Allen's Interval Calculus

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theory *xor-cal*

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Main

begin
definition xor::$bool \Rightarrow bool \Rightarrow bool$ (**infixl** $\oplus$ *60*)
where $xor\ A\ B \equiv (A \wedge \neg B) \vee (\neg A \wedge B)$

declare *xor-def* [*simp*]

interpretation *bool*:*semigroup* $(\oplus)$
proof
{ **fix** $a\ b\ c$ **show** $a \oplus b \oplus c = a \oplus (b \oplus c)$ **by** *auto***}**
qed

lemma *xor-distr-L* [*simp*]:$A \oplus (B \oplus C) = (A\wedge\neg B\wedge\neg C)\vee(A\wedge B\wedge C)\vee(\neg A\wedge B\wedge\neg C)\vee(\neg A\wedge\neg B\wedge C)$
by *auto*

lemma *xor-distr-R* [*simp*]:$(A \oplus B) \oplus C = A \oplus (B \oplus C)$
by *auto*

end

theory *axioms*

imports
Main *xor-cal*

begin

1 Axioms

We formalize Allen's definition of theory of time in term of intervals (Allen, 1983). Two relations, namely meets and equality, are defined between intervals. Two interval meets if they are adjacent A set of 5 axioms ((M1) ∼ (M5)) are then defined based on relation meets.

We define a class interval whose assumptions are (i) properties of relations meets and, (ii) axioms (M1) ∼ (M5).

class *interval* =
fixes
$meets$::$'a \Rightarrow\ 'a \Rightarrow bool$ (**infixl** $\|$ *60*) **and**
$\mathcal{I}$::$'a \Rightarrow bool$
assumes
meets-atrans:$[\![(p\|q);(q\|r)]\!] \Longrightarrow \neg(p\|r)$ **and**

meets-irrefl:$\mathcal{I}\ p \Longrightarrow \neg(p\|p)$ **and**
meets-asym:$(p\|q) \Longrightarrow \neg(q\|p)$ **and**
meets-wd:$p\|q \Longrightarrow \mathcal{I}\ p \wedge \mathcal{I}\ q$ **and**

M1:$[\![(p\|q);\ (p\|s);\ (r\|q)]\!] \Longrightarrow (r\|s)$ **and**
M2:$[\![(p\|q)\ ;\ (r\|s)]\!] \Longrightarrow p\|s \oplus ((\exists\, t.\ (p\|t)\wedge(t\|s)) \oplus (\exists\, t.\ (r\|t)\wedge(t\|q)))$ **and**
M3:$\mathcal{I}\ p \Longrightarrow (\exists\, q\ r.\ q\|p \wedge p\|r)$ **and**
M4:$[\![p\|q\ ;\ q\|s\ ;\ p\|r\ ;\ r\|s]\!] \Longrightarrow q = r$ **and**
M5exist:$p\|q \Longrightarrow (\exists\, r\ s\ t.\ r\|p \wedge p\|q \wedge q\|s \wedge r\|t \wedge t\|s)$

lemma (**in** *interval*) *trans2*:$[\![p\|t;\ t\|r;\ r\|q]\!] \Longrightarrow \neg p\|q$
using *M1 meets-asym* **by** *blast*

lemma (**in** *interval*) *nontrans1*: $u\|r \Longrightarrow \neg\ (\exists\, t.\ u\|t \wedge t\|r)$
using *meets-atrans* **by** *blast*

lemma (**in** *interval*) *nontrans2*:$u\|r \Longrightarrow \neg\ (\exists\, t.\ r\|t \wedge t\|u)$
using *M1 M5exist trans2* **by** *blast*

lemma (**in** *interval*) *nonmeets1*:$\neg\ (u\|r \wedge r\|u)$
using *meets-asym* **by** *blast*

lemma (**in** *interval*) *nonmeets2*: $[\![\mathcal{I}\ u\ ;\ \mathcal{I}\ r\ ]\!] \Longrightarrow \neg\ (u\|r \wedge u = r)$
using *meets-irrefl* **by** *blast*

lemma (**in** *interval*) *nonmeets3*: $\neg\ (u\|r \wedge (\exists\, p.\ u\|p \wedge p\|r))$
using *nontrans1* **by** *blast*

lemma (**in** *interval*) *nonmeets4*: $\neg(u\|r \wedge (\exists\, p.\ r\|p \wedge p\|u))$
using *nontrans2* **by** *blast*

lemma (**in** *interval*) *elimmeets*: $(p \parallel s \wedge (\exists\, t.\ p \parallel t \wedge t \parallel s) \wedge (\exists\, t.\ r \parallel t \wedge t \parallel q))$ $=$ *False*
using *meets-atrans* **by** *blast*

lemma (**in** *interval*) *M5exist-var*:
assumes $x\|y\ y\|z\ z\|w$
shows $\exists\, t.\ x\|t \wedge t\|w$
proof $-$
from *assms*(*1*,*3*) **have** *a*:$x\|w \oplus (\exists\, t.\ x\|t \wedge t\|w) \oplus (\exists\, t.\ z\|t \wedge t\|y)$ **using** *M2*[*of* *x y z w*] **by** *auto*
from *assms* **have** *b1*:$\neg x\|w$ **using** *trans2* **by** *blast*
from *assms*(*2*) **have** $\neg\ (\exists\, t.\ z\|t \wedge t\|y)$ **by** (*simp add*: *nontrans2*)
with *b1 a* **have** $(\exists\, t.\ x\|t \wedge t\|w)$ **by** *simp*
thus *?thesis* **by** *simp*
qed

lemma (**in** *interval*) *M5exist-var2*:

assumes $p\|q$
shows $\exists\, r1\ r2\ r3\ s\ t.\ r1\|r2 \wedge r2\|r3 \wedge r3\|p \wedge p\|q \wedge q\|s \wedge r1\|t \wedge t\|s$
proof −
 from *assms* **obtain** *r3 k1 s* **where** *r3p*:$r3\|p$ **and** *qs*:$q\|s$ **and** *r3k1*:$r3\ \|k1$ **and** *k1s*:$k1\|s$ **using** *M5exist* **by** *blast*
 from *r3p* **obtain** *r2* **where** *r2r3*:$r2\|r3$ **using** *M3*[*of r3*] *meets-wd* **by** *auto*
 from *r2r3* **obtain** *r1* **where** *r1r2*:$r1\|r2$ **using** *M3*[*of r2*] *meets-wd* **by** *auto*
 with *assms r2r3 r3p qs* **obtain** *t* **where** *r1t1*:$r1\|t$ **and** *t1q*:$t\|s$ **using** *M5exist-var* **by** *blast*
 with *assms r1r2 r2r3 r3p qs* **show** *?thesis* **by** *blast*
qed

lemma (**in** *interval*) *M5exist-var3*:
assumes $k\|l$ **and** $l\|q$ **and** $q\|t$ **and** $t\|r$
shows $\exists\, lqt.\ k\|lqt \wedge lqt\|r$
proof −
 from *assms*(*1−3*) **obtain** *lq* **where** $k\|lq$ **and** $lq\|t$
 using *M5exist-var* **by** *blast*
 with *assms*(*4*) **obtain** *lqt* **where** $k\|lqt$ **and** $lqt\|r$
 using *M5exist-var* **by** *blast*
 thus *?thesis* **by** *auto*
qed

end

2 Time interval relations

theory *allen*

imports

 Main axioms
 HOL−Eisbach.Eisbach-Tools

begin

3 Basic relations

We define 7 binary relations between time intervals. Relations e, m, b, ov, d, s and f stand for equal, meets, before, overlaps, during, starts and finishes, respectively.

class *arelations* = *interval* +
 fixes
 e::(′*a*×′*a*) *set* **and**
 m::(′*a*×′*a*) *set* **and**

b::('a×'a) *set* **and**
ov::('a×'a) *set* **and**
d::('a×'a) *set* **and**
s::('a×'a) *set* **and**
f::('a×'a) *set*
assumes
e:$(p,q) \in e = (p = q)$ **and**
m:$(p,q) \in m = p\|q$ **and**
b:$(p,q) \in b = (\exists\, t::'a.\ p\|t \wedge t\|q)$ **and**
ov:$(p,q) \in ov = (\exists\, k\ l\ u\ v\ t::'a.$
$(k\|p \wedge p\|u \wedge u\|v) \wedge (k\|l \wedge l\|q \wedge q\|v) \wedge (l\|t \wedge t\|u))$ **and**
s:$(p,q) \in s = (\exists\, k\ u\ v::'a.\ k\|p \wedge p\|u \wedge u\|v \wedge k\|q \wedge q\|v)$ **and**
f:$(p,q) \in f = (\exists\, k\ l\ \ u ::'a.\ k\|l \wedge l\|p \wedge p\|u \wedge k\|q \wedge q\|u)$ **and**
d:$(p,q) \in d = (\exists\, k\ l\ u\ v::'a.\ k\|l \wedge l\|p \wedge p\|u \wedge u\|v \wedge k\|q \wedge q\|v)$

3.1 e-composition

Relation e is the identity relation for composition.

lemma *cer*:
assumes $r \in \{e,m,b,ov,s,f,d,m\hat{}{-1},b\hat{}{-1},ov\hat{}{-1},s\hat{}{-1},f\hat{}{-1},d\hat{}{-1}\}$
shows $e\ O\ r = r$
proof −
{ **fix** x y **assume** a:$(x,y) \in e\ O\ r$
then obtain z **where** $(x,z) \in e$ **and** $(z,y) \in r$ **by** *auto*
from ‹$(x,z) \in e$› **have** $x = z$ **using** e **by** *auto*
with ‹$(z,y) \in r$› **have** $(x,y) \in r$ **by** *simp*} **note** *c1* = *this*

{ **fix** x y **assume** a:$(x,y) \in r$
have $(x,x) \in e$ **using** e **by** *auto*
with a **have** $(x,y) \in e\ O\ r$ **by** *blast*} **note** *c2* = *this*

from *c1 c2* **show** *?thesis* **by** *auto*
qed

lemma *cre*:
assumes $r \in \{e,m,b,ov,s,f,d,m\hat{}{-1},b\hat{}{-1},ov\hat{}{-1},s\hat{}{-1},f\hat{}{-1},d\hat{}{-1}\}$
shows $r\ O\ e = r$
proof −
{ **fix** x y **assume** a:$(x,y) \in r\ O\ e$
then obtain z **where** $(x,z) \in r$ **and** $(z,y) \in e$ **by** *auto*
from ‹$(z,y) \in e$› **have** $z = y$ **using** e **by** *auto*
with ‹$(x,z) \in r$› **have** $(x,y) \in r$ **by** *simp*} **note** *c1* = *this*

