Allen's Interval Calculus

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May 21, 2019

Contents

1 Axioms **2**

2 Time interval relations **4**

3 Basic relations **4**
- 3.1 e-composition . . . 4
- 3.2 r-composition . . . 5
- 3.3 α-composition . . . 8
- 3.4 β-composition . . . 9
- 3.5 γ-composition . . . 10
- 3.6 γ-composition . . . 10
- 3.7 The rest of the composition table . . . 11
- 3.8 Composition rules . . . 15

4 PD property **18**
- 4.1 Intersection rules . . . 24

5 JE property **25**

6 Examples **26**
- 6.1 Compositions of non-basic relations . . . 26
- 6.2 Intersection of non-basic relations . . . 28

7 Nests **29**
- 7.1 Definitions . . . 29
- 7.2 Properties of Nests . . . 29
- 7.3 Ordering of nests . . . 31

theory *xor-cal*

imports

Main

begin
definition $xor::bool \Rightarrow bool \Rightarrow bool$ (**infixl** $\oplus$ *60*)
where $xor\ A\ B \equiv (A \wedge \neg B) \vee (\neg A \wedge B)$

declare *xor-def* [*simp*]

interpretation *bool*:*semigroup* $(\oplus)$
$\langle proof \rangle$

lemma *xor-distr-L* [*simp*]:$A \oplus (B \oplus C) = (A\wedge\neg B\wedge\neg C)\vee(A\wedge B\wedge C)\vee(\neg A\wedge B\wedge\neg C)\vee(\neg A\wedge\neg B\wedge C)$
$\langle proof \rangle$

lemma *xor-distr-R* [*simp*]:$(A \oplus B) \oplus C = A \oplus (B \oplus C)$
$\langle proof \rangle$

end

theory *axioms*

imports
Main xor-cal

begin

1 Axioms

We formalize Allen's definition of theory of time in term of intervals (Allen, 1983). Two relations, namely meets and equality, are defined between intervals. Two interval meets if they are adjacent A set of 5 axioms ((M1) $\sim$ (M5)) are then defined based on relation meets.

We define a class interval whose assumptions are (i) properties of relations meets and, (ii) axioms (M1) $\sim$ (M5).

class *interval* =
fixes
$meets::'a \Rightarrow 'a \Rightarrow bool$ (**infixl** $\|$ *60*) **and**
$\mathcal{I}::'a \Rightarrow bool$
assumes
meets-atrans:$[\![(p\|q);(q\|r)]\!] \Longrightarrow \neg(p\|r)$ **and**
meets-irrefl:$\mathcal{I}\ p \Longrightarrow \neg(p\|p)$ **and**
meets-asym:$(p\|q) \Longrightarrow \neg(q\|p)$ **and**

meets-wd:$p \| q \Longrightarrow \mathcal{I}\ p \wedge \mathcal{I}\ q$ **and**

M1:$[\![(p\|q);\ (p\|s);\ (r\|q)]\!] \Longrightarrow (r\|s)$ **and**
M2:$[\![(p\|q)\ ;\ (r\|s)]\!] \Longrightarrow p\|s \oplus ((\exists\, t.\ (p\|t)\wedge(t\|s)) \oplus (\exists\, t.\ (r\|t)\wedge(t\|q)))$ **and**
M3:$\mathcal{I}\ p \Longrightarrow (\exists\, q\ r.\ q\|p \wedge p\|r)$ **and**
M4:$[\![p\|q\ ;\ q\|s\ ;\ p\|r\ ;\ r\|s]\!] \Longrightarrow q = r$ **and**
M5exist:$p\|q \Longrightarrow (\exists\, r\ s\ t.\ r\|p \wedge p\|q \wedge q\|s \wedge r\|t \wedge t\|s)$

lemma (**in** *interval*) *trans2*:$[\![p\|t;\ t\|r;\ r\|q]\!] \Longrightarrow \neg p\|q$
⟨*proof*⟩

lemma (**in** *interval*) *nontrans1*: $u\|r \Longrightarrow \neg\ (\exists\, t.\ u\|t \wedge t\|r)$
⟨*proof*⟩

lemma (**in** *interval*) *nontrans2*:$u\|r \Longrightarrow \neg\ (\exists\, t.\ r\|t \wedge t\|u)$
⟨*proof*⟩

lemma (**in** *interval*) *nonmeets1*:$\neg\ (u\|r \wedge r\|u)$
⟨*proof*⟩

lemma (**in** *interval*) *nonmeets2*: $[\![\mathcal{I}\ u\ ;\ \mathcal{I}\ r\ ]\!] \Longrightarrow \neg\ (u\|r \wedge u = r)$
⟨*proof*⟩

lemma (**in** *interval*) *nonmeets3*: $\neg\ (u\|r \wedge (\exists\, p.\ u\|p \wedge p\|r))$
⟨*proof*⟩

lemma (**in** *interval*) *nonmeets4*: $\neg(u\|r \wedge (\exists\, p.\ r\|p \wedge p\|u))$
⟨*proof*⟩

lemma (**in** *interval*) *elimmeets*: $(p\ \|\ s \wedge (\exists\, t.\ p\ \|\ t \wedge t\ \|\ s) \wedge (\exists\, t.\ r\ \|\ t \wedge t\ \|\ q)) = \mathit{False}$
⟨*proof*⟩

lemma (**in** *interval*) *M5exist-var*:
assumes $x\|y\ y\|z\ z\|w$
shows $\exists\, t.\ x\|t \wedge t\|w$
⟨*proof*⟩

lemma (**in** *interval*) *M5exist-var2*:
assumes $p\|q$
shows $\exists\, r1\ r2\ r3\ s\ t.\ r1\|r2 \wedge r2\|r3 \wedge r3\|p \wedge p\|q \wedge q\|s \wedge r1\|t \wedge t\|s$
⟨*proof*⟩

lemma (**in** *interval*) *M5exist-var3*:
assumes $k\|l$ **and** $l\|q$ **and** $q\|t$ **and** $t\|r$
shows $\exists\, lqt.\ k\|lqt \wedge lqt\|r$
⟨*proof*⟩

end

2 Time interval relations

theory *allen*

imports

Main axioms
HOL−Eisbach.Eisbach-Tools

begin

3 Basic relations

We define 7 binary relations between time intervals. Relations e, m, b, ov, d, s and f stand for equal, meets, before, overlaps, during, starts and finishes, respectively.

class *arelations* = *interval* +
fixes
e::($'a \times 'a$) *set* **and**
m::($'a \times 'a$) *set* **and**
b::($'a \times 'a$) *set* **and**
ov::($'a \times 'a$) *set* **and**
d::($'a \times 'a$) *set* **and**
s::($'a \times 'a$) *set* **and**
f::($'a \times 'a$) *set*
assumes
e:$(p,q) \in e = (p = q)$ **and**
m:$(p,q) \in m = p \| q$ **and**
b:$(p,q) \in b = (\exists\, t::'a.\ p \| t \wedge t \| q)$ **and**
ov:$(p,q) \in ov = (\exists\, k\ l\ u\ v\ t::'a.$
$(k \| p \wedge p \| u \wedge u \| v) \wedge (k \| l \wedge l \| q \wedge q \| v) \wedge (l \| t \wedge t \| u))$ **and**
s:$(p,q) \in s = (\exists\, k\ u\ v::'a.\ k \| p \wedge p \| u \wedge u \| v \wedge k \| q \wedge q \| v)$ **and**
f:$(p,q) \in f = (\exists\, k\ l\ u::'a.\ k \| l \wedge l \| p \wedge p \| u \wedge k \| q \wedge q \| u)$ **and**
d:$(p,q) \in d = (\exists\, k\ l\ u\ v::'a.\ k \| l \wedge l \| p \wedge p \| u \wedge u \| v \wedge k \| q \wedge q \| v)$

3.1 e-composition

Relation e is the identity relation for composition.

lemma *cer*:
assumes $r \in \{e,m,b,ov,s,f,d,m\hat{}{-1},b\hat{}{-1},ov\hat{}{-1},s\hat{}{-1},f\hat{}{-1},d\hat{}{-1}\}$
shows $e\ O\ r = r$
$\langle proof \rangle$

lemma *cre*:
assumes $r \in \{e,m,b,ov,s,f,d,m\hat{}{-1},b\hat{}{-1},ov\hat{}{-1},s\hat{}{-1},f\hat{}{-1},d\hat{}{-1}\}$
shows $r\ O\ e = r$
⟨*proof*⟩

lemmas *ceb* = *cer*[*of b*]
lemmas *cebi* = *cer*[*of b ˆ−1*]
lemmas *cem* = *cer*[*of m*]
lemmas *cemi* = *cer*[*of m ˆ−1*]
lemmas *cee* = *cer*[*of e*]
lemmas *ces* = *cer*[*of s*]
lemmas *cesi* = *cer*[*of s ˆ−1*]
lemmas *cef* = *cer*[*of f*]
lemmas *cefi* = *cer*[*of f ˆ−1*]
lemmas *ceov* = *cer*[*of ov*]
lemmas *ceovi* = *cer*[*of ov ˆ−1*]
lemmas *ced* = *cer*[*of d*]
lemmas *cedi* = *cer*[*of d ˆ−1*]
lemmas *cbe* = *cre*[*of b*]
lemmas *cbie* = *cre*[*of b ˆ−1*]
lemmas *cme* = *cre*[*of m*]
lemmas *cmie* = *cre*[*of m ˆ−1*]
lemmas *cse* = *cre*[*of s*]
lemmas *csie* = *cre*[*of s ˆ−1*]
lemmas *cfe* = *cre*[*of f*]
lemmas *cfie* = *cre*[*of f ˆ−1*]
lemmas *cove* = *cre*[*of ov*]
lemmas *covie* = *cre*[*of ov ˆ−1*]
lemmas *cde* = *cre*[*of d*]
lemmas *cdie* = *cre*[*of d ˆ−1*]

