∈ *END i*
⟨*proof*⟩

lemma *NEST-nonempty*:
assumes *NEST S*
shows *S* ≠ {}
⟨*proof*⟩

lemma *NEST-BEGIN*:
assumes $\mathcal{I}$ *i*
shows *NEST* (*BEGIN i*)

⟨*proof*⟩

lemma *NEST-END*:
assumes $\mathcal{I}$ *i*
shows *NEST* (*END i*)
⟨*proof*⟩

lemma *before*:
assumes *a*:$\mathcal{I}$ *i*
shows *BEGIN i* $\ll$ *END i*
⟨*proof*⟩

lemma *meets*:
fixes *i j*
assumes $\mathcal{I}$ *i* **and** $\mathcal{I}$ *j*
shows (*i*,*j*) $\in$ *m* = ((*END i*) = (*BEGIN j*))
⟨*proof*⟩

lemma *starts*:
fixes *i j*
assumes $\mathcal{I}$ *i* **and** $\mathcal{I}$ *j*
shows ((*i*,*j*) $\in$ *s* $\cup$ *s*^−*1* $\cup$ *e*) = (*BEGIN i* = *BEGIN j*)
⟨*proof*⟩

lemma *xj-set*:$x \in \{a \mid a.\ (a, j) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}\} = ((x,j) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1})$
⟨*proof*⟩

lemma *ends*:
fixes *i j*
assumes $\mathcal{I}$ *i* **and** $\mathcal{I}$ *j*
shows ((*i*,*j*) $\in$ *f* $\cup$ *f*^−*1* $\cup$ *e*) = (*END i* = *END j*)
⟨*proof*⟩

lemma *before-irrefl*:
fixes *a*
shows $\neg$ *a* $\ll$ *a*
⟨*proof*⟩

lemma *BEGIN-before*:
fixes *i j*
assumes $\mathcal{I}$ *i* **and** $\mathcal{I}$ *j*
shows *BEGIN i* $\ll$ *BEGIN j* = $((i,j) \in b \cup m \cup ov \cup f^{-1} \cup d^{-1})$
⟨*proof*⟩

lemma *BEGIN-END-before*:
fixes *i j*
assumes $\mathcal{I}$ *i* **and** $\mathcal{I}$ *j*

shows *BEGIN i* ≪ *END j* = ((*i*,*j*) ∈ *e* ∪ *b* ∪ *m* ∪ *ov* ∪ *ov*^−*1* ∪ *s* ∪ *s*^−*1* ∪ *f* ∪ f^{-1} ∪ *d* ∪ d^{-1})
⟨*proof*⟩

lemma *END-BEGIN-before*:
fixes *i j*
assumes $\mathcal{I}$ *i* **and** $\mathcal{I}$ *j*
shows *END i* ≪ *BEGIN j* = ((*i*,*j*) ∈ *b*)
⟨*proof*⟩

lemma *END-END-before*:
fixes *i j*
assumes $\mathcal{I}$ *i* **and** $\mathcal{I}$ *j*
shows *END i* ≪ *END j* = ((*i*,*j*) ∈ *b* ∪ *m* ∪ *ov* ∪ *s* ∪ *d*)
⟨*proof*⟩

lemma *overlaps*:
assumes $\mathcal{I}$ *i* **and** $\mathcal{I}$ *j*
shows (*i*,*j*) ∈ *ov* = ((*BEGIN i* ≪ *BEGIN j*) ∧ (*BEGIN j* ≪ *END i*) ∧ (*END i* ≪ *END j*))
⟨*proof*⟩

7.3 Ordering of nests

class *strict-order* =
fixes *ls*::′*a nest* ⇒ ′*a nest* ⇒ *bool*
assumes
irrefl:¬ *ls a a* **and**
trans:*ls a c* ⟹ *ls c g* ⟹ *ls a g* **and**
asym:*ls a c* ⟹ ¬ *ls c a*

class *total-strict-order* = *strict-order* +
assumes *trichotomy*: *a* = *c* ⟹ (¬ (*ls a c*) ∧ ¬ (*ls c a*))

interpretation *nest*:*total-strict-order* (≪)
⟨*proof*⟩

end