{ **fix** x y **assume** a:$(x,y) \in r$
have $(y,y) \in e$ **using** e **by** *auto*
with a **have** $(x,y) \in r\ O\ e$ **by** *blast*} **note** *c2* = *this*

from *c1 c2* **show** *?thesis* **by** *auto*
qed

lemmas *ceb* = *cer*[*of b*]
lemmas *cebi* = *cer*[*of b*^−*1*]
lemmas *cem* = *cer*[*of m*]
lemmas *cemi* = *cer*[*of m*^−*1*]
lemmas *cee* = *cer*[*of e*]
lemmas *ces* = *cer*[*of s*]
lemmas *cesi* = *cer*[*of s*^−*1*]
lemmas *cef* = *cer*[*of f*]
lemmas *cefi* = *cer*[*of f*^−*1*]
lemmas *ceov* = *cer*[*of ov*]
lemmas *ceovi* = *cer*[*of ov*^−*1*]
lemmas *ced* = *cer*[*of d*]
lemmas *cedi* = *cer*[*of d*^−*1*]
lemmas *cbe* = *cre*[*of b*]
lemmas *cbie* = *cre*[*of b*^−*1*]
lemmas *cme* = *cre*[*of m*]
lemmas *cmie* = *cre*[*of m*^−*1*]
lemmas *cse* = *cre*[*of s*]
lemmas *csie* = *cre*[*of s*^−*1*]
lemmas *cfe* = *cre*[*of f*]
lemmas *cfie* = *cre*[*of f*^−*1*]
lemmas *cove* = *cre*[*of ov*]
lemmas *covie* = *cre*[*of ov*^−*1*]
lemmas *cde* = *cre*[*of d*]
lemmas *cdie* = *cre*[*of d*^−*1*]

3.2 r-composition

We prove compositions of the form $r_1 \circ r_2 \subseteq r$, where r is a basic relation.

method (**in** *arelations*) *r-compose* **uses** *r1 r2 r3* = ((*auto*, (*subst* (*asm*) *r1*), (*subst* (*asm*) *r2*), (*subst r3*)) , (*meson M5exist-var*))

lemma (**in** *arelations*) *cbb*:*b O b* ⊆ *b*
by (*r-compose r1*:*b r2*:*b r3*:*b*)

lemma (**in** *arelations*) *cbm*:*b O m* ⊆ *b*
by (*r-compose r1*:*b r2*:*m r3*:*b*)

lemma *cbov*:*b O ov* ⊆ *b*
apply (*auto simp*:*b ov*)
using *M1 M5exist-var* **by** *blast*

lemma *cbfi*:*b O f*^−*1* ⊆ *b*
apply (*auto simp*:*b f*)
by (*meson M1 M5exist-var*)

lemma *cbdi*:*b O d*^−*1* ⊆ *b*

apply (*auto simp*: *b d*)
by (*meson M1 M5exist-var*)

lemma *cbs*:$b\ O\ s \subseteq b$
apply (*auto simp*: *b s*)
by (*meson M1 M5exist-var*)

lemma *cbsi*:$b\ O\ s\hat{}{-1} \subseteq b$
apply (*auto simp*: *b s*)
by (*meson M1 M5exist-var*)

lemma (**in** *arelations*) *cmb*:$m\ O\ b \subseteq b$
by (*r-compose r1*:*m r2*:*b r3*:*b*)

lemma *cmm*:$m\ O\ m \subseteq b$
by (*auto simp*: *b m*)

lemma *cmov*:$m\ O\ ov \subseteq b$
apply (*auto simp*:*b m ov*)
using *M1 M5exist-var* **by** *blast*

lemma *cmfi*:$m\ O\ f\hat{}{-1} \subseteq b$
apply (*r-compose r1*:*m r2*:*f r3*:*b*)
by (*meson M1*)

lemma *cmdi*:$m\ O\ d\hat{}{-1} \subseteq b$
apply (*auto simp add*:*m d b*)
using *M1* **by** *blast*

lemma *cms*:$m\ O\ s \subseteq m$
apply (*auto simp add*:*m s*)
using *M1* **by** *auto*

lemma *cmsi*:$m\ O\ s\hat{}{-1} \subseteq m$
apply (*auto simp add*:*m s*)
using *M1* **by** *blast*

lemma *covb*:$ov\ O\ b \subseteq b$
apply (*auto simp*:*ov b*)
using *M1 M5exist-var* **by** *blast*

lemma *covm*:$ov\ O\ m \subseteq b$
apply (*auto simp*:*ov m b*)
using *M1* **by** *blast*

lemma *covs*:$ov\ O\ s \subseteq ov$
proof
fix p::$'a \times 'a$ **assume** $p \in ov\ O\ s$ **then obtain** $x\ y\ z$ **where** p:$p = (x,z)$ **and** *xyov*:$(x,y) \in ov$ **and** *yzs*:$(y,z) \in s$ **by** *auto*

from *xyov* **obtain** *r u v t k* **where** *rx*:$r\|x$ **and** *xu*:$x\|u$ **and** *uv*:$u\|v$ **and** *rt*:$r\|t$ **and** *tk*:$t\|k$ **and** *ty*:$t\|y$ **and** *yv*:$y\|v$ **and** *ku*:$k\|u$ **using** *ov* **by** *blast*
from *yzs* **obtain** *l1 l2* **where** *yl1*:$y\|l1$ **and** *l1l2*:$l1\|l2$ **and** *zl2*:$z\|l2$ **using** *s* **by** *blast*
from *uv yl1 yv* **have** $u\|l1$ **using** *M1* **by** *blast*
with *xu l1l2* **obtain** *ul1* **where** *xul1*:$x\|ul1$ **and** *ul1l2*:$ul1\|l2$ **using** *M5exist-var* **by** *blast*
from *ku xu xul1 l1l2* **have** *kul1*:$k\|ul1$ **using** *M1* **by** *blast*
from *ty yzs* **have** $t\|z$ **using** *s M1* **by** *blast*
with *rx rt xul1 ul1l2 zl2 tk kul1* **have** $(x,z) \in ov$ **using** *ov* **by** *blast*
with *p* **show** $p \in ov$ **by** *simp*
qed

lemma *cfib*:$f\hat{}{-1}\ O\ b \subseteq b$
apply (*auto simp*:*f b*)
using *M1* **by** *blast*

lemma *cfim*:$f\hat{}{-1}\ O\ m \subseteq m$
apply (*auto simp*:*f m*)
using *M1* **by** *auto*

lemma *cfiov*:$f\hat{}{-1}\ O\ ov \subseteq ov$
proof
fix p::$'a\times 'a$ **assume** $p \in f\hat{}{-1}\ O\ ov$ **then obtain** *x y z* **where** *p*:$p = (x,z)$ **and** *xyfi*:$(x,y)\in f\hat{}{-1}$ **and** *yzov*:$(y,z) \in ov$ **by** *auto*
from *xyfi yzov* **obtain** t' *r u* **where** *tpr*:$t'\|r$ **and** *ry*:$r\|y$ **and** *yu*:$y\|u$ **and** *tpx*:$t'\|x$ **and** *xu*:$x\|u$ **using** *f* **by** *blast*
from *yzov ry* **obtain** *v k t* u' **where** *yup*:$y\|u'$ **and** *upv*:$u'\|v$ **and** *rk*:$r\|k$ **and** *kz*:$k\|z$ **and** *zv*:$z\|v$ **and** *kt*:$k\|t$ **and** *tup*:$t\|u'$
using *ov* **using** *M1* **by** *blast*
from *yu xu yup* **have** *xup*:$x\|u'$ **using** *M1* **by** *blast*
from *tpr rk kt* **obtain** r' **where** *tprp*:$t'\|r'$ **and** *rpt*:$r'\|t$ **using** *M5exist-var* **by** *blast*
from *kt rpt kz* **have** *rpz*:$r'\|z$ **using** *M1* **by** *blast*
from *tprp rpz rpt tpx xup zv upv tup* **have** $(x,z) \in ov$ **using** *ov* **by** *blast*
with *p* **show** $p \in ov$ **by** *simp*
qed

lemma *cfifi*:$f\hat{}{-1}\ O\ f\hat{}{-1} \subseteq f\hat{}{-1}$
proof
fix x::$'a\times 'a$ **assume** $x \in f\hat{}{-1}\ O\ f\hat{}{-1}$ **then obtain** *p q z* **where** *x*:$x = (p,\ q)$ **and** $(p,z) \in f\hat{}{-1}$ **and** $(z,q) \in f\hat{}{-1}$ **by** *auto*
from ‹$(p,z) \in f\hat{}{-1}$› **obtain** *k l u* **where** *kp*:$k\|p$ **and** *kl*:$k\|l$ **and** *lz*:$l\|z$ **and** *pu*:$p\|u$ **and** *zu*:$z\|u$ **using** *f* **by** *blast*
from ‹$(z,q) \in f\hat{}{-1}$› **obtain** k' u' l' **where** *kpz*:$k'\|z$ **and** *kplp*:$k'\|l'$ **and** *lpq*:$l'\|q$ **and** *qup*:$q\|u'$ **and** *zup*:$z\|u'$ **using** *f* **by** *blast*
from *zu zup pu* **have** $p\|u'$ **using** *M1* **by** *blast*
from *lz kpz kplp* **have** $l\|l'$ **using** *M1* **by** *blast*
with *kl lpq* **obtain** *ll* **where** $k\|ll$ **and** $ll\|q$ **using** *M5exist-var* **by** *blast*

with *kp* ‹$p\|u'$› *qup* **show** $x \in f\widehat{\ }{-1}$ **using** *x f* **by** *blast*
qed