3.2 r-composition

We prove compositions of the form $r_1 \circ r_2 \subseteq r$, where r is a basic relation.

method (**in** *arelations*) *r-compose* **uses** *r1 r2 r3* = ((*auto*, (*subst* (*asm*) *r1*), (*subst* (*asm*) *r2*), (*subst r3*)) , (*meson M5exist-var*))

lemma (**in** *arelations*) *cbb*:$b\ O\ b \subseteq b$
⟨*proof*⟩

lemma (**in** *arelations*) *cbm*:$b\ O\ m \subseteq b$
⟨*proof*⟩

lemma *cbov*:$b\ O\ ov \subseteq b$
⟨*proof*⟩

lemma *cbfi*:$b\ O\ f\hat{}{-1} \subseteq b$
⟨*proof*⟩

lemma *cbdi*:$b\ O\ d\hat{}{-1} \subseteq b$
⟨*proof*⟩

lemma *cbs*:$b\ O\ s \subseteq b$
⟨*proof*⟩

lemma *cbsi*:$b\ O\ s\hat{}{-1} \subseteq b$
⟨*proof*⟩

lemma (**in** *arelations*) *cmb*:$m\ O\ b \subseteq b$
⟨*proof*⟩

lemma *cmm*:$m\ O\ m \subseteq b$
⟨*proof*⟩

lemma *cmov*:$m\ O\ ov \subseteq b$
⟨*proof*⟩

lemma *cmfi*:$m\ O\ f\hat{}{-1} \subseteq b$
⟨*proof*⟩

lemma *cmdi*:$m\ O\ d\hat{}{-1} \subseteq b$
⟨*proof*⟩

lemma *cms*:$m\ O\ s \subseteq m$
⟨*proof*⟩

lemma *cmsi*:$m\ O\ s\hat{}{-1} \subseteq m$
⟨*proof*⟩

lemma *covb*:$ov\ O\ b \subseteq b$
⟨*proof*⟩

lemma *covm*:$ov\ O\ m \subseteq b$
⟨*proof*⟩

lemma *covs*:$ov\ O\ s \subseteq ov$
⟨*proof*⟩

lemma *cfib*:$f\hat{}{-1}\ O\ b \subseteq b$
⟨*proof*⟩

lemma *cfim*:$f\hat{}{-1}\ O\ m \subseteq m$
⟨*proof*⟩

lemma *cfiov*:$f\hat{}{-1}\ O\ ov \subseteq ov$
⟨*proof*⟩

lemma *cfifi*:$f\hat{}{-1}\ O\ f\hat{}{-1} \subseteq f\hat{}{-1}$
⟨*proof*⟩

lemma *cfidi*:$f\hat{}{-1}\ O\ d\hat{}{-1} \subseteq d\hat{}{-1}$
⟨*proof*⟩

lemma *cfis*:$f\hat{}{-1}\ O\ s \subseteq ov$
⟨*proof*⟩

lemma *cfisi*:$f\hat{}{-1}\ O\ s\hat{}{-1} \subseteq d\hat{}{-1}$
⟨*proof*⟩

lemma *cdifi*:$d\hat{}{-1}\ O\ f\hat{}{-1} \subseteq d\hat{}{-1}$
⟨*proof*⟩

lemma *cdidi*:$d\hat{}{-1}\ O\ d\hat{}{-1} \subseteq d\hat{}{-1}$
⟨*proof*⟩

lemma *cdisi*:$d\hat{}{-1}\ O\ s\hat{}{-1} \subseteq d\hat{}{-1}$
⟨*proof*⟩

lemma *csb*:$s\ O\ b \subseteq b$
⟨*proof*⟩

lemma *csm*:$s\ O\ m \subseteq b$
⟨*proof*⟩

lemma *css*:$s\ O\ s \subseteq s$
⟨*proof*⟩

lemma *csifi*:$s\hat{}{-1}\ O\ f\hat{}{-1} \subseteq d\hat{}{-1}$
⟨*proof*⟩

lemma *csidi*:$s\hat{}{-1}\ O\ d\hat{}{-1} \subseteq d\hat{}{-1}$
⟨*proof*⟩

lemma *cdb*:$d\ O\ b \subseteq b$
⟨*proof*⟩

lemma *cdm*:$d\ O\ m \subseteq b$
⟨*proof*⟩

lemma *cfb*:$f\ O\ b \subseteq b$
⟨*proof*⟩

lemma *cfm*:$f\ O\ m \subseteq m$
⟨*proof*⟩

3.3 α-composition

We prove compositions of the form $r_1 \circ r_2 \subseteq s \cup ov \cup d$.

lemma (**in** *arelations*) *cmd*:$m\ O\ d \subseteq s \cup ov \cup d$
⟨*proof*⟩

lemma (**in** *arelations*) *cmf*:$m\ O\ f \subseteq s \cup ov \cup d$
⟨*proof*⟩

lemma *cmovi*:$m\ O\ ov\hat{}{-1} \subseteq s \cup ov \cup d$
⟨*proof*⟩

lemma *covd*:$ov\ O\ d \subseteq s \cup ov \cup d$
⟨*proof*⟩

lemma *covf*:$ov\ O\ f \subseteq s \cup ov \cup d$
⟨*proof*⟩

lemma *cfid*:$f\hat{}{-1}\ O\ d \subseteq s \cup ov \cup d$
⟨*proof*⟩

lemma *cfov*:$f\ O\ ov \subseteq ov \cup s \cup d$
⟨*proof*⟩

We prove compositions of the form $r_1 \circ r_2 \subseteq ov \cup f^{-1} \cup d^{-1}$.

lemma *covsi*:$ov\ O\ s\hat{}{-1} \subseteq ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *cdim*:$d\hat{}{-1}\ O\ m \subseteq ov \cup d\hat{}{-1} \cup f\hat{}{-1}$
⟨*proof*⟩

lemma *cdiov*:$d\hat{}{-1}\ O\ ov \subseteq ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *cdis*:$d\hat{}{-1}\ O\ s \subseteq ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *csim*:$s\hat{}{-1}\ O\ m \subseteq ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *csiov*:$s\hat{}{-1}\ O\ ov \subseteq ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *covim*:$ov\hat{}{-1}\ O\ m \subseteq ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

We prove compositions of the form $r_1 \circ r_2 \subseteq b \cup m \cup ov$.

lemma *covov*:*ov O ov* ⊆ *b* ∪ *m* ∪ *ov*
⟨*proof*⟩

lemma *covfi*:*ov O f*ˆ−*1* ⊆ *b* ∪ *m* ∪ *ov*
⟨*proof*⟩

lemma *csov*:*s O ov* ⊆ *b* ∪ *m* ∪ *ov*
⟨*proof*⟩

lemma *csfi*:*s O f*ˆ−*1* ⊆ *b* ∪ *m* ∪ *ov*
⟨*proof*⟩

We prove compositions of the form $r_1 \circ r_2 \subseteq f \cup f^{-1} \cup e$.

lemma *cmmi*:*m O m*ˆ−*1* ⊆ *f* ∪ *f*ˆ−*1* ∪ *e*
⟨*proof*⟩

lemma *cfif*:*f*ˆ−*1 O f* ⊆ *e* ∪ *f*ˆ−*1* ∪ *f*
⟨*proof*⟩

lemma *cffi*:*f O f*ˆ−*1* ⊆ *e* ∪ *f* ∪ *f*ˆ−*1*
⟨*proof*⟩

We prove compositions of the form $r_1 \circ r_2 \subseteq e \cup s \cup s^{-1}$.

lemma *cssi*:*s O s*ˆ−*1* ⊆ *e* ∪ *s* ∪ *s*ˆ−*1*
⟨*proof*⟩

lemma *csis*:*s*ˆ−*1 O s* ⊆ *e* ∪ *s* ∪ *s*ˆ−*1*
⟨*proof*⟩

lemma *cmim*:*m*ˆ−*1 O m* ⊆ *s* ∪ *s*ˆ−*1* ∪ *e*
⟨*proof*⟩

3.4 β-composition

We prove compositions of the form $r_1 \circ r_2 \subseteq b \cup m \cup ov \cup s \cup d$.

lemma *cbd*:*b O d* ⊆ *b* ∪ *m* ∪ *ov* ∪ *s* ∪ *d*
⟨*proof*⟩

lemma *cbf*:*b O f* ⊆ *b* ∪ *m* ∪ *ov* ∪ *s* ∪ *d*
⟨*proof*⟩

lemma *cbovi*:*b O ov*ˆ−*1* ⊆ *b* ∪ *m* ∪ *ov* ∪ *s* ∪ *d*
⟨*proof*⟩

lemma *cbmi*:*b O m*ˆ−*1* ⊆ *b* ∪ *m* ∪ *ov* ∪ *s* ∪ *d*

⟨*proof*⟩

lemma *cdov*:*d O ov* ⊆*b* ∪ *m* ∪ *ov* ∪ *s* ∪ *d*
⟨*proof*⟩

lemma *cdfi*:*d O f ˆ−1* ⊆ *b* ∪ *m* ∪ *ov* ∪ *s* ∪ *d*
⟨*proof*⟩

We prove compositions of the form $r_1 \circ r_2 \subseteq b \cup m \cup ov \cup f^{-1} \cup d^{-1}$.