lemma *cfidi*:$f\widehat{\ }{-1}\ O\ d\widehat{\ }{-1} \subseteq d\widehat{\ }{-1}$
proof
fix $x{::}'a{\times}'a$ **assume** $x : f\widehat{\ }{-1}\ O\ d\widehat{\ }{-1}$ **then obtain** *p q z* **where** $x{:}x = (p,q)$ **and** $(p,z) \in f\widehat{\ }{-1}$ **and** $(z,q) \in d\widehat{\ }{-1}$ **by** *auto*
then obtain *k l u* **where** $kp{:}k \parallel p$ **and** $kl{:}k\|l$ **and** $lz{:}l\|z$ **and** $pu{:}p \parallel u$ **and** $zu{:}z\|u$ **using** *f* **by** *blast*
obtain $k'\ l'\ u'\ v'$ **where** $kpz{:}k' \parallel z$ **and** $kplp{:}k' \parallel l'$ **and** $lpq{:}l' \parallel q$ **and** $qup{:}q \parallel u'$ **and** $upvp{:}u'\|v'$ **and** $zvp{:}z\|v'$ **using** *d* ‹$(z,q){\in}d\widehat{\ }{-1}$› **by** *blast*
from *lz kpz kplp* **have** $l\|l'$ **using** *M1* **by** *blast*
with *kl lpq* **obtain** *ll* **where** $k\|ll$ **and** $ll\|q$ **using** *M5exist-var* **by** *blast*
moreover from *zu zvp upvp* **have** $u' \parallel u$ **using** *M1* **by** *blast*
ultimately show $x \in d\widehat{\ }{-1}$ **using** *x kp pu qup d* **by** *blast*
qed

lemma *cfis*:$f\widehat{\ }{-1}\ O\ s \subseteq ov$
proof
fix $x{::}'a{\times}'a$ **assume** $x \in f\widehat{\ }{-1}\ O\ s$ **then obtain** *p q z* **where** $x{:}x = (p,q)$ **and** $(p,z){\in}\ f\widehat{\ }{-1}$ **and** $(z,q) \in s$ **by** *auto*
from ‹$(p,z){\in}\ f\widehat{\ }{-1}$› **obtain** *k l u* **where** $kp{:}k\|p$ **and** $kl{:}k\|l$ **and** $lz{:}l\|z$ **and** $pu{:}p\|u$ **and** $zu{:}z\|u$ **using** *f* **by** *blast*
from ‹$(z,q){\in}\ s$› **obtain** $k'\ u'\ v'$ **where** $kpz{:}k'\|z$ **and** $kpq{:}k'\|q$ **and** $zup{:}z\|u'$ **and** $upvp{:}u'\|v'$ **and** $qvp{:}q\|v'$ **using** *s M1* **by** *blast*
from *pu zu zup* **have** $pup{:}p\|u'$ **using** *M1* **by** *blast*
moreover from *lz kpz kpq* **have** $lq{:}l\|q$ **using** *M1* **by** *blast*
ultimately show $x \in ov$ **using** *x lz zup kp kl upvp upvp ov qvp* **by** *blast*
qed

lemma *cfisi*:$f\widehat{\ }{-1}\ O\ s\widehat{\ }{-1} \subseteq d\widehat{\ }{-1}$
proof
fix $x{::}'a{\times}'a$ **assume** $x \in f\widehat{\ }{-1}\ O\ s\widehat{\ }{-1}$ **then obtain** *p q z* **where** $x{:}x = (p,q)$ **and** $(p,z) \in f\widehat{\ }{-1}$ **and** $(z,q) \in s\widehat{\ }{-1}$ **by** *auto*
then obtain *k l u* **where** $kp{:}k \parallel p$ **and** $kl{:}k\|l$ **and** $lz{:}l\|z$ **and** $pu{:}p \parallel u$ **and** $zu{:}z\|u$ **using** *f* **by** *blast*
obtain $k'\ u'\ v'$ **where** $kpz{:}k' \parallel z$ **and** $kpq{:}k' \parallel q$ **and** $qup{:}q \parallel u'$ **and** $upvp{:}u'\|v'$ **and** $zvp{:}z\|v'$ **using** *s* ‹$(z,q){:}\ s\widehat{\ }{-1}$› **by** *blast*
from *zu zvp upvp* **have** $u'\|u$ **using** *M1* **by** *blast*
moreover from *lz kpz kpq* **have** $l \parallel q$ **using** *M1* **by** *blast*
ultimately show $x \in d\widehat{\ }{-1}$ **using** *x d kl kp qup pu* **by** *blast*
qed

lemma *cdifi*:$d\widehat{\ }{-1}\ O\ f\widehat{\ }{-1} \subseteq d\widehat{\ }{-1}$
proof
fix $x{::}'a{\times}'a$ **assume** $x : d\widehat{\ }{-1}\ O\ f\widehat{\ }{-1}$ **then obtain** *p q z* **where** $x{:}x = (p,q)$ **and** $(p,z) \in d\widehat{\ }{-1}$ **and** $(z,q) \in f\widehat{\ }{-1}$ **by** *auto*
then obtain *k l u v* **where** $kp{:}k \parallel p$ **and** $kl{:}k\|l$ **and** $lz{:}l\|z$ **and** $zu{:}z \parallel u$ **and** $uv{:}u\|v$ **and** $pv{:}p\|v$ **using** *d* **by** *blast*

obtain k' l' u' **where** *kpz*:$k' \| z$ **and** *kplp*:$k' \| l'$ **and** *lpq*:$l' \| q$ **and** *qup*:$q \| u'$ **and** *zup*:$z \| u'$ **using** f ‹(z,q): $f\widehat{\ }{-1}$› **by** *blast*
from *lz kpz kplp* **have** $l \| l'$ **using** *M1* **by** *blast*
with *kl lpq* **obtain** *ll* **where** $k \| ll$ **and** $ll \| q$ **using** *M5exist-var* **by** *blast*
moreover from *zu qup zup* **have** $q \| u$ **using** *M1* **by** *blast*
ultimately show $x \in d\widehat{\ }{-1}$ **using** *x d kp uv pv* **by** *blast*
qed

lemma *cdidi*:$d\widehat{\ }{-1}$ O $d\widehat{\ }{-1} \subseteq d\widehat{\ }{-1}$
proof
fix $x::'a \times 'a$ **assume** x : $d\widehat{\ }{-1}$ O $d\widehat{\ }{-1}$ **then obtain** p q z **where** x:$x = (p,q)$ **and** $(p,z) \in d\widehat{\ }{-1}$ **and** $(z,q) \in d\widehat{\ }{-1}$ **by** *auto*
then obtain k l u v **where** *kp*:$k \| p$ **and** *kl*:$k \| l$ **and** *lz*:$l \| z$ **and** *zu*:$z \| u$ **and** *uv*:$u \| v$ **and** *pv*:$p \| v$ **using** d **by** *blast*
obtain k' l' u' v' **where** *kpz*:$k' \| z$ **and** *kplp*:$k' \| l'$ **and** *lpq*:$l' \| q$ **and** *qup*:$q \| u'$ **and** *upvp*:$u' \| v'$ **and** *zvp*:$z \| v'$ **using** d ‹(z,q): $d\widehat{\ }{-1}$› **by** *blast*
from *lz kpz kplp* **have** $l \| l'$ **using** *M1* **by** *blast*
with *kl lpq* **obtain** *ll* **where** $k \| ll$ **and** $ll \| q$ **using** *M5exist-var* **by** *blast*
moreover from *zvp zu upvp* **have** $u' \| u$ **using** *M1* **by** *blast*
moreover with *qup uv* **obtain** *uu* **where** $q \| uu$ **and** $uu \| v$ **using** *M5exist-var* **by** *blast*
ultimately show $x \in d\widehat{\ }{-1}$ **using** *x d kp pv* **by** *blast*
qed

lemma *cdisi*:$d\widehat{\ }{-1}$ O $s\widehat{\ }{-1} \subseteq d\widehat{\ }{-1}$
proof
fix $x::'a \times 'a$ **assume** x : $d\widehat{\ }{-1}$ O $s\widehat{\ }{-1}$ **then obtain** p q z **where** x:$x = (p,q)$ **and** $(p,z) \in d\widehat{\ }{-1}$ **and** $(z,q) \in s\widehat{\ }{-1}$ **by** *auto*
then obtain k l u v **where** *kp*:$k \| p$ **and** *kl*:$k \| l$ **and** *lz*:$l \| z$ **and** *zu*:$z \| u$ **and** *uv*:$u \| v$ **and** *pv*:$p \| v$ **using** d **by** *blast*
obtain k' u' v' **where** *kpz*:$k' \| z$ **and** *kpq*:$k' \| q$ **and** *qup*:$q \| u'$ **and** *upvp*:$u' \| v'$ **and** *zvp*:$z \| v'$ **using** s ‹(z,q): $s\widehat{\ }{-1}$› **by** *blast*
from *upvp zvp zu* **have** $u' \| u$ **using** *M1* **by** *blast*
with *qup uv* **obtain** *uu* **where** $q \| uu$ **and** $uu \| v$ **using** *M5exist-var* **by** *blast*
moreover from *kpz lz kpq* **have** $l \| q$ **using** *M1* **by** *blast*
ultimately show $x \in d\widehat{\ }{-1}$ **using** *x d kp kl pv* **by** *blast*
qed

lemma *csb*:s O $b \subseteq b$
apply (*auto simp*:*s b*)
using *M1 M5exist-var* **by** *blast*

lemma *csm*:s O $m \subseteq b$
apply (*auto simp*:*s m b*)
using *M1* **by** *blast*

lemma *css*:s O $s \subseteq s$
proof
fix $x::'a \times 'a$ **assume** $x \in s$ O s **then obtain** p q z **where** x:$x = (p,q)$ **and** (p,z)

$\in s$ **and** $(z,q) \in s$ **by** *auto*
from ‹$(p,z) \in s$› **obtain** $k\ u\ v$ **where** $kp{:}k \| p$ **and** $kz{:}k \| z$ **and** $pu{:}p \| u$ **and** $uv{:}u \| v$ **and** $zv{:}z \| v$ **using** s **by** *blast*
from ‹$(z,q) \in s$› **obtain** $k'\ u'\ v'$ **where** $kpq{:}k' \| q$ **and** $kpz{:}k' \| z$ **and** $zup{:}z \| u'$ **and** $upvp{:}u' \| v'$ **and** $qvp{:}q \| v'$ **using** s **by** *blast*
from $kp\ kpz\ kz$ **have** $k' \| p$ **using** *M1* **by** *blast*
moreover from $uv\ zup\ zv$ **have** $u \| u'$ **using** *M1* **by** *blast*
moreover with $pu\ upvp$ **obtain** uu **where** $p \| uu$ **and** $uu \| v'$ **using** *M5exist-var* **by** *blast*
ultimately show $x \in s$ **using** $x\ s\ kpq\ qvp$ **by** *blast*
qed

lemma *csifi*: $s\widehat{\ }{-1}\ O\ f\widehat{\ }{-1} \subseteq d\widehat{\ }{-1}$
proof
fix $x{::}'a \times 'a$ **assume** $x : s\widehat{\ }{-1}\ O\ f\widehat{\ }{-1}$ **then obtain** $p\ q\ z$ **where** $x{:}x = (p,q)$ **and** $(p,z) \in s\widehat{\ }{-1}$ **and** $(z,q) \in f\widehat{\ }{-1}$ **by** *auto*
then obtain $k\ u\ v$ **where** $kp{:}k \| p$ **and** $kz{:}k \| z$ **and** $zu{:}z \| u$ **and** $uv{:}u \| v$ **and** $pv{:}p \| v$ **using** s **by** *blast*
obtain $k'\ l'\ u'$ **where** $kpz{:}k' \| z$ **and** $kplp{:}k' \| l'$ **and** $lpq{:}l' \| q$ **and** $zup{:}z \| u'$ **and** $qup{:}q \| u'$ **using** f ‹$(z,q){:}\ f\widehat{\ }{-1}$› **by** *blast*
from $kz\ kpz\ kplp$ **have** $k \| l'$ **using** *M1* **by** *blast*
moreover from $qup\ zup\ zu$ **have** $q \| u$ **using** *M1* **by** *blast*
ultimately show $x \in d\widehat{\ }{-1}$ **using** $x\ d\ kp\ lpq\ pv\ uv$ **by** *blast*
qed

lemma *csidi*: $s\widehat{\ }{-1}\ O\ d\widehat{\ }{-1} \subseteq d\widehat{\ }{-1}$
proof
fix $x{::}'a \times 'a$ **assume** $x : s\widehat{\ }{-1}\ O\ d\widehat{\ }{-1}$ **then obtain** $p\ q\ z$ **where** $x{:}x = (p,q)$ **and** $(p,z) \in s\widehat{\ }{-1}$ **and** $(z,q) \in d\widehat{\ }{-1}$ **by** *auto*
then obtain $k\ u\ v$ **where** $kp{:}k \| p$ **and** $kz{:}k \| z$ **and** $zu{:}z \| u$ **and** $uv{:}u \| v$ **and** $pv{:}p \| v$ **using** s **by** *blast*
obtain $k'\ l'\ u'\ v'$ **where** $kpz{:}k' \| z$ **and** $kplp{:}k' \| l'$ **and** $lpq{:}l' \| q$ **and** $qup{:}q \| u'$ **and** $upvp{:}u' \| v'$ **and** $zvp{:}z \| v'$ **using** d ‹$(z,q){:}\ d\widehat{\ }{-1}$› **by** *blast*
from $zvp\ upvp\ zu$ **have** $u' \| u$ **using** *M1* **by** *blast*
with $qup\ uv$ **obtain** uu **where** $q \| uu$ **and** $uu \| v$ **using** *M5exist-var* **by** *blast*
moreover from $kz\ kpz\ kplp$ **have** $k \| l'$ **using** *M1* **by** *blast*
ultimately show $x \in d\widehat{\ }{-1}$ **using** $x\ d\ kp\ lpq\ pv$ **by** *blast*
qed

lemma *cdb*: $d\ O\ b \subseteq b$
apply (*auto simp*: $d\ b$)
using *M1 M5exist-var* **by** *blast*

lemma *cdm*: $d\ O\ m \subseteq b$
apply (*auto simp*: $d\ m\ b$)
using *M1* **by** *blast*

lemma *cfb*: $f\ O\ b \subseteq b$
apply (*auto simp*: $f\ b$)

using *M1* **by** *blast*

lemma *cfm*:$f\ O\ m \subseteq m$
proof
fix x::$'a\times'a$ **assume** $x \in f\ O\ m$ **then obtain** $p\ q\ z$ **where** x:$x = (p,q)$ **and** 1:$(p,z) \in f$ **and** 2:$(z,q) \in m$ **by** *auto*
from *1* **obtain** u **where** *pu*:$p\|u$ **and** *zu*:$z\|u$ **using** f **by** *auto*
with *2* **have** $(p,q) \in m$ **using** *M1 m* **by** *blast*
thus $x\in m$ **using** x **by** *auto*
qed

3.3 α-composition

We prove compositions of the form $r_1 \circ r_2 \subseteq s \cup ov \cup d$.

lemma (**in** *arelations*) *cmd*:$m\ O\ d \subseteq s \cup ov \cup d$
proof
fix x::$'a\times'a$ **assume** a:$x \in m\ O\ d$ **then obtain** $p\ q\ z$ **where** x:$x = (p,q)$ **and** 1:$(p,z) \in m$ **and** 2:$(z,q) \in d$ **by** *auto*
then obtain $k\ l\ u\ v$ **where** *pz*:$p\|z$ **and** *kq*:$k\|q$ **and** *kl*:$k\|l$ **and** *lz*:$l\|z$ **and** *zu*:$z\|u$ **and** *uv*:$u\|v$ **and** *qv*:$q\|v$ **using** *m d* **by** *blast*
obtain k' **where** *kpp*:$k'\|p$ **using** *M3 meets-wd pz* **by** *blast*
from *pz zu uv* **obtain** *zu* **where** *pzu*:$p\|zu$ **and** *zuv*:$zu\|v$ **using** *M5exist-var* **by** *blast*
from *kpp kq* **have** $k'\|q \oplus ((\exists t.\ k'\|t \wedge t\|q) \oplus (\exists t.\ k\|t \wedge t\|p))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A\wedge\neg ?B\wedge\neg ?C)\vee(\neg ?A\wedge ?B\wedge\neg ?C)\vee(\neg ?A\wedge\neg ?B\wedge ?C)$ **using** *local.meets-atrans xor-distr-L*[*of ?A ?B ?C*] **by** *blast*
thus $x \in s \cup ov \cup d$
proof (*elim disjE*)
{**assume** $(?A\wedge\neg ?B\wedge\neg ?C)$ **then have** *?A* **by** *simp*
then have $(p,q) \in s$ **using** *s qv kpp pzu zuv* **by** *blast*
thus *?thesis* **using** x **by** *simp* }
next
{**assume** $(\neg ?A\wedge ?B\wedge\neg ?C)$ **then have** *?B* **by** *simp*
then obtain t **where** *kpt*:$k'\|t$ **and** *tq*:$t\|q$ **by** *auto*
moreover from *kq kl tq* **have** $t\|l$ **using** *M1* **by** *blast*
moreover from *lz pz pzu* **have** $l\|zu$ **using** *M1* **by** *blast*
ultimately have $(p,q) \in ov$ **using** *ov kpp qv pzu zuv* **by** *blast*
thus *?thesis* **using** x **by** *simp*}
next
{**assume** $(\neg ?A\wedge\neg ?B\wedge ?C)$ **then have** *?C* **by** *simp*
then obtain t **where** *kt*:$k\|t$ **and** *tp*:$t\|p$ **by** *auto*
with *kq pzu zuv qv* **have** $(p,q)\in d$ **using** d **by** *blast*
thus *?thesis* **using** x **by** *simp*}
qed
qed

lemma (**in** *arelations*) *cmf*:$m\ O\ f \subseteq s \cup ov \cup d$
proof

fix $x::'a \times 'a$ **assume** $a{:}x \in m\ O\ f$ **then obtain** $p\ q\ z$ **where** $x{:}x = (p,q)$ **and** $1{:}(p,z) \in m$ **and** $2{:}(z,q) \in f$ **by** *auto*
then obtain $k\ l\ u$ **where** $pz{:}p \| z$ **and** $kq{:}k \| q$ **and** $kl{:}k \| l$ **and** $lz{:}l \| z$ **and** $zu{:}z \| u$ **and** $qu{:}q \| u$ **using** $m\ f$ **by** *blast*
obtain k' **where** $kpp{:}k' \| p$ **using** *M3 meets-wd pz* **by** *blast*
from *kpp kq* **have** $k' \| q \oplus ((\exists\, t.\ k' \| t \wedge t \| q) \oplus (\exists\, t.\ k \| t \wedge t \| p))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee (\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C)$ **using** *local.meets-atrans xor-distr-L*[*of ?A ?B ?C*] **by** *blast*
thus $x \in s \cup ov \cup d$
proof (*elim disjE*)
{**assume** $(?A \wedge \neg ?B \wedge \neg ?C)$ **then have** $?A$ **by** *simp*
then have $(p,q) \in s$ **using** *s qu kpp pz zu* **by** *blast*
thus *?thesis* **using** x **by** *simp* }
next
{**assume** $(\neg ?A \wedge ?B \wedge \neg ?C)$ **then have** $?B$ **by** *simp*
then obtain t **where** $kpt{:}k' \| t$ **and** $tq{:}t \| q$ **by** *auto*
moreover from *kq kl tq* **have** $t \| l$ **using** *M1* **by** *blast*
moreover from *lz pz pz* **have** $l \| z$ **using** *M1* **by** *blast*
ultimately have $(p,q) \in ov$ **using** *ov kpp qu pz zu* **by** *blast*
thus *?thesis* **using** x **by** *simp*}
next
{**assume** $(\neg ?A \wedge \neg ?B \wedge ?C)$ **then have** $?C$ **by** *simp*
then obtain t **where** $kt{:}k \| t$ **and** $tp{:}t \| p$ **by** *auto*
with *kq pz zu qu* **have** $(p,q) \in d$ **using** d **by** *blast*
thus *?thesis* **using** x **by** *simp*}
qed
qed