lemma *covdi*:*ov O d ˆ−1* ⊆ *b* ∪ *m* ∪ *ov* ∪ *f ˆ−1* ∪ *d ˆ−1*
⟨*proof*⟩

lemma *cdib*:*d ˆ−1 O b* ⊆ *b* ∪ *m* ∪ *ov* ∪ *f ˆ−1* ∪ *d ˆ−1*
⟨*proof*⟩

lemma *csdi*:*s O d ˆ−1* ⊆ *b* ∪ *m* ∪ *ov* ∪ *f ˆ−1* ∪ *d ˆ−1*
⟨*proof*⟩

lemma *csib*:*s ˆ−1 O b* ⊆ *b* ∪ *m* ∪ *ov* ∪ *f ˆ−1* ∪ *d ˆ−1*
⟨*proof*⟩

lemma *covib*:*ov ˆ−1 O b* ⊆ *b* ∪ *m* ∪ *ov* ∪ *f ˆ−1* ∪ *d ˆ−1*
⟨*proof*⟩

lemma *cmib*:*m ˆ−1 O b* ⊆ *b* ∪ *m* ∪ *ov* ∪ *f ˆ−1* ∪ *d ˆ−1*
⟨*proof*⟩

3.5 γ-composition

We prove compositions of the form $r_1 \circ r_2 \subseteq ov \cup s \cup d \cup f \cup e \cup f^{-1} \cup d^{-1} \cup s^{-1} \cup ov^{-1}$.

lemma *covovi*:*ov O ov ˆ−1* ⊆ *e* ∪ *ov* ∪ *ov ˆ−1* ∪ *d* ∪ *d ˆ−1* ∪ *s* ∪ *s ˆ−1* ∪ *f* ∪ *f ˆ−1*
⟨*proof*⟩

lemma *cdid*:*d ˆ−1 O d* ⊆ *e* ∪ *ov* ∪ *ov ˆ−1* ∪ *d* ∪ *d ˆ−1* ∪ *s* ∪ *s ˆ−1* ∪ *f* ∪ *f ˆ−1*
⟨*proof*⟩

lemma *coviov*:*ov ˆ−1 O ov* ⊆ *e* ∪ *ov* ∪ *ov ˆ−1* ∪ *d* ∪ *d ˆ−1* ∪ *s* ∪ *s ˆ−1* ∪ *f* ∪ *f ˆ−1*
⟨*proof*⟩

3.6 γ-composition

We prove compositions of the form $r_1 \circ r_2 \subseteq b \cup m \cup ov \cup s \cup d \cup f \cup e \cup f^{-1} \cup d^{-1} \cup s^{-1} \cup ov^{-1} \cup b^{-1} \cup m^{-1}$.

lemma *cbbi*:$b\ O\ b\hat{}{-1} \subseteq b \cup b\hat{}{-1} \cup m \cup m\hat{}{-1} \cup e \cup ov \cup ov\hat{}{-1} \cup s \cup s\hat{}{-1} \cup d \cup d\hat{}{-1} \cup f \cup f\hat{}{-1}$ (**is** $b\ O\ b\hat{}{-1} \subseteq ?R$)
⟨*proof*⟩

lemma *cbib*:$b\hat{}{-1}\ O\ b \subseteq b \cup b\hat{}{-1} \cup m \cup m\hat{}{-1} \cup e \cup ov \cup ov\hat{}{-1} \cup s \cup s\hat{}{-1} \cup d \cup d\hat{}{-1} \cup f \cup f\hat{}{-1}$ (**is** $b\hat{}{-1}\ O\ b \subseteq ?R$)
⟨*proof*⟩

lemma *cddi*:$d\ O\ d\hat{}{-1} \subseteq b \cup b\hat{}{-1} \cup m \cup m\hat{}{-1} \cup e \cup ov \cup ov\hat{}{-1} \cup s \cup s\hat{}{-1} \cup d \cup d\hat{}{-1} \cup f \cup f\hat{}{-1}$ (**is** $d\ O\ d\hat{}{-1} \subseteq ?R$)
⟨*proof*⟩

3.7 The rest of the composition table

Because of the symmetry $(r_1 \circ r_2)^{-1} = r_2^{-1} \circ r_1^{-1}$, the rest of the compositions is easily deduced.

lemma *cmbi*:$m\ O\ b\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup s\hat{}{-1} \cup ov\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *covmi*:$ov\ O\ m\hat{}{-1} \subseteq ov\hat{}{-1} \cup d\hat{}{-1} \cup s\hat{}{-1}$
⟨*proof*⟩

lemma *covbi*:$ov\ O\ b\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup s\hat{}{-1} \cup ov\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *cfiovi*:$f\hat{}{-1}\ O\ ov\hat{}{-1} \subseteq ov\hat{}{-1} \cup s\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *cfimi*:$(f\hat{}{-1}\ O\ m\hat{}{-1}) \subseteq s\hat{}{-1} \cup ov\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *cfibi*:$f\hat{}{-1}\ O\ b\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1} \cup s\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *cdif*:$d\hat{}{-1}\ O\ f \subseteq ov\hat{}{-1} \cup s\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *cdiovi*:$d\hat{}{-1}\ O\ ov\ \hat{}{-1} \subseteq ov\hat{}{-1} \cup s\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *cdimi*:$d\hat{}{-1}\ O\ m\hat{}{-1} \subseteq s\hat{}{-1} \cup ov\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *cdibi*:$d\hat{}{-1}\ O\ b\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1} \cup s\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *csd*:$s\ O\ d \subseteq d$
⟨*proof*⟩

lemma *csf*:$s\ O\ f \subseteq d$
⟨*proof*⟩

lemma *csovi*:$s\ O\ ov\hat{}{-1} \subseteq ov\hat{}{-1} \cup f \cup d$
⟨*proof*⟩

lemma *csmi*:$s\ O\ m\hat{}{-1} \subseteq m\hat{}{-1}$
⟨*proof*⟩

lemma *csbi*:$s\ O\ b\hat{}{-1} \subseteq b\hat{}{-1}$
⟨*proof*⟩

lemma *csisi*:$s\hat{}{-1}\ O\ s\hat{}{-1} \subseteq s\hat{}{-1}$
⟨*proof*⟩

lemma *csid*:$s\hat{}{-1}\ O\ d \subseteq ov\hat{}{-1} \cup f \cup d$
⟨*proof*⟩

lemma *csif*:$s\hat{}{-1}\ O\ f \subseteq ov\hat{}{-1}$
⟨*proof*⟩

lemma *csiovi*:$s\hat{}{-1}\ O\ ov\hat{}{-1} \subseteq ov\hat{}{-1}$
⟨*proof*⟩

lemma *csimi*:$s\hat{}{-1}\ O\ m\hat{}{-1} \subseteq m\hat{}{-1}$
⟨*proof*⟩

lemma *csibi*:$s\hat{}{-1}\ O\ b\hat{}{-1} \subseteq b\hat{}{-1}$
⟨*proof*⟩

lemma *cds*:$d\ O\ s \subseteq d$
⟨*proof*⟩

lemma *cdsi*:$d\ O\ s\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1} \cup f \cup d$
⟨*proof*⟩

lemma *cdd*:$d\ O\ d \subseteq d$
⟨*proof*⟩

lemma *cdf*:$d\ O\ f \subseteq d$
⟨*proof*⟩

lemma *cdovi*:$d\ O\ ov\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1} \cup f \cup d$
⟨*proof*⟩

lemma *cdmi*:$d\ O\ m\hat{}{-1} \subseteq b\hat{}{-1}$

⟨*proof*⟩

lemma *cdbi*:$d\ O\ b\hat{}{-1} \subseteq b\hat{}{-1}$
⟨*proof*⟩

lemma *cfdi*:$f\ O\ d\hat{}{-1} \subseteq\ b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1} \cup s\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *cfs*:$f\ O\ s \subseteq d$
⟨*proof*⟩

lemma *cfsi*:$f\ O\ s\hat{}{-1} \subseteq b\ \hat{}{-1} \cup m\hat{}{-1} \cup ov\ \hat{}{-1}$
⟨*proof*⟩

lemma *cfd*:$f\ O\ d \subseteq d$
⟨*proof*⟩

lemma *cff*:$f\ O\ f \subseteq f$
⟨*proof*⟩

lemma *cfovi*:$f\ O\ ov\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1}$
⟨*proof*⟩

lemma *cfmi*:$f\ O\ m\hat{}{-1} \subseteq b\hat{}{-1}$
⟨*proof*⟩

lemma *cfbi*:$f\ O\ b\hat{}{-1} \subseteq b\hat{}{-1}$
⟨*proof*⟩

lemma *covifi*:$ov\hat{}{-1}\ O\ f\hat{}{-1} \subseteq ov\hat{}{-1} \cup s\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *covidi*:$ov\hat{}{-1}\ O\ d\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup s\hat{}{-1} \cup ov\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *covis*:$ov\hat{}{-1}\ O\ s \subseteq ov\hat{}{-1} \cup f \cup d$
⟨*proof*⟩

lemma *covisi*:$ov\hat{}{-1}\ O\ s\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1}$
⟨*proof*⟩

lemma *covid*:$ov\hat{}{-1}\ O\ d \subseteq ov\hat{}{-1} \cup f \cup d$
⟨*proof*⟩

lemma *covif*:$ov\hat{}{-1}\ O\ f \subseteq ov\hat{}{-1}$
⟨*proof*⟩

lemma *coviovi*:$ov\hat{}{-1}\ O\ ov\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1}$