lemma *cmovi*:$m\ O\ ov\hat{}{-1} \subseteq s \cup ov \cup d$
proof
fix $x::'a \times 'a$ **assume** $a{:}x \in m\ O\ ov\hat{}{-1}$ **then obtain** $p\ q\ z$ **where** $x{:}x = (p,q)$ **and** $1{:}(p,z) \in m$ **and** $2{:}(z,q) \in ov\hat{}{-1}$ **by** *auto*
then obtain $k\ l\ c\ u\ v$ **where** $pz{:}p \| z$ **and** $kq{:}k \| q$ **and** $kl{:}k \| l$ **and** $lz{:}l \| z$ **and** $qu{:}q \| u$ **and** $uv{:}u \| v$ **and** $zv{:}z \| v$ **and** $lc{:}l \| c$ **and** $cu{:}c \| u$ **using** *m ov* **by** *blast*
obtain k' **where** $kpp{:}k' \| p$ **using** *M3 meets-wd pz* **by** *blast*
from *lz lc pz* **have** $pc{:}p \| c$ **using** *M1* **by** *auto*
from *kpp kq* **have** $k' \| q \oplus ((\exists\, t.\ k' \| t \wedge t \| q) \oplus (\exists\, t.\ k \| t \wedge t \| p))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee (\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C)$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in s \cup ov \cup d$
proof (*elim disjE*)
{**assume** $(?A \wedge \neg ?B \wedge \neg ?C)$ **then have** $?A$ **by** *simp*
then have $(p,q) \in s$ **using** *s kpp qu cu pc* **by** *blast*
thus *?thesis* **using** x **by** *simp* }
next
{**assume** $(\neg ?A \wedge ?B \wedge \neg ?C)$ **then have** $?B$ **by** *simp*
then obtain t **where** $kpt{:}k' \| t$ **and** $tq{:}t \| q$ **by** *auto*

moreover from *kq kl tq* **have** $t\|l$ **using** *M1* **by** *auto*
ultimately have $(p,q) \in ov$ **using** *ov kpp qu cu lc pc* **by** *blast*
thus *?thesis* **using** *x* **by** *simp***}**
next
{assume $(\neg ?A \wedge \neg ?B \wedge ?C)$ **then have** *?C* **by** *simp*
then obtain *t* **where** $kt{:}k\|t$ **and** $tp{:}t\|p$ **by** *auto*
then have $(p,q) \in d$ **using** *d kq cu qu pc* **by** *blast*
thus *?thesis* **using** *x* **by** *simp***}**
qed
qed

lemma *covd*:$ov\ O\ d \subseteq s \cup ov \cup d$
proof
fix $x{::}'a \times 'a$ **assume** $x \in ov\ O\ d$ **then obtain** *p q z* **where** $x{:}x=(p,q)$ **and** $(p,z) \in ov$ **and** $(z,q) \in d$ **by** *auto*
from ‹$(p,z) \in ov$› **obtain** *k u v l c* **where** $kp{:}k\|p$ **and** $pu{:}p\|u$ **and** $uv{:}u\|v$ **and** $zv{:}z\|v$ **and** $lc{:}l\|c$ **and** $cu{:}c\|u$ **and** $kl{:}k\|l$ **and** $lz{:}l\|z$ **and** $cu{:}c\|u$ **using** *ov* **by** *blast*
from ‹$(z,q) \in d$› **obtain** $k'\ l'\ u'\ v'$ **where** $kpq{:}k'\|q$ **and** $kplp{:}k'\|l'$ **and** $lpz{:}l'\|z$ **and** $qvp{:}q\|v'$ **and** $zup{:}z\|u'$ **and** $upvp{:}u'\|v'$ **using** *d* **by** *blast*
from *uv zv zup* **have** $u\|u'$ **using** *M1* **by** *auto*
from *pu upvp* **obtain** *uu* **where** $puu{:}p\|uu$ **and** $uuvp{:}uu\|v'$ **using** ‹$u\|u'$› **using** *M5exist-var* **by** *blast*
from *kp kpq* **have** $k\|q \oplus ((\exists\, t.\ k\|t \wedge t\|q) \oplus (\exists\, t.\ k'\|t \wedge t\|p))$ **(is** $?A \oplus (?B \oplus ?C)$**)** **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in s \cup ov \cup d$
proof (*elim disjE*)
{ assume $?A \wedge \neg ?B \wedge \neg ?C$ **then have** *?A* **by** *simp*
then have $(p,q) \in s$ **using** *s kp qvp puu uuvp* **by** *blast*
thus *?thesis* **using** *x* **by** *blast***}**
next
{ assume $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** *?B* **by** *simp*
then obtain *t* **where** $kt{:}k\|t$ **and** $tq{:}t\|q$ **by** *auto*
from *cu pu puu* **have** $c\|uu$ **using** *M1* **by** *auto*
moreover from *kpq tq kplp* **have** $t\|l'$ **using** *M1* **by** *auto*
moreover from *lpz lz lc* **have** $lpc{:}l'\|c$ **using** *M1* **by** *auto*
ultimately obtain *lc* **where** $t\|lc$ **and** $lc\|uu$ **using** *M5exist-var* **by** *blast*
then have $(p,q) \in ov$ **using** *ov kp kt tq puu uuvp qvp* **by** *blast*
thus *?thesis* **using** *x* **by** *auto***}**
next
{ assume $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** *?C* **by** *simp*
then obtain *t* **where** $k'\|t$ **and** $t\|p$ **by** *auto*
with *puu uuvp qvp kpq* **have** $(p,q) \in d$ **using** *d* **by** *blast*
thus *?thesis* **using** *x* **by** *auto***}**
qed
qed

lemma $covf{:}ov\ O\ f \subseteq s \cup ov \cup d$
proof
fix $x{::}'a{\times}'a$ **assume** $x \in ov\ O\ f$ **then obtain** $p\ q\ z$ **where** $x{:}x{=}(p,q)$ **and** $(p,z) \in ov$ **and** $(z,q) \in f$ **by** *auto*
from $\langle(p,z) \in ov\rangle$ **obtain** $k\ u\ v\ l\ c$ **where** $kp{:}k\|p$ **and** $pu{:}p\|u$ **and** $uv{:}u\|v$ **and** $zv{:}z\|v$ **and** $lc{:}l\|c$ **and** $cu{:}c\|u$ **and** $kl{:}k\|l$ **and** $lz{:}l\|z$ **and** $cu{:}c\|u$ **using** *ov* **by** *blast*
from $\langle(z,q) \in f\rangle$ **obtain** $k'\ l'\ u'$ **where** $kpq{:}k'\|q$ **and** $kplp{:}k'\|l'$ **and** $lpz{:}l'\|z$ **and** $qup{:}q\|u'$ **and** $zup{:}z\|u'$ **using** *f* **by** *blast*
from *uv zv zup* **have** $uu{:}u\|u'$ **using** *M1* **by** *auto*
from *kp kpq* **have** $k\|q \oplus ((\exists\, t.\ k\|t \wedge t\|q) \oplus (\exists\, t.\ k'\|t \wedge t\|p))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A{\wedge}\neg ?B{\wedge}\neg ?C) \vee ((\neg ?A{\wedge}?B{\wedge}\neg ?C) \vee (\neg ?A{\wedge}\neg ?B{\wedge}?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in\ s \cup ov \cup d$
proof (*elim disjE*)
{ **assume** $?A{\wedge}\neg ?B{\wedge}\neg ?C$ **then have** $?A$ **by** *simp*
then have $(p,q) \in s$ **using** *s kp qup uu pu* **by** *blast*
thus *?thesis* **using** *x* **by** *blast*}
next
{ **assume** $\neg ?A{\wedge}?B{\wedge}\neg ?C$ **then have** $?B$ **by** *simp*
then obtain t **where** $kt{:}k\|t$ **and** $tq{:}t\|q$ **by** *auto*
moreover from *kpq tq kplp* **have** $t\|l'$ **using** *M1* **by** *auto*
moreover from *lpz lz lc* **have** $lpc{:}l'\|c$ **using** *M1* **by** *auto*
ultimately obtain *lc* **where** $t\|lc$ **and** $lc\|u$ **using** *cu M5exist-var* **by** *blast*
then have $(p,q) \in ov$ **using** *ov kp kt tq pu uu qup* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
then obtain t **where** $k'\|t$ **and** $t\|p$ **by** *auto*
with *pu uu qup kpq* **have** $(p,q) \in d$ **using** *d* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
qed
qed

lemma $cfid{:}f\hat{\ }{-1}\ O\ d \subseteq s \cup ov \cup d$
proof
fix $x{::}'a{\times}'a$ **assume** $x \in f\hat{\ }{-1}\ O\ d$ **then obtain** $p\ q\ z$ **where** $x{:}x = (p,q)$ **and** $(p,z) \in f\hat{\ }{-1}$ **and** $(z,q){\in}\ d$ **by** *auto*
from $\langle(p,z) \in f\hat{\ }{-1}\rangle$ **obtain** $k\ l\ u$ **where** $k\|l$ **and** $l\|z$ **and** $kp{:}k\|p$ **and** $pu{:}p\|u$ **and** $zu{:}z\|u$ **using** *f* **by** *blast*
from $\langle(z,q) \in d\rangle$ **obtain** $k'\ l'\ u'\ v$ **where** $kplp{:}k'\|l'$ **and** $kpq{:}k'\|q$ **and** $lpz{:}l'\|z$ **and** $zup{:}z\|u'$ **and** $upv{:}u'\|v$ **and** $qv{:}q\|v$ **using** *d* **by** *blast*
from *pu zu zup* **have** $pup{:}p\|u'$ **using** *M1* **by** *blast*
from *kp kpq* **have** $k\|q \oplus ((\exists\, t.\ k\|t \wedge t\|q) \oplus (\exists\, t.\ k'\|t \wedge t\|p))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A{\wedge}\neg ?B{\wedge}\neg ?C) \vee ((\neg ?A{\wedge}?B{\wedge}\neg ?C) \vee (\neg ?A{\wedge}\neg ?B{\wedge}?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in s \cup ov \cup d$
proof (*elim disjE*)