⟨*proof*⟩

lemma *covimi*:$ov\hat{}-1\ O\ m\hat{}-1 \subseteq b\hat{}-1$
⟨*proof*⟩

lemma *covibi*:$ov\hat{}-1\ O\ b\hat{}-1 \subseteq b\hat{}-1$
⟨*proof*⟩

lemma *cmiov*:$m\hat{}-1\ O\ ov \subseteq ov\hat{}-1 \cup d \cup f$
⟨*proof*⟩

lemma *cmifi*:$m\hat{}-1\ O\ f\hat{}-1 \subseteq m\hat{}-1$
⟨*proof*⟩

lemma *cmidi*:$m\hat{}-1\ O\ d\hat{}-1 \subseteq b\hat{}-1$
⟨*proof*⟩

lemma *cmis*:$m\hat{}-1\ O\ s \subseteq ov\hat{}-1 \cup d \cup f$
⟨*proof*⟩

lemma *cmisi*:$m\hat{}-1\ O\ s\hat{}-1 \subseteq b\hat{}-1$
⟨*proof*⟩

lemma *cmid*:$m\hat{}-1\ O\ d \subseteq ov\hat{}-1 \cup d \cup f$
⟨*proof*⟩

lemma *cmif*:$m\hat{}-1\ O\ f \subseteq m\hat{}-1$
⟨*proof*⟩

lemma *cmiovi*:$m\hat{}-1\ O\ ov\hat{}-1 \subseteq b\hat{}-1$
⟨*proof*⟩

lemma *cmimi*:$m\hat{}-1\ O\ m\hat{}-1 \subseteq b\hat{}-1$
⟨*proof*⟩

lemma *cmibi*:$m\hat{}-1\ O\ b\hat{}-1 \subseteq b\hat{}-1$
⟨*proof*⟩

lemma *cbim*:$b\hat{}-1\ O\ m \subseteq b\hat{}-1 \cup m\hat{}-1 \cup ov\hat{}-1 \cup f \cup d$
⟨*proof*⟩

lemma *cbiov*:$b\hat{}-1\ O\ ov \subseteq b\hat{}-1 \cup m\hat{}-1 \cup ov\hat{}-1 \cup f \cup d$
⟨*proof*⟩

lemma *cbifi*:$b\hat{}-1\ O\ f\hat{}-1 \subseteq b\hat{}-1$
⟨*proof*⟩

lemma *cbidi*:$b\hat{}-1\ O\ d\hat{}-1 \subseteq b\hat{}-1$
⟨*proof*⟩

lemma *cbis*:$b\hat{}{-1}\ O\ s \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1} \cup f \cup d$
⟨*proof*⟩

lemma *cbisi*:$b\hat{}{-1}\ O\ s\hat{}{-1} \subseteq b\hat{}{-1}$
⟨*proof*⟩

lemma *cbid*:$b\hat{}{-1}\ O\ d \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1} \cup f \cup d$
⟨*proof*⟩

lemma *cbif*:$b\hat{}{-1}\ O\ f \subseteq b\hat{}{-1}$
⟨*proof*⟩

lemma *cbiovi*:$b\hat{}{-1}\ O\ ov\hat{}{-1} \subseteq b\hat{}{-1}$
⟨*proof*⟩

lemma *cbimi*:$b\hat{}{-1}\ O\ m\hat{}{-1} \subseteq b\hat{}{-1}$
⟨*proof*⟩

lemma *cbibi*:$b\hat{}{-1}\ O\ b\hat{}{-1} \subseteq b\hat{}{-1}$
⟨*proof*⟩

3.8 Composition rules

named-theorems *ce-rules* **declare** *cem*[*ce-rules*] **and** *ceb*[*ce-rules*] **and** *ceov*[*ce-rules*] **and** *ces*[*ce-rules*] **and** *cef*[*ce-rules*] **and** *ced*[*ce-rules*] **and** *cemi*[*ce-rules*] **and** *cebi*[*ce-rules*] **and** *ceovi*[*ce-rules*] **and** *cesi*[*ce-rules*] **and** *cefi*[*ce-rules*] **and** *cedi*[*ce-rules*]

named-theorems *cm-rules* **declare** *cme*[*cm-rules*] **and** *cmb*[*cm-rules*] **and** *cmm*[*cm-rules*] **and** *cmov*[*cm-rules*] **and** *cms* [*cm-rules*] **and** *cmd*[*cm-rules*] **and** *cmf*[*cm-rules*] **and** *cmbi*[*cm-rules*] **and** *cmmi*[*cm-rules*] **and** *cmovi*[*cm-rules*] **and** *cmsi*[*cm-rules*] **and** *cmdi*[*cm-rules*] **and** *cmfi*[*cm-rules*]

named-theorems *cb-rules* **declare** *cbe*[*cb-rules*] **and** *cbm*[*cb-rules*] **and** *cbb*[*cb-rules*] **and** *cbov*[*cb-rules*] **and** *cbs* [*cb-rules*] **and** *cbd*[*cb-rules*] **and** *cbf*[*cb-rules*] **and** *cbbi*[*cb-rules*] **and** *cbbi*[*cb-rules*] **and** *cbovi*[*cb-rules*] **and** *cbsi*[*cb-rules*] **and** *cbdi*[*cb-rules*] **and** *cbfi*[*cb-rules*]

named-theorems *cov-rules* **declare** *cove*[*cov-rules*] **and** *covb*[*cov-rules*] **and** *covb*[*cov-rules*] **and** *covov*[*cov-rules*] **and** *covs* [*cov-rules*] **and** *covd*[*cov-rules*] **and** *covf*[*cov-rules*] **and** *covbi*[*cov-rules*] **and** *covbi*[*cov-rules*] **and** *covovi*[*cov-rules*] **and** *covsi*[*cov-rules*] **and** *covdi*[*cov-rules*] **and** *covfi*[*cov-rules*]

named-theorems *cs-rules* **declare** *cse*[*cs-rules*] **and** *csb*[*cs-rules*] **and** *csb*[*cs-rules*] **and** *csov*[*cs-rules*] **and** *css* [*cs-rules*] **and** *csd*[*cs-rules*] **and** *csf*[*cs-rules*] **and** *csbi*[*cs-rules*] **and** *csbi*[*cs-rules*] **and** *csovi*[*cs-rules*] **and** *cssi*[*cs-rules*] **and** *csdi*[*cs-rules*]

and *csfi*[*cs-rules*]

named-theorems *cf-rules* **declare** *cfe*[*cf-rules*] **and** *cfb*[*cf-rules*] **and** *cfb*[*cf-rules*] **and** *cfov*[*cf-rules*] **and** *cfs* [*cf-rules*] **and** *cfd*[*cf-rules*] **and** *cff*[*cf-rules*] **and** *cfbi*[*cf-rules*] **and** *cfbi*[*cf-rules*] **and** *cfovi*[*cf-rules*] **and** *cfsi*[*cf-rules*] **and** *cfdi*[*cf-rules*] **and** *cffi*[*cf-rules*]

named-theorems *cd-rules* **declare** *cde*[*cd-rules*] **and** *cdb*[*cd-rules*] **and** *cdb*[*cd-rules*] **and** *cdov*[*cd-rules*] **and** *cds* [*cd-rules*] **and** *cdd*[*cd-rules*] **and** *cdf*[*cd-rules*] **and** *cdbi*[*cd-rules*] **and** *cdbi*[*cd-rules*] **and** *cdovi*[*cd-rules*] **and** *cdsi*[*cd-rules*] **and** *cddi*[*cd-rules*] **and** *cdfi*[*cd-rules*]

named-theorems *cmi-rules* **declare** *cmie*[*cmi-rules*] **and** *cmib*[*cmi-rules*] **and** *cmib*[*cmi-rules*] **and** *cmiov*[*cmi-rules*] **and** *cmis* [*cmi-rules*] **and** *cmid*[*cmi-rules*] **and** *cmif*[*cmi-rules*] **and**
cmibi[*cmi-rules*] **and** *cmibi*[*cmi-rules*] **and** *cmiovi*[*cmi-rules*] **and** *cmisi*[*cmi-rules*] **and** *cmidi*[*cmi-rules*] **and** *cmifi*[*cmi-rules*]

named-theorems *cbi-rules* **declare** *cbie*[*cbi-rules*] **and** *cbim*[*cbi-rules*] **and** *cbib*[*cbi-rules*] **and** *cbiov*[*cbi-rules*] **and** *cbis* [*cbi-rules*] **and** *cbid*[*cbi-rules*] **and** *cbif*[*cbi-rules*] **and**
cbimi[*cbi-rules*] **and** *cbibi*[*cbi-rules*] **and** *cbiovi*[*cbi-rules*] **and** *cbisi*[*cbi-rules*] **and** *cbidi*[*cbi-rules*] **and** *cbifi*[*cbi-rules*]