{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
with *pup upv kp qv* **have** $(p,q) \in s$ **using** s **by** *blast*
thus *?thesis* **using** x **by** *auto*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
then obtain t **where** $kt{:}k \| t$ **and** $tq{:}t \| q$ **by** *auto*
from *tq kpq kplp* **have** $t \| l'$ **using** *M1* **by** *blast*
with *lpz zup* **obtain** *lpz* **where** $t \| lpz$ **and** $lpz \| u'$ **using** *M5exist-var* **by** *blast*
with *kp pup upv kt tq qv* **have** $(p,q) \in ov$ **using** *ov* **by** *blast*
thus *?thesis* **using** x **by** *blast*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
then obtain t **where** $k' \| t$ **and** $t \| p$ **by** *auto*
with *pup upv kpq qv* **have** $(p,q) \in d$ **using** d **by** *blast*
thus *?thesis* **using** x **by** *auto*}
qed
qed

lemma *cfov*:$f\ O\ ov \subseteq ov \cup s \cup d$
proof
fix $x{::}'a \times 'a$ **assume** $x \in f\ O\ ov$ **then obtain** $p\ q\ z$ **where** $x{:}x = (p,q)$ **and** $(p,z) \in f$ **and** $(z,q) \in ov$ **by** *auto*
from $\langle (p,z) \in f \rangle$ **obtain** $k\ l\ u$ **where** $k \| l$ **and** $kz{:}k \| z$ **and** $lp{:}l \| p$ **and** $pu{:}p \| u$ **and** $zu{:}z \| u$ **using** f **by** *blast*
from $\langle (z,q) \in ov \rangle$ **obtain** $k'\ l'\ c\ \ u'\ v$ **where** $k' \| l'$ **and** $kpz{:}k' \| z$ **and** $lpq{:}l' \| q$ **and** $zup{:}z \| u'$ **and** $upv{:}u' \| v$ **and** $qv{:}q \| v$ **and** $lpc{:}l' \| c$ **and** $cup{:}c \| u'$ **using** *ov* **by** *blast*
from *pu zu zup* **have** $pup{:}p \| u'$ **using** *M1* **by** *blast*
from *lp lpq* **have** $l \| q \oplus ((\exists\, t.\ l \| t \wedge t \| q) \oplus (\exists\, t.\ l' \| t \wedge t \| p))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in ov \cup s \cup d$
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
with *lp pup upv qv* **have** $(p,q) \in s$ **using** s **by** *blast*
thus *?thesis* **using** x **by** *auto*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
then obtain t **where** $lt{:}l \| t$ **and** $tq{:}t \| q$ **by** *auto*
from *tq lpq lpc* **have** $t \| c$ **using** *M1* **by** *blast*
with *lp lt tq pup upv qv cup* **have** $(p,q) \in ov$ **using** *ov* **by** *blast*
thus *?thesis* **using** x **by** *blast*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
then obtain t **where** $l' \| t$ **and** $t \| p$ **by** *auto*
with *lpq pup upv qv* **have** $(p,q) \in d$ **using** d **by** *blast*
thus *?thesis* **using** x **by** *auto*}
qed

qed

We prove compositions of the form $r_1 \circ r_2 \subseteq ov \cup f^{-1} \cup d^{-1}$.

lemma *covsi*:*ov O s^−1* ⊆ *ov* ∪ *f^−1* ∪ *d^−1*
proof
fix *x*::′*a*×′*a* **assume** *x* ∈ *ov O s^−1* **then obtain** *p q z* **where** *x*:*x* = (*p*,*q*) **and** (*p*,*z*) ∈ *ov* **and** (*z*,*q*) ∈ *s^−1* **by** *auto*
from ‹(*p*,*z*) ∈ *ov*› **obtain** *k l c u* **where** *kp*:*k*‖*p* **and** *pu*:*p*‖*u* **and** *kl*:*k*‖*l* **and** *lz*:*l*‖*z* **and** *lc*:*l*‖*c* **and** *cu*:*c*‖*u* **using** *ov* **by** *blast*
from ‹(*z*,*q*) ∈ *s^−1*› **obtain** *k′ u′ v′* **where** *kpz*:*k′*‖*z* **and** *kpq*:*k′*‖*q* **and** *kpz*:*k′*‖*z* **and** *zup*:*z*‖*u′* **and** *qvp*:*q*‖*v′* **using** *s* **by** *blast*
from *lz kpz kpq* **have** *lq*:*l*‖*q* **using** *M1* **by** *blast*
from *pu qvp* **have** *p*‖*v′* ⊕ ((∃ *t*. *p*‖*t* ∧ *t*‖*v′*) ⊕ (∃ *t*. *q*‖*t* ∧ *t*‖*u*)) (**is** *?A* ⊕ (*?B* ⊕ *?C*)) **using** *M2* **by** *blast*
then have (*?A*∧¬*?B*∧¬*?C*) ∨ ((¬*?A*∧*?B*∧¬*?C*) ∨ (¬*?A*∧¬*?B*∧*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *x* ∈ *ov* ∪ *f^−1* ∪ *d^−1*
proof (*elim disjE*)
{ **assume** *?A*∧¬*?B*∧¬*?C* **then have** *?A* **by** *simp*
with *qvp kp kl lq* **have** (*p*,*q*) ∈ *f^−1* **using** *f* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
next
{ **assume** ¬*?A*∧*?B*∧¬*?C* **then have** *?B* **by** *simp*
then obtain *t* **where** *ptp*:*p*‖*t* **and** *t*‖*v′* **by** *auto*
moreover with *pu cu* **have** *c*‖*t* **using** *M1* **by** *blast*
ultimately have (*p*,*q*)∈ *ov* **using** *kp kl lc cu lq qvp ov* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
next
{ **assume** ¬*?A* ∧ ¬*?B* ∧ *?C* **then have** *?C* **by** *simp*
then obtain *t* **where** *qt*:*q*‖*t* **and** *t*‖*u* **by** *auto*
with *kp kl lq pu* **have** (*p*,*q*) ∈ *d^−1* **using** *d* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
qed
qed

lemma *cdim*:*d^−1 O m* ⊆ *ov* ∪ *d^−1* ∪ *f^−1*
proof
fix *x*::′*a*×′*a* **assume** *x* ∈ *d^−1 O m* **then obtain** *p q z* **where** *x*:*x* = (*p*,*q*) **and** (*p*,*z*) ∈ *d^−1* **and** (*z*,*q*) ∈ *m* **by** *auto*
from ‹(*p*,*z*) ∈ *d^−1*› **obtain** *k l u v* **where** *kp*:*k*‖*p* **and** *pv*:*p*‖*v* **and** *kl*:*k*‖*l* **and** *lz*:*l*‖*z* **and** *zu*:*z*‖*u* **and** *uv*:*u*‖*v* **using** *d* **by** *blast*
from ‹(*z*,*q*) ∈ *m*› **have** *zq*:*z*‖*q* **using** *m* **by** *blast*
obtain *v′* **where** *qvp*:*q*‖*v′* **using** *M3 meets-wd zq* **by** *blast*
from *kl lz zq* **obtain** *lz* **where** *klz*:*k*‖*lz* **and** *lzq*:*lz*‖*q* **using** *M5exist-var* **by** *blast*
from *pv qvp* **have** *p*‖*v′* ⊕ ((∃ *t*. *p*‖*t* ∧ *t*‖*v′*) ⊕ (∃ *t*. *q*‖*t* ∧ *t*‖*v*)) (**is** *?A* ⊕ (*?B* ⊕ *?C*)) **using** *M2* **by** *blast*
then have (*?A*∧¬*?B*∧¬*?C*) ∨ ((¬*?A*∧*?B*∧¬*?C*) ∨ (¬*?A*∧¬*?B*∧*?C*)) **by** (*insert*

xor-distr-L[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
  thus $x \in\ ov \cup d^{\frown}{-1} \cup f^{\frown}{-1}$
  proof (*elim disjE*)
   { assume $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
    with *qvp kp klz lzq*‹$?A$› **have** $(p,q) \in f^{\frown}{-1}$ **using** f **by** *blast*
    thus *?thesis* **using** x **by** *auto***}**
   next
   { assume $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
    then obtain t **where** $pt{:}p\|t$ **and** $tvp{:}t\|v'$ **by** *auto*
    from *zq lzq zu* **have** $lz\|u$ **using** *M1* **by** *auto*
    moreover from *pt pv uv* **have** $u\|t$ **using** *M1* **by** *auto*
    ultimately have $(p,q) \in ov$ **using** *kp klz lzq pt tvp qvp ov* **by** *blast*
    thus *?thesis* **using** x **by** *auto***}**
  next
   { assume $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
    then obtain t **where** $qt{:}q\|t$ **and** $t\|v$ **by** *auto*
    with *kp klz lzq pv* **have** $(p,q) \in d^{\frown}{-1}$ **using** d **by** *blast*
    thus *?thesis* **using** x **by** *auto***}**
   qed
qed

lemma *cdiov*:$d^{\frown}{-1}\ O\ ov \subseteq ov \cup f^{\frown}{-1} \cup d^{\frown}{-1}$
proof
  fix $x{::}'a \times 'a$ **assume** $x \in d^{\frown}{-1}\ O\ ov$ **then obtain** $p\ q\ r$ **where** $x{:}x = (p,r)$ **and** $(p,q) \in d^{\frown}{-1}$ **and** $(q,r) \in ov$ **by** *auto*
  from ‹$(p,q) \in d^{\frown}{-1}$› **obtain** $u\ v\ k\ l$ **where** $kp{:}k\|p$ **and** $pv{:}p\|v$ **and** $kl{:}k\|l$ **and** $lq{:}l\|q$ **and** $qu{:}q\|u$ **and** $uv{:}u\|v$ **using** d **by** *blast*
  from ‹$(q,r) \in ov$› **obtain** $k'\ l'\ t\ u'\ v'$ **where** $lpr{:}l'\|r$ **and** $kpq{:}k'\|q$ **and** $kplp{:}k'\|l'$ **and** $qup{:}q\|u'$ **and** $u'\|v'$ **and** $rvp{:}r\|v'$ **and** $lpt{:}l'\|t$ **and** $tup{:}t\|u'$ **using** *ov* **by** *blast*
  from *lq kplp kpq* **have** $l\|l'$ **using** *M1* **by** *blast*
  with *kl lpr* **obtain** *ll* **where** $kll{:}k\|ll$ **and** $llr{:}ll\|r$ **using** *M5exist-var* **by** *blast*
  from *pv rvp* **have** $p\|v' \oplus ((\exists t'.\ p\|t' \wedge t'\|v') \oplus (\exists t'.\ r\|t' \wedge t'\|v))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
  then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
  thus $x \in ov \cup f^{\frown}{-1} \cup d^{\frown}{-1}$
  proof (*elim disjE*)
   { assume $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
    with *rvp llr kp kll* **have** $(p,r) \in f^{\frown}{-1}$ **using** f **by** *blast*
    thus *?thesis* **using** x **by** *auto***}**
   next
   { assume $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
    then obtain t' **where** $ptp{:}p\|t'$ **and** $tpvp{:}t'\|v'$ **by** *auto*
    moreover from *lpt lpr llr* **have** $llt{:}ll\|t$ **using** *M1* **by** *blast*
    moreover from *ptp uv pv* **have** $utp{:}u\|t'$ **using** *M1* **by** *blast*
    moreover from *qu tup qup* **have** $t\|u$ **using** *M1* **by** *blast*
   moreover with *utp llt* **obtain** *tu* **where** $ll\|tu$ **and** $tu\|t'$ **using** *M5exist-var* **by** *blast*
    with *kp ptp tpvp kll llr rvp* **have** $(p,r) \in ov$ **using** *ov* **by** *blast*