named-theorems *covi-rules* **declare** *covie*[*covi-rules*] **and** *covib*[*covi-rules*] **and** *covib*[*covi-rules*] **and** *coviov*[*covi-rules*] **and** *covis* [*covi-rules*] **and** *covid*[*covi-rules*] **and** *covif*[*covi-rules*] **and**
covibi[*covi-rules*] **and** *covibi*[*covi-rules*] **and** *coviovi*[*covi-rules*] **and** *covisi*[*covi-rules*] **and** *covidi*[*covi-rules*] **and** *covifi*[*covi-rules*]

named-theorems *csi-rules* **declare** *csie*[*csi-rules*] **and** *csib*[*csi-rules*] **and** *csib*[*csi-rules*] **and** *csiov*[*csi-rules*] **and** *csis* [*csi-rules*] **and** *csid*[*csi-rules*] **and** *csif*[*csi-rules*] **and** *csibi*[*csi-rules*] **and** *csibi*[*csi-rules*] **and** *csiovi*[*csi-rules*] **and** *csisi*[*csi-rules*] **and** *csidi*[*csi-rules*] **and** *csifi*[*csi-rules*]

named-theorems *cfi-rules* **declare** *cfie*[*cfi-rules*] **and** *cfib*[*cfi-rules*] **and** *cfib*[*cfi-rules*] **and** *cfiov*[*cfi-rules*] **and** *cfis* [*cfi-rules*] **and** *cfid*[*cfi-rules*] **and** *cfif*[*cfi-rules*] **and** *cfibi*[*cfi-rules*] **and** *cfibi*[*cfi-rules*] **and** *cfiovi*[*cfi-rules*] **and** *cfisi*[*cfi-rules*] **and** *cfidi*[*cfi-rules*] **and** *cfifi*[*cfi-rules*]

named-theorems *cdi-rules* **declare** *cdie*[*cdi-rules*] **and** *cdib*[*cdi-rules*] **and** *cdib*[*cdi-rules*] **and** *cdiov*[*cdi-rules*] **and** *cdis* [*cdi-rules*] **and** *cdid*[*cdi-rules*] **and** *cdif*[*cdi-rules*] **and**
cdibi[*cdi-rules*] **and** *cdibi*[*cdi-rules*] **and** *cdiovi*[*cdi-rules*] **and** *cdisi*[*cdi-rules*] **and** *cdidi*[*cdi-rules*] **and** *cdifi*[*cdi-rules*]

named-theorems *cre-rules* **declare** *cee*[*cre-rules*] **and** *cme*[*cre-rules*] **and** *cbe*[*cre-rules*] **and** *cove*[*cre-rules*] **and** *cse*[*cre-rules*] **and** *cfe*[*cre-rules*] **and** *cde*[*cre-rules*] **and** *cmie*[*cre-rules*] **and** *cbie*[*cre-rules*] **and** *covie*[*cre-rules*] **and** *csie*[*cre-rules*] **and**

cfie[*cre-rules*] **and** *cdie*[*cre-rules*]

named-theorems *crm-rules* **declare** *cem*[*crm-rules*] **and** *cbm*[*crm-rules*] **and** *cmm*[*crm-rules*] **and** *covm*[*crm-rules*] **and** *csm*[*crm-rules*] **and** *cfm*[*crm-rules*] **and** *cdm*[*crm-rules*] **and**
cmim[*crm-rules*] **and** *cbim*[*crm-rules*] **and** *covim*[*crm-rules*] **and** *csim*[*crm-rules*] **and** *cfim*[*crm-rules*] **and** *cdim*[*crm-rules*]

named-theorems *crmi-rules* **declare** *cemi*[*crmi-rules*] **and** *cbmi*[*crmi-rules*] **and** *cmmi*[*crmi-rules*] **and** *covmi*[*crmi-rules*] **and** *csmi*[*crmi-rules*] **and** *cfmi*[*crmi-rules*] **and** *cdmi*[*crmi-rules*] **and**
cmimi[*crmi-rules*] **and** *cbimi*[*crmi-rules*] **and** *covimi*[*crmi-rules*] **and** *csimi*[*crmi-rules*] **and** *cfimi*[*crmi-rules*] **and** *cdimi*[*crmi-rules*]

named-theorems *crs-rules* **declare** *ces*[*crs-rules*] **and** *cbs*[*crs-rules*] **and** *cms*[*crs-rules*] **and** *covs*[*crs-rules*] **and** *css*[*crs-rules*] **and** *cfs*[*crs-rules*] **and** *cds*[*crs-rules*] **and** *cmis*[*crs-rules*] **and** *cbis*[*crs-rules*] **and** *covis*[*crs-rules*] **and** *csis*[*crs-rules*] **and** *cfis*[*crs-rules*] **and** *cdis*[*crs-rules*]

named-theorems *crsi-rules* **declare** *cesi*[*crsi-rules*] **and** *cbsi*[*crsi-rules*] **and** *cmsi*[*crsi-rules*] **and** *covsi*[*crsi-rules*] **and** *cssi*[*crsi-rules*] **and** *cfsi*[*crsi-rules*] **and** *cdsi*[*crsi-rules*] **and**
cmisi[*crsi-rules*] **and** *cbisi*[*crsi-rules*] **and** *covisi*[*crsi-rules*] **and** *csisi*[*crsi-rules*] **and** *cfisi*[*crsi-rules*] **and** *cdisi*[*crsi-rules*]

named-theorems *crb-rules* **declare** *ceb*[*crb-rules*] **and** *cbb*[*crb-rules*] **and** *cmb*[*crb-rules*] **and** *covb*[*crb-rules*] **and** *csb*[*crb-rules*] **and** *cfb*[*crb-rules*] **and** *cdb*[*crb-rules*] **and** *cmib*[*crb-rules*] **and** *cbib*[*crb-rules*] **and** *covib*[*crb-rules*] **and** *csib*[*crb-rules*] **and** *cfib*[*crb-rules*] **and** *cdib*[*crb-rules*]

named-theorems *crbi-rules* **declare** *cebi*[*crbi-rules*] **and** *cbbi*[*crbi-rules*] **and** *cmbi*[*crbi-rules*] **and** *covbi*[*crbi-rules*] **and** *csbi*[*crbi-rules*] **and** *cfbi*[*crbi-rules*] **and** *cdbi*[*crbi-rules*] **and**
cmibi[*crbi-rules*] **and** *cbibi*[*crbi-rules*] **and** *covibi*[*crbi-rules*] **and** *csibi*[*crbi-rules*] **and** *cfibi*[*crbi-rules*] **and** *cdibi*[*crbi-rules*]

named-theorems *crov-rules* **declare** *ceov*[*crov-rules*] **and** *cbov*[*crov-rules*] **and** *cmov*[*crov-rules*] **and** *covov*[*crov-rules*] **and** *csov*[*crov-rules*] **and** *cfov*[*crov-rules*] **and** *cdov*[*crov-rules*] **and**
cmiov[*crov-rules*] **and** *cbiov*[*crov-rules*] **and** *coviov*[*crov-rules*] **and** *csiov*[*crov-rules*] **and** *cfiov*[*crov-rules*] **and** *cdiov*[*crov-rules*]

named-theorems *crovi-rules* **declare** *ceovi*[*crovi-rules*] **and** *cbovi*[*crovi-rules*] **and** *cmovi*[*crovi-rules*] **and** *covovi*[*crovi-rules*] **and** *csovi*[*crovi-rules*] **and** *cfovi*[*crovi-rules*] **and** *cdovi*[*crovi-rules*] **and**
cmiovi[*crovi-rules*] **and** *cbiovi*[*crovi-rules*] **and** *coviovi*[*crovi-rules*] **and** *csiovi*[*crovi-rules*] **and** *cfiovi*[*crovi-rules*] **and** *cdiovi*[*crovi-rules*]

named-theorems *crf-rules* **declare** *cef*[*crf-rules*] **and** *cbf*[*crf-rules*] **and** *cmf*[*crf-rules*]

and *covf* [*crf-rules*] **and** *csf* [*crf-rules*] **and** *cff* [*crf-rules*] **and** *cdf* [*crf-rules*] **and** *cmif* [*crf-rules*] **and** *cbif* [*crf-rules*] **and** *covif* [*crf-rules*] **and** *csif* [*crf-rules*] **and** *cfif* [*crf-rules*] **and** *cdif* [*crf-rules*]

named-theorems *crfi-rules* **declare** *cefi*[*crfi-rules*] **and** *cbfi*[*crfi-rules*] **and** *cmfi*[*crfi-rules*] **and** *covfi*[*crfi-rules*] **and** *csfi*[*crfi-rules*] **and** *cffi*[*crfi-rules*] **and** *cdfi*[*crfi-rules*] **and**

cmifi[*crfi-rules*] **and** *cbifi*[*crfi-rules*] **and** *covifi*[*crfi-rules*] **and** *csifi*[*crfi-rules*] **and** *cfifi*[*crfi-rules*] **and** *cdifi*[*crfi-rules*]

named-theorems *crd-rules* **declare** *ced*[*crd-rules*] **and** *cbd*[*crd-rules*] **and** *cmd*[*crd-rules*] **and** *covd*[*crd-rules*] **and** *csd*[*crd-rules*] **and** *cfd*[*crd-rules*] **and** *cdd*[*crd-rules*] **and**

cmid[*crd-rules*] **and** *cbid*[*crd-rules*] **and** *covid*[*crd-rules*] **and** *csid*[*crd-rules*] **and** *cfid*[*crd-rules*] **and** *cdid*[*crd-rules*]

named-theorems *crdi-rules* **declare** *cedi*[*crdi-rules*] **and** *cbdi*[*crdi-rules*] **and** *cmdi*[*crdi-rules*] **and** *covdi*[*crdi-rules*] **and** *csdi*[*crdi-rules*] **and** *cfdi*[*crdi-rules*] **and** *cddi*[*crdi-rules*] **and**
cmidi[*crdi-rules*] **and** *cbidi*[*crdi-rules*] **and** *covidi*[*crdi-rules*] **and** *csidi*[*crdi-rules*] **and** *cfidi*[*crdi-rules*] **and** *cdidi*[*crdi-rules*]

end

theory *disjoint-relations*

imports
allen

begin

4 PD property

The 13 time interval relations (i.e. e, b, m, s, f, d, ov and their inverse relations) are pairwise disjoint.