thus *?thesis* **using** *x* **by** *auto*}
next
{ **assume** ¬*?A* ∧ ¬*?B* ∧ *?C* **then have** *?C* **by** *simp*
then obtain t' **where** *rtp*:$r \| t'$ **and** $t' \| v$ **by** *auto*
with *kll llr kp pv* **have** $(p,r) \in d\widehat{\ }{-1}$ **using** *d* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
qed
qed

lemma *cdis*:$d\widehat{\ }{-1}\ O\ s \subseteq ov \cup f\widehat{\ }{-1} \cup d\widehat{\ }{-1}$
proof
fix $x{::}'a{\times}'a$ **assume** $x \in d\widehat{\ }{-1}\ O\ s$ **then obtain** *p q z* **where** $x{:}x = (p,q)$ **and** $(p,z) \in d\widehat{\ }{-1}$ **and** $(z,q) \in s$ **by** *auto*
from ‹$(p,z){\in}d\widehat{\ }{-1}$› **obtain** *k l u v* **where** *kl*:$k\|l$ **and** *lz*:$l\|z$ **and** *kp*:$k\|p$ **and** *zu*:$z\|u$ **and** *uv*:$u\|v$ **and** *pv*:$p\|v$ **using** *d* **by** *blast*
from ‹$(z,q) \in s$› **obtain** l' v' **where** *lpz*:$l'\|z$ **and** *lpq*:$l'\|q$ **and** *qvp*:$q\|v'$ **using** *s* **by** *blast*
from *lz lpz lpq* **have** *lq*:$l\|q$ **using** *M1* **by** *blast*
from *pv qvp* **have** $p\|v' \oplus ((\exists\, t.\ p\|t \wedge t\|v') \oplus (\exists\, t.\ q\|t \wedge t\|v))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A{\wedge}\neg?B{\wedge}\neg?C) \vee ((\neg?A{\wedge}?B{\wedge}\neg?C) \vee (\neg?A{\wedge}\neg?B{\wedge}?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in ov \cup f\widehat{\ }{-1} \cup d\widehat{\ }{-1}$
proof (*elim disjE*)
{ **assume** $?A{\wedge}\neg?B{\wedge}\neg?C$ **then have** *?A* **by** *simp*
with *kl lq qvp kp* **have** $(p,q) \in f\widehat{\ }{-1}$ **using** *f* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
next
{ **assume** $\neg?A{\wedge}?B{\wedge}\neg?C$ **then have** *?B* **by** *simp*
then obtain *t* **where** *pt*:$p\|t$ **and** *tvp*:$t\|v'$ **by** *auto*
from *pt pv uv* **have** $u\|t$ **using** *M1* **by** *blast*
with *lz zu* **obtain** *zu* **where** $l\|zu$ **and** $zu\|t$ **using** *M5exist-var* **by** *blast*
with *kp pt tvp kl lq qvp* **have** $(p,q) \in ov$ **using** *ov* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
next
{ **assume** ¬*?A* ∧ ¬*?B* ∧ *?C* **then have** *?C* **by** *simp*
then obtain *t* **where** $q\|t$ **and** $t\|v$ **by** *auto*
with *kl lq kp pv* **have** $(p,q){\in}d\widehat{\ }{-1}$ **using** *d* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
qed
qed

lemma *csim*:$s\widehat{\ }{-1}\ O\ m \subseteq ov \cup f\widehat{\ }{-1} \cup d\widehat{\ }{-1}$
proof
fix $x{::}'a{\times}'a$ **assume** $x \in s\widehat{\ }{-1}\ O\ m$ **then obtain** *p q z* **where** $x{:}x = (p,q)$ **and** $(p,z) \in s\widehat{\ }{-1}$ **and** $(z,q) \in m$ **by** *auto*
from ‹$(p,z){\in}s\widehat{\ }{-1}$› **obtain** *k u v* **where** *kp*:$k\|p$ **and** *kz*:$k\|z$ **and** *zu*:$z\|u$ **and** *uv*:$u\|v$ **and** *pv*:$p\|v$ **using** *s* **by** *blast*
from ‹$(z,q) \in m$› **have** *zq*:$z\|q$ **using** *m* **by** *auto*

obtain v' **where** $qvp{:}q\|v'$ **using** *M3 meets-wd zq* **by** *blast*
from *pv qvp* **have** $p\|v' \oplus ((\exists\, t.\ p\|t \wedge t\|v') \oplus (\exists\, t.\ q\|t \wedge t\|v))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A\wedge\neg ?B\wedge\neg ?C) \vee ((\neg ?A\wedge ?B\wedge\neg ?C) \vee (\neg ?A\wedge\neg ?B\wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in ov \cup f\widehat{\ }{-1} \cup d\widehat{\ }{-1}$
proof (*elim disjE*)
{ assume $?A\wedge\neg ?B\wedge\neg ?C$ **then have** $?A$ **by** *simp*
with *kp kz zq qvp* **have** $(p,q) \in f\widehat{\ }{-1}$ **using** *f* **by** *blast*
thus *?thesis* **using** *x* **by** *auto***}**
next
{ assume $\neg ?A\wedge ?B\wedge\neg ?C$ **then have** $?B$ **by** *simp*
then obtain t **where** $pt{:}p\|t$ **and** $tvp{:}t\|v'$ **by** *auto*
from *pt pv uv* **have** $u\|t$ **using** *M1* **by** *blast*
with *kp pt tvp kz zq qvp zu* **have** $(p,q) \in ov$ **using** *ov* **by** *blast*
thus *?thesis* **using** *x* **by** *auto***}**
next
{ assume $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
then obtain t **where** $q\|t$ **and** $t\|v$ **by** *auto*
with *kp kz zq pv* **have** $(p,q){\in}d\widehat{\ }{-1}$ **using** *d* **by** *blast*
thus *?thesis* **using** *x* **by** *auto***}**
qed
qed

lemma $csiov{:}s\widehat{\ }{-1}\ O\ ov \subseteq ov \cup f\widehat{\ }{-1} \cup d\widehat{\ }{-1}$
proof
fix $x{::}'a{\times}'a$ **assume** $x \in s\widehat{\ }{-1}\ O\ ov$ **then obtain** $p\ q\ z$ **where** $x{:}x = (p,q)$ **and** $(p,z) \in s\widehat{\ }{-1}$ **and** $(z,q) \in ov$ **by** *auto*
from ‹$(p,z){\in}s\widehat{\ }{-1}$› **obtain** $k\ u\ v$ **where** $kp{:}k\|p$ **and** $kz{:}k\|z$ **and** $zu{:}z\|u$ **and** $uv{:}u\|v$ **and** $pv{:}p\|v$ **using** *s* **by** *blast*
from ‹$(z,q) \in ov$› **obtain** $k'\ l'\ u'\ v'\ c$ **where** $kpz{:}k'\|z$ **and** $zup{:}z\|u'$ **and** $upvp{:}u'\|v'$ **and** $kplp{:}k'\|l'$ **and** $lpq{:}l'\|q$ **and** $qvp{:}q\|v'$ **and** $lpc{:}l'\|c$ **and** $cup{:}c\|u'$ **using** *ov* **by** *blast*
from *kz kpz kplp* **have** $klp{:}k\|l'$ **using** *M1* **by** *auto*
from *pv qvp* **have** $p\|v' \oplus ((\exists\, t.\ p\|t \wedge t\|v') \oplus (\exists\, t.\ q\|t \wedge t\|v))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A\wedge\neg ?B\wedge\neg ?C) \vee ((\neg ?A\wedge ?B\wedge\neg ?C) \vee (\neg ?A\wedge\neg ?B\wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in ov \cup f\widehat{\ }{-1} \cup d\widehat{\ }{-1}$
proof (*elim disjE*)
{ assume $?A\wedge\neg ?B\wedge\neg ?C$ **then have** $?A$ **by** *simp*
with *kp kplp lpq qvp klp* **have** $(p,q) \in f\widehat{\ }{-1}$ **using** *f* **by** *blast*
thus *?thesis* **using** *x* **by** *auto***}**
next
{ assume $\neg ?A\wedge ?B\wedge\neg ?C$ **then have** $?B$ **by** *simp*
then obtain t **where** $pt{:}p\|t$ **and** $tvp{:}t\|v'$ **by** *auto*
from *pt pv uv* **have** $u\|t$ **using** *M1* **by** *blast*
moreover from *cup zup zu* **have** $cu{:}c\|u$ **using** *M1* **by** *auto*
ultimately obtain cu **where** $l'\|cu$ **and** $cu\|t$ **using** *lpc M5exist-var* **by**

blast

  with *kp pt tvp klp lpq qvp* **have** $(p,q) \in ov$ **using** *ov* **by** *blast*

  thus *?thesis* **using** *x* **by** *auto*}

 next

 { **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** *?C* **by** *simp*

  then obtain *t* **where** $q\|t$ **and** $t\|v$ **by** *auto*

  with *kp klp lpq pv* **have** $(p,q) \in d\widehat{\ }{-1}$ **using** *d* **by** *blast*

  thus *?thesis* **using** *x* **by** *auto*}

 qed

qed

lemma *covim*:$ov\widehat{\ }{-1}\ O\ m \subseteq ov \cup f\widehat{\ }{-1} \cup d\widehat{\ }{-1}$

proof

 fix $x::'a\times'a$ **assume** $x \in ov\widehat{\ }{-1}\ O\ m$ **then obtain** *p q z* **where** $x{:}x = (p,q)$ **and** $(p,z) \in ov\widehat{\ }{-1}$ **and** $(z,q) \in m$ **by** *auto*

 from ‹$(p,z) \in ov\widehat{\ }{-1}$› **obtain** *k l c u v* **where** $kz{:}k\|z$ **and** $zu{:}z\|u$ **and** $kl{:}k\|l$ **and** $lp{:}l\|p$ **and** $lc{:}l\|c$ **and** $cu{:}c\|u$ **and** $pv{:}p\|v$ **and** $uv{:}u\|v$ **using** *ov* **by** *blast*

 from ‹$(z,q) \in m$› **have** $zq{:}z\|q$ **using** *m* **by** *auto*

 obtain v' **where** $qvp{:}q\|v'$ **using** *M3 meets-wd zq* **by** *blast*

 from *zu zq cu* **have** $cq{:}c\|q$ **using** *M1* **by** *blast*

 from *pv qvp* **have** $p\|v' \oplus ((\exists\, t.\ p\|t \wedge t\|v') \oplus (\exists\, t.\ q\|t \wedge t\|v))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*

 then have $(?A\wedge\neg?B\wedge\neg?C) \vee ((\neg?A\wedge?B\wedge\neg?C) \vee (\neg?A\wedge\neg?B\wedge?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)

 thus $x \in ov \cup f\widehat{\ }{-1} \cup d\widehat{\ }{-1}$

 proof (*elim disjE*)