lemma *em* :$e \cap m = \{\}$
⟨*proof*⟩

lemma *eb* :$e \cap b = \{\}$
⟨*proof*⟩

lemma *eov* :$e \cap ov = \{\}$
⟨*proof*⟩

lemma *es* :$e \cap s = \{\}$
⟨*proof*⟩

lemma *ef* :$e \cap f = \{\}$
⟨*proof*⟩

lemma *ed* :$e \cap d = \{\}$
⟨*proof*⟩

lemma *emi* :$e \cap m\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *ebi* :$e \cap b\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *eovi* :$e \cap ov\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *esi* :$e \cap s\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *efi* :$e \cap f\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *edi* :$e \cap d\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *mb* :$m \cap b = \{\}$
⟨*proof*⟩

lemma *mov* : $m \cap ov = \{\}$
⟨*proof*⟩

lemma *ms* :$m \cap s = \{\}$
⟨*proof*⟩

lemma *mf* :$m \cap f = \{\}$
⟨*proof*⟩

lemma *md* :$m \cap d = \{\}$
⟨*proof*⟩

lemma *mi* :$m \cap m\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *mbi* :$m \cap b\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *movi* :$m \cap ov\hat{}-1 = \{\}$
⟨*proof*⟩

lemma *msi* :$m \cap s\hat{}-1 = \{\}$
⟨*proof*⟩

lemma *mfi* :$m \cap f\hat{}-1 = \{\}$
⟨*proof*⟩

lemma *mdi* :$m \cap d\hat{}-1 = \{\}$
⟨*proof*⟩

lemma *bov* :$b \cap ov = \{\}$
⟨*proof*⟩

lemma *bs* :$b \cap s = \{\}$
⟨*proof*⟩

lemma *bf* :$b \cap f = \{\}$
⟨*proof*⟩

lemma *bd* :$b \cap d = \{\}$
⟨*proof*⟩

lemma *bmi* :$b \cap m\hat{}-1 = \{\}$
⟨*proof*⟩

lemma *bi* :$b \cap b\hat{}-1 = \{\}$
⟨*proof*⟩

lemma *bovi* :$b \cap ov\hat{}-1 = \{\}$
⟨*proof*⟩

lemma *bsi* :$b \cap s\hat{}-1 = \{\}$
⟨*proof*⟩

lemma *bfi* :$b \cap f\hat{}-1 = \{\}$
⟨*proof*⟩

lemma *bdi* :$b \cap d\hat{}-1 = \{\}$
⟨*proof*⟩

lemma *ovs* :$ov \cap s = \{\}$
⟨*proof*⟩

lemma *ovf* :$ov \cap f = \{\}$
⟨*proof*⟩

lemma *ovd* :$ov\cap\ d = \{\}$
⟨*proof*⟩

lemma *ovmi* :$ov \cap m\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *ovbi* :$ov \cap b\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *ovi* :$ov \cap ov\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *ovsi* :$ov \cap s\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *ovfi* :$ov \cap f\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *ovdi* :$ov \cap d\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *sf* :$s \cap f = \{\}$
⟨*proof*⟩

lemma *sd* :$s \cap d = \{\}$
⟨*proof*⟩

lemma *smi* :$s \cap m\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *sbi* :$s \cap b\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *sovi* :$s \cap ov\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *si* :$s \cap s\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *sfi* :$s \cap f\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *sdi* :$s\cap\ d\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *fd* :$f \cap d = \{\}$
⟨*proof*⟩

lemma *fmi* :$f \cap m\hat{\ }{-1} = \{\}$
⟨*proof*⟩

lemma *fbi* :$f \cap b\hat{\ }{-1} = \{\}$
⟨*proof*⟩

lemma *fovi* :$f \cap ov\hat{\ }{-1} = \{\}$
⟨*proof*⟩

lemma *fsi* :$f \cap s\hat{\ }{-1} = \{\}$
⟨*proof*⟩

lemma *fi* :$f \cap f\hat{\ }{-1} = \{\}$
⟨*proof*⟩

lemma *fdi* :$f \cap d\hat{\ }{-1} = \{\}$
⟨*proof*⟩

lemma *dmi* :$d \cap m\hat{\ }{-1} = \{\}$
⟨*proof*⟩

lemma *dbi* :$d \cap b\hat{\ }{-1} = \{\}$
⟨*proof*⟩

lemma *dovi* :$d \cap ov\hat{\ }{-1} = \{\}$
⟨*proof*⟩

lemma *dsi* :$d \cap s\hat{\ }{-1} = \{\}$
⟨*proof*⟩

lemma *dfi* :$d \cap f\hat{\ }{-1} = \{\}$
⟨*proof*⟩

lemma *di* :$d \cap d\hat{\ }{-1} = \{\}$
⟨*proof*⟩

lemma *mibi* :$m\hat{\ }{-1} \cap b\hat{\ }{-1} = \{\}$
⟨*proof*⟩

lemma *miovi* :$m\hat{\ }{-1} \cap ov\hat{\ }{-1} = \{\}$
⟨*proof*⟩

lemma *misi* $:m\hat{\ }{-1} \cap s\hat{\ }{-1} = \{\}$
$\langle proof \rangle$

lemma *mifi* $:m\hat{\ }{-1} \cap f\hat{\ }{-1} = \{\}$
$\langle proof \rangle$

lemma *midi* $:m\hat{\ }{-1} \cap d\hat{\ }{-1} = \{\}$
$\langle proof \rangle$

lemma *bid* $:b\hat{\ }{-1} \cap d = \{\}$
$\langle proof \rangle$

lemma *bimi* $:b\hat{\ }{-1} \cap m\hat{\ }{-1} = \{\}$
$\langle proof \rangle$

lemma *biovi* $:b\hat{\ }{-1} \cap ov\hat{\ }{-1} = \{\}$
$\langle proof \rangle$

lemma *bisi* $:b\hat{\ }{-1} \cap s\hat{\ }{-1} = \{\}$
$\langle proof \rangle$

lemma *bifi* $:b\hat{\ }{-1} \cap f\hat{\ }{-1} = \{\}$
$\langle proof \rangle$

lemma *bidi* $:b\hat{\ }{-1} \cap d\hat{\ }{-1} = \{\}$
$\langle proof \rangle$

lemma *ovisi* $:ov\hat{\ }{-1} \cap s\hat{\ }{-1} = \{\}$
$\langle proof \rangle$

lemma *ovifi* $:ov\hat{\ }{-1} \cap f\hat{\ }{-1} = \{\}$
$\langle proof \rangle$

lemma *ovidi* $:ov\hat{\ }{-1} \cap d\hat{\ }{-1} = \{\}$
$\langle proof \rangle$

lemma *sifi* $:s\hat{\ }{-1} \cap f\hat{\ }{-1} = \{\}$
$\langle proof \rangle$

lemma *sidi* $:s\hat{\ }{-1} \cap d\hat{\ }{-1} = \{\}$
$\langle proof \rangle$

lemma *fidi* :$f\hat{}{-1} \cap d\hat{}{-1} = \{\}$
⟨*proof*⟩

lemma *eei*[*simp*]:$e\hat{}{-1} = e$
⟨*proof*⟩

lemma *rdisj-sym*:$A \cap B = \{\} \Longrightarrow B \cap A = \{\}$
⟨*proof*⟩

4.1 Intersection rules

named-theorems *e-rules* **declare** *em*[*e-rules*] **and** *eb*[*e-rules*] **and** *eov*[*e-rules*] **and** *es*[*e-rules*] **and** *ef*[*e-rules*] **and** *ed*[*e-rules*] **and** *emi*[*e-rules*] **and** *ebi*[*e-rules*] **and** *eovi*[*e-rules*]
and *esi*[*e-rules*] **and** *efi*[*e-rules*] **and** *edi*[*e-rules*]

named-theorems *m-rules* **declare** *em*[*THEN rdisj-sym*, *m-rules*] **and** *mb* [*m-rules*] **and** *ms* [*m-rules*] **and** *mov* [*m-rules*] **and** *mf*[*m-rules*] **and**
md[*m-rules*] **and** *mi* [*m-rules*] **and** *mbi* [*m-rules*] **and** *movi* [*m-rules*] **and** *msi* [*m-rules*] **and** *mfi* [*m-rules*] **and** *mdi* [*m-rules*] **and** *emi*[*m-rules*]

named-theorems *b-rules* **declare** *eb*[*THEN rdisj-sym*, *b-rules*] **and** *mb* [*THEN rdisj-sym*, *b-rules*] **and** *bs* [*b-rules*] **and** *bov* [*b-rules*] **and** *bf*[*b-rules*] **and**
bd[*b-rules*] **and** *bmi* [*b-rules*] **and** *bi* [*b-rules*] **and** *bovi* [*b-rules*] **and** *bsi* [*b-rules*] **and** *bfi* [*b-rules*] **and** *bdi* [*b-rules*] **and** *ebi*[*b-rules*]

named-theorems *ov-rules* **declare** *eov*[*THEN rdisj-sym*, *ov-rules*] **and** *mov* [*THEN rdisj-sym*, *ov-rules*] **and** *ovs* [*ov-rules*] **and** *bov* [*THEN rdisj-sym*,*ov-rules*] **and** *ovf*[*ov-rules*] **and**
ovd[*ov-rules*] **and** *ovmi* [*ov-rules*] **and** *ovi* [*ov-rules*] **and** *ovsi* [*ov-rules*] **and** *ovfi* [*ov-rules*] **and** *ovdi* [*ov-rules*] **and** *eovi*[*ov-rules*]

named-theorems *s-rules* **declare** *es*[*THEN rdisj-sym*, *s-rules*] **and** *ms* [*THEN rdisj-sym*, *s-rules*] **and** *ovs* [*THEN rdisj-sym*, *s-rules*] **and** *bs* [*THEN rdisj-sym*,*s-rules*] **and** *sf*[*s-rules*] **and**
sd[*s-rules*] **and** *smi* [*s-rules*] **and** *sovi* [*s-rules*] **and** *si* [*s-rules*] **and** *sfi* [*s-rules*] **and** *sdi* [*s-rules*]

named-theorems *d-rules* **declare** *ed*[*THEN rdisj-sym*, *d-rules*] **and** *md* [*THEN rdisj-sym*, *d-rules*] **and** *sd* [*THEN rdisj-sym*, *d-rules*] **and** *fd*[*THEN rdisj-sym*, *d-rules*] **and**
ovd[*THEN rdisj-sym*,*d-rules*] **and** *dmi* [*d-rules*] **and** *dovi* [*d-rules*] **and** *dsi* [*d-rules*] **and** *dfi* [*d-rules*] **and** *di* [*d-rules*]

named-theorems *f-rules* **declare** *ef*[*THEN rdisj-sym*, *f-rules*] **and** *mf* [*THEN rdisj-sym*, *f-rules*] **and** *sf* [*THEN rdisj-sym*, *f-rules*] **and** *ovf* [*THEN rdisj-sym*,*f-rules*] **and** *fd*[*f-rules*] **and**
fmi [*f-rules*] **and** *fovi* [*f-rules*] **and** *fsi* [*f-rules*] **and** *fi* [*f-rules*] **and** *fdi* [*f-rules*]

end

theory *jointly-exhaustive*

imports
allen

begin

5 JE property

The 13 time interval relations are jointly exhaustive. For any two intervals x and y, we can find a basic relation r such that $(x, y) \in r$.

lemma (**in** *arelations*) *jointly-exhaustive*:
assumes $\mathcal{I}\ p\ \mathcal{I}\ q$
shows $(p{::}'a,q{::}'a) \in b \vee (p,q) \in m \vee (p,q) \in ov \vee (p,q) \in s \vee (p,q) \in d \vee (p,q) \in f\hat{}{-1} \vee (p,q) \in e \vee$
$(p,q) \in f \vee (p,q) \in s\hat{}{-1} \vee (p,q) \in d\hat{}{-1} \vee (p,q) \in ov\hat{}{-1} \vee (p,q) \in m\hat{}{-1} \vee (p,q) \in b\hat{}{-1}$ (**is** *?R*)
⟨*proof*⟩

lemma (**in** *arelations*) *JE*:
assumes $\mathcal{I}\ p\ \mathcal{I}\ q$
shows $(p{::}'a,q{::}'a) \in b \cup m \cup ov \cup s \cup d \cup f\hat{}{-1} \cup e \cup f \cup s\hat{}{-1} \cup d\hat{}{-1} \cup ov\hat{}{-1} \cup m\hat{}{-1} \cup b\hat{}{-1}$
⟨*proof*⟩

end

theory *examples*

imports

disjoint-relations

begin

6 Examples

6.1 Compositions of non-basic relations

Basic relations are the 13 time interval relations. The unions of basic relations are also relations and their compositions is the union of compositions. We prove few of these compositions that are required in theory nest.thy.

method (**in** *arelations*) *e-compose* = (*match* **conclusion in** *e O b* ⊆ - ⇒ ⟨*insert ceb*, *blast*⟩
| - ⇒ ⟨*match conclusion in e O m* ⊆ - ⇒ ⟨*insert cem*, *blast*⟩ | - ⇒ ⟨*fail*⟩⟩)

declare [[*simp-trace-depth-limit=4*]]

lemma *eovisidifmifiOm*:$(e \cup\ ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup m^{-1} \cup f\hat{}{-1})\ O\ m \subseteq m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1} \cup s \cup s^{-1} \cup e$
⟨*proof*⟩

lemma *ovsmfidiesiOmi*:$(ov \cup s \cup m\ \cup f\hat{}{-1} \cup d\hat{}{-1} \cup e \cup s\hat{}{-1})\ O\ m\hat{}{-1} \subseteq d\hat{}{-1} \cup s\hat{}{-1} \cup\ ov\hat{}{-1} \cup m\hat{}{-1} \cup f\hat{}{-1} \cup f \cup e$
⟨*proof*⟩

lemma *ovsmfidiesiOm*:$(ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1})\ O\ m \subseteq b \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1} \cup m$
⟨*proof*⟩

lemma *ovsmfidiesiOssie*:$(ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1})\ O\ (s \cup s\hat{}{-1} \cup e) \subseteq ov \cup f\hat{}{-1} \cup d\hat{}{-1} \cup s \cup e \cup s\hat{}{-1} \cup m$
⟨*proof*⟩

lemma $(b \cup m \cup ov \cup s \cup d)\ O\ (b \cup m \cup ov \cup s \cup d) \subseteq\ b \cup m \cup ov \cup s \cup d$
⟨*proof*⟩

lemma *ebmovovissifsiddib*:$(e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1})\ O\ b \subseteq b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
⟨*proof*⟩

lemma *ebmovovissiffiddibmovsd*:$(e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1})\ O\ (\ b \cup m \cup ov \cup s \cup d) \subseteq (b \cup m \cup ov \cup s \cup d \cup f\hat{}{-1} \cup d\hat{}{-1} \cup ov\hat{}{-1} \cup s^{-1} \cup f \cup e)$
⟨*proof*⟩

lemma *difimov*:$(d\hat{}{-1} \cup f\hat{}{-1} \cup ov \cup e \cup f \cup m \cup b \cup s\hat{}{-1} \cup s)\ O\ (\ m \cup ov \cup s \cup d \cup b \cup f\hat{}{-1} \cup f \cup e) \subseteq (\ e \cup b \cup m \cup ov \cup ov\hat{}{-1} \cup s \cup s\hat{}{-1} \cup f \cup f^{-1} \cup d \cup d^{-1})$

⟨*proof*⟩

lemma *difibs*:$(d^{-1} \cup f^{-1} \cup ov \cup e \cup f \cup m \cup b \cup s^{-1} \cup s)\ O\ (b \cup s \cup m) \subseteq (b \cup m \cup ov \cup f^{-1} \cup d^{-1} \cup d \cup e \cup s \cup s^{-1})$
⟨*proof*⟩

lemma *bebmovovissiffiddi*:$b\ O\ (e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1}) \subseteq (b \cup m \cup ov \cup s \cup d)$
⟨*proof*⟩

lemma *ovsmfidiesi*:$(((ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1})\ O\ (ov\hat{\ }{-1} \cup s\hat{\ }{-1} \cup m\hat{\ }{-1} \cup f \cup d \cup e \cup s)) \subseteq (s \cup s\hat{\ }{-1} \cup f \cup f\hat{\ }{-1} \cup d \cup d\hat{\ }{-1} \cup e \cup ov \cup ov\hat{\ }{-1} \cup m \cup m\hat{\ }{-1}))$
⟨*proof*⟩

lemma *piiq*:$(p,i) \in ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1} \Longrightarrow (i,q) \in ov\hat{\ }{-1} \cup s\hat{\ }{-1} \cup m\hat{\ }{-1} \cup f \cup d \cup e \cup s \Longrightarrow (p,q) \in s \cup s\hat{\ }{-1} \cup f \cup f\hat{\ }{-1} \cup d \cup d\hat{\ }{-1} \cup e \cup ov \cup ov\hat{\ }{-1} \cup m \cup m\hat{\ }{-1}$
⟨*proof*⟩

lemma *ceovisidiffimi-ffie*:$(e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1})\ O\ (f \cup f^{-1} \cup e) \subseteq e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$
⟨*proof*⟩

lemma *ceovisidiffimi-ffie-simp*:$(p,i) \in (e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}) \Longrightarrow (i,\ q) \in (f \cup f^{-1} \cup e) \Longrightarrow (p,q) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$