 { **assume** $?A\wedge\neg?B\wedge\neg?C$ **then have** *?A* **by** *simp*

  with *lp lc cq qvp* **have** $(p,q) \in f\widehat{\ }{-1}$ **using** *f* **by** *blast*

  thus *?thesis* **using** *x* **by** *auto*}

 next

 { **assume** $\neg?A\wedge?B\wedge\neg?C$ **then have** *?B* **by** *simp*

  then obtain *t* **where** $ptp{:}p\|t$ **and** $t\|v'$ **by** *auto*

  moreover with *pv uv* **have** $u\|t$ **using** *M1* **by** *blast*

  ultimately have $(p,q) \in ov$ **using** *lp lc cq qvp cu ov* **by** *blast*

  thus *?thesis* **using** *x* **by** *auto*}

 next

 { **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** *?C* **by** *simp*

  then obtain *t* **where** $qt{:}q\|t$ **and** $t\|v$ **by** *auto*

  with *lp lc cq pv* **have** $(p,q) \in d\widehat{\ }{-1}$ **using** *d* **by** *blast*

  thus *?thesis* **using** *x* **by** *auto*}

 qed

qed

We prove compositions of the form $r_1 \circ r_2 \subseteq b \cup m \cup ov$.

lemma *covov*:$ov\ O\ ov \subseteq b \cup m \cup ov$

proof

 fix $x::'a\times'a$ **assume** $x \in ov\ O\ ov$ **then obtain** *p q z* **where** $x{:}x = (p,q)$ **and** $(p,z) \in ov$ **and** $(z,q) \in ov$ **by** *auto*

 from ‹$(p,z) \in ov$› **obtain** *k u l t v* **where** $kp{:}k\|p$ **and** $pu{:}p\|u$ **and** $kl{:}k\|l$ **and**

lz:$l\|z$ **and** $l\|t$ **and** $t\|u$ **and** uv:$u\|v$ **and** zv:$z\|v$ **using** *ov* **by** *blast*
from ‹$(z,q) \in ov$› **obtain** k' l' y u' v' **where** $kplp$:$k'\|l'$ **and** kpz:$k'\|z$ **and** lpq:$l'\|q$ **and** lpy:$l'\|y$ **and** $y\|u'$ **and** zup:$z\|u'$ **and** $upvp$:$u'\|v'$ **and** qvp:$q\|v'$ **using** *ov* **by** *blast*
from *lz kplp kpz* **have** llp:$l\|l'$ **using** *M1* **by** *blast*
from *uv zv zup* **have** $u\|u'$ **using** *M1* **by** *blast*
with *pu upvp* **obtain** *uu* **where** puu:$p\|uu$ **and** uuv:$uu\|v'$ **using** *M5exist-var* **by** *blast*
from *puu lpq* **have** $p\|q \oplus ((\exists\, t'.\ p\|t' \wedge t'\|q) \oplus (\exists\, t'.\ l'\|t' \wedge t'\|uu))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A\wedge\neg ?B\wedge\neg ?C) \vee ((\neg ?A\wedge ?B\wedge\neg ?C) \vee (\neg ?A\wedge\neg ?B\wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov$
proof (*elim disjE*)
{ **assume** $?A\wedge\neg ?B\wedge\neg ?C$ **then have** $?A$ **by** *simp*
then have $(p,q) \in m$ **using** *m* **by** *auto*
thus *?thesis* **using** *x* **by** *auto*}
next
{ **assume** $\neg ?A\wedge ?B\wedge\neg ?C$ **then have** $?B$ **by** *simp*
then have $(p,q) \in b$ **using** *b* **by** *auto*
thus *?thesis* **using** *x* **by** *auto*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
then obtain t' **where** $lptp$:$l'\|t'$ **and** $t'\|uu$ **by** *auto*
from *kl llp lpq* **obtain** *ll* **where** kll:$k\|ll$ **and** llq:$ll\|q$ **using** *M5exist-var* **by** *blast*
with *lpq lptp* **have** $ll\|t'$ **using** *M1* **by** *blast*
with *kp puu uuv kll llq qvp* ‹$t'\|uu$› **have** $(p,q) \in ov$ **using** *ov* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
qed
qed

lemma *covfi*:$ov\ O\ f\hat{}{-1} \subseteq b \cup m \cup ov$
proof
fix x::$'a\times 'a$ **assume** $x \in ov\ O\ f\hat{}{-1}$ **then obtain** p q z **where** x:$x = (p,q)$ **and** $(p,z) \in ov$ **and** $(z,q)\in f\hat{}{-1}$ **by** *auto*
from ‹$(p,z) \in ov$› **obtain** k u l c v **where** kp:$k\|p$ **and** pu:$p\|u$ **and** kl:$k\|l$ **and** lz:$l\|z$ **and** $l\|c$ **and** $c\|u$ **and** uv:$u\|v$ **and** zv:$z\|v$ **using** *ov* **by** *blast*
from ‹$(z,q) \in f\hat{}{-1}$› **obtain** k' l' v' **where** $kplp$:$k'\|l'$ **and** kpz:$k'\|z$ **and** lpq:$l'\|q$ **and** qvp:$q\|v'$ **and** zvp:$z\|v'$ **using** *f* **by** *blast*
from *lz kplp kpz* **have** llp:$l\|l'$ **using** *M1* **by** *blast*
from *zv qvp zvp* **have** qv:$q\|v$ **using** *M1* **by** *blast*
from *pu lpq* **have** $p\|q \oplus ((\exists\, t.\ p\|t \wedge t\|q) \oplus (\exists\, t.\ l'\|t \wedge t\|u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A\wedge\neg ?B\wedge\neg ?C) \vee ((\neg ?A\wedge ?B\wedge\neg ?C) \vee (\neg ?A\wedge\neg ?B\wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov$
proof (*elim disjE*)
{ **assume** $?A\wedge\neg ?B\wedge\neg ?C$ **then have** $?A$ **by** *simp*

then have $(p,q) \in m$ **using** m **by** *auto*
thus *?thesis* **using** x **by** *auto*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** *?B* **by** *simp*
then have $(p,q) \in b$ **using** b **by** *auto*
thus *?thesis* **using** x **by** *auto*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** *?C* **by** *simp*
then obtain t **where** $lptp{:}l'\|t$ **and** $t\|u$ **by** *auto*
from *kl llp lpq* **obtain** *ll* **where** $kll{:}k\|ll$ **and** $llr{:}ll\|q$ **using** *M5exist-var* **by** *blast*
with *lpq lptp* **have** $ll\|t$ **using** *M1* **by** *blast*
with *kp pu uv kll llr qv* $\langle t\|u\rangle$ **have** $(p,q) \in ov$ **using** *ov* **by** *blast*
thus *?thesis* **using** x **by** *auto*}
qed
qed

lemma *csov*:$s\ O\ ov \subseteq b \cup m \cup ov$
proof
fix $x{::}'a \times 'a$ **assume** $x \in s\ O\ ov$ **then obtain** $p\ q\ z$ **where** $x{:}x = (p,q)$ **and** $(p,z) \in s$ **and** $(z,q) \in ov$ **by** *auto*
from $\langle (p,z) \in s\rangle$ **obtain** $k\ u\ v$ **where** $kp{:}k\|p$ **and** $kz{:}k\|z$ **and** $pu{:}p\|u$ **and** $uv{:}u\|v$ **and** $zv{:}z\|v$ **using** s **by** *blast*
from $\langle (z,q) \in ov\rangle$ **obtain** $k'\ l'\ u'\ v'$ **where** $kpz{:}k'\|z$ **and** $kplp{:}k'\|l'$ **and** $lpq{:}l'\|q$ **and** $zup{:}z\|u'$ **and** $qvp{:}q\|v'$ **and** $upvp{:}u'\|v'$ **using** *ov* **by** *blast*
from *kz kpz kplp* **have** $klp{:}k\|l'$ **using** *M1* **by** *blast*
from *uv zv zup* **have** $uup{:}u\|u'$ **using** *M1* **by** *blast*
with *pu upvp* **obtain** *uu* **where** $puu{:}p\|uu$ **and** $uuvp{:}uu\|v'$ **using** *M5exist-var* **by** *blast*
from *pu lpq* **have** $p\|q \oplus ((\exists\, t.\ p\|t \wedge t\|q) \oplus (\exists\, t.\ l'\|t \wedge t\|u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov$
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** *?A* **by** *simp*
then have $(p,q) \in m$ **using** m **by** *auto*
thus *?thesis* **using** x **by** *auto*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** *?B* **by** *simp*
then have $(p,q) \in b$ **using** b **by** *auto*
thus *?thesis* **using** x **by** *auto*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** *?C* **by** *simp*
then obtain t **where** $lpt{:}l'\|t$ **and** $t\|u$ **by** *auto*
with *pu puu* **have** $t\|uu$ **using** *M1* **by** *blast*
with *lpt kp puu uuvp klp lpq qvp* **have** $(p,q) \in ov$ **using** *ov* **by** *blast*
thus *?thesis* **using** x **by** *auto*}

qed
qed

lemma *csfi*:$s\ O\ f\widehat{\ }{-1} \subseteq b \cup m \cup ov$
proof
fix $x{::}'a{\times}'a$ **assume** $x \in s\ O\ f\widehat{\ }{-1}$ **then obtain** $p\ q\ r$ **where** $x{:}x = (p,r)$ **and** $(p,q) \in s$ **and** $(q,r){\in}\ f\widehat{\ }{-1}$ **by** *auto*
from ‹$(p,q) \in s$› **obtain** $k\ u\