⟨*proof*⟩

lemma *ceovisidiffimi-fife*: $(e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1})\ O\ (f^{-1} \cup f \cup e) \subseteq e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$
⟨*proof*⟩

lemma $(x,\ j) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1} \Longrightarrow (j,\ i) \in f^{-1} \cup f \cup e \Longrightarrow (x,\ i) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$
⟨*proof*⟩

lemma *m-ovsmfidiesi*:$m\ O\ (ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}) \subseteq b \cup s \cup m$
⟨*proof*⟩

lemma *ovsmfidiesi-d*:$(ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1})\ O\ d \subseteq e \cup s \cup d \cup ov \cup ov\hat{\ }{-1} \cup s\hat{\ }{-1} \cup f \cup f\hat{\ }{-1} \cup d\hat{\ }{-1}$
⟨*proof*⟩

lemma *cbi-esdovovisiffidi*:$b\hat{\ }{-1}\ O\ (e \cup s \cup d \cup ov \cup ov^{-1} \cup s^{-1} \cup f \cup f^{-1} \cup d^{-1}) \subseteq b\hat{\ }{-1} \cup m\hat{\ }{-1} \cup ov\hat{\ }{-1} \cup f \cup d$
⟨*proof*⟩

lemma *cm-alpha1ialpha4mi*:$m\ O\ \ (e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}) \subseteq m \cup ov \cup s \cup d \cup b \cup f\hat{\ }{-1} \cup f \cup e$
⟨*proof*⟩

lemma *cbi-alpha1ialpha4mi*:$b\hat{\ }{-1}\ O\ (\ e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}) \subseteq\ \ b\hat{\ }{-1}$
⟨*proof*⟩

lemma *cbeta2-beta2*:$(\ b \cup m \cup ov \cup f^{-1} \cup d^{-1})\ O\ (\ b \cup m \cup ov \cup f^{-1} \cup d^{-1}) \subseteq\ \ b \cup m \cup ov \cup f^{-1} \cup d^{-1}$
⟨*proof*⟩

lemma *cbeta2-gammabm*: $(b \cup m \cup ov \cup f^{-1} \cup d^{-1})\ O\ (\ e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1}) \subseteq (\ e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1})$
⟨*proof*⟩

lemma *calpha1-alpha1*:$(b \cup m \cup ov \cup s \cup d)\ O\ (\ b \cup m \cup ov \cup s \cup d) \subseteq (\ b \cup m \cup ov \cup s \cup d)$
⟨*proof*⟩

6.2 Intersection of non-basic relations

lemma *inter-ov*:
assumes $(i,j) \in\ (b \cup m \cup ov \cup f^{-1} \cup d^{-1}) \cap (e \cup b\hat{\ }{-1} \cup m\hat{\ }{-1} \cup ov\hat{\ }{-1} \cup ov \cup s\hat{\ }{-1} \cup s \cup f\hat{\ }{-1} \cup f \cup d\hat{\ }{-1} \cup d) \cap (b \cup m \cup ov \cup s \cup d)$
shows $(i,j) \in ov$
⟨*proof*⟩

lemma *neq-beta2i-alpha2alpha5m*:
assumes $(q,\ j) \in b^{-1} \cup d \cup f \cup ov^{-1} \cup m^{-1}$ **and** $(q,\ j) \in ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}$
shows *False*
⟨*proof*⟩

lemma *neq-bi-alpha1ialpha4mi*:
assumes $(q,i) \in b\hat{\ }{-1}$ **and** $(q,\ i) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$
shows *False*
⟨*proof*⟩

end

theory *nest*

imports
Main jointly-exhaustive examples
HOL−Eisbach.Eisbach-Tools

begin

7 Nests

Nests are sets of intervals that share a meeting point. We define relation before between nests that give the ordering properties of points.

7.1 Definitions

type-synonym $'a\ nest = 'a\ set$

definition (**in** *arelations*) $BEGIN :: 'a \Rightarrow 'a\ nest$
where $BEGIN\ i\ = \{j \mid j.\ (j,i) \in ov \cup s \cup m \cup f\hat{}{-1} \cup d\hat{}{-1} \cup e \cup s\hat{}{-1}\}$

definition (**in** *arelations*) $END :: 'a \Rightarrow 'a\ nest$
where $END\ i\ = \{j \mid j.\ (j,i) \in e \cup ov\hat{}{-1} \cup s\hat{}{-1} \cup d\hat{}{-1} \cup f \cup f\hat{}{-1} \cup m\hat{}{-1}\}$

definition (**in** *arelations*) $NEST ::'a\ nest \Rightarrow bool$
where $NEST\ S \equiv \exists\, i.\ \mathcal{I}\ i \wedge (S = BEGIN\ i \vee S = END\ i)$

definition (**in** *arelations*) $before :: 'a\ nest \Rightarrow 'a\ nest \Rightarrow bool$ (**infix** $\ll$ *100*)
where $before\ N\ M \equiv NEST\ N \wedge NEST\ M \wedge (\exists\, n\ m.\ \cancel{\mathcal{I}\ m \wedge \mathcal{I}\ n \wedge}\ n \in N \wedge m \in M \wedge (n,m) \in b)$

7.2 Properties of Nests

lemma *intv1*:
assumes $\mathcal{I}\ i$
shows $i \in BEGIN\ i$
⟨*proof*⟩

lemma *intv2*:
assumes $\mathcal{I}\ i$
shows $i \in END\ i$
⟨*proof*⟩

lemma *NEST-nonempty*:
assumes $NEST\ S$
shows $S \neq \{\}$
⟨*proof*⟩

lemma *NEST-BEGIN*:
assumes $\mathcal{I}\ i$
shows $NEST\ (BEGIN\ i)$

⟨*proof*⟩

lemma *NEST-END*:
assumes $\mathcal{I}\ i$
shows *NEST* (*END* i)
⟨*proof*⟩

lemma *before*:
assumes $a{:}\mathcal{I}\ i$
shows *BEGIN* $i \ll$ *END* i
⟨*proof*⟩

lemma *meets*:
fixes $i\ j$
assumes $\mathcal{I}\ i$ **and** $\mathcal{I}\ j$
shows $(i,j) \in m = ((END\ i) = (BEGIN\ j))$
⟨*proof*⟩

lemma *starts*:
fixes $i\ j$
assumes $\mathcal{I}\ i$ **and** $\mathcal{I}\ j$
shows $((i,j) \in s \cup s\hat{}{-1} \cup e) = (BEGIN\ i = BEGIN\ j)$
⟨*proof*⟩

lemma *xj-set*:$x \in \{a\ |a.\ (a,\ j) \in\ e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}\} = ((x,j) \in\ e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1})$
⟨*proof*⟩

lemma *ends*:
fixes $i\ j$
assumes $\mathcal{I}\ i$ **and** $\mathcal{I}\ j$
shows $((i,j) \in f \cup f\hat{}{-1} \cup e) = (END\ i = END\ j)$
⟨*proof*⟩

lemma *before-irrefl*:
fixes a
shows $\neg\ a \ll a$
⟨*proof*⟩

lemma *BEGIN-before*:
fixes $i\ j$
assumes $\mathcal{I}\ i$ **and** $\mathcal{I}\ j$
shows $BEGIN\ i \ll BEGIN\ j = ((i,j) \in b \cup m \cup ov \cup f^{-1} \cup d^{-1})$
⟨*proof*⟩

lemma *BEGIN-END-before*:
fixes $i\ j$
assumes $\mathcal{I}\ i$ **and** $\mathcal{I}\ j$

shows $BEGIN\ i \ll END\ j = ((i,j) \in e \cup b \cup m \cup ov \cup ov\hat{}-1 \cup s \cup s\hat{}-1 \cup f \cup f^{-1} \cup d \cup d^{-1})$
⟨*proof*⟩

lemma *END-BEGIN-before*:
fixes $i\ j$
assumes $\mathcal{I}\ i$ **and** $\mathcal{I}\ j$
shows $END\ i \ll BEGIN\ j = ((i,j) \in b)$
⟨*proof*⟩

lemma *END-END-before*:
fixes $i\ j$
assumes $\mathcal{I}\ i$ **and** $\mathcal{I}\ j$
shows $END\ i \ll END\ j = ((i,j) \in b \cup m \cup ov \cup s \cup d)$
⟨*proof*⟩

lemma *overlaps*:
assumes $\mathcal{I}\ i$ **and** $\mathcal{I}\ j$
shows $(i,j) \in ov = ((BEGIN\ i \ll BEGIN\ j) \wedge (BEGIN\ j \ll END\ i) \wedge (END\ i \ll END\ j))$
⟨*proof*⟩

7.3 Ordering of nests

class *strict-order* =
fixes $ls::'a\ nest \Rightarrow 'a\ nest \Rightarrow bool$
assumes
  irrefl: $\neg\ ls\ a\ a$ **and**
  trans: $ls\ a\ c \Longrightarrow ls\ c\ g \Longrightarrow ls\ a\ g$ **and**
  asym: $ls\ a\ c \Longrightarrow \neg\ ls\ c\ a$

class *total-strict-order* = *strict-order* +
assumes *trichotomy*: $a = c \Longrightarrow (\neg\ (ls\ a\ \ c) \wedge \neg\ (ls\ c\ \ a))$

interpretation *nest*:*total-strict-order* $(\ll)$
⟨*proof*⟩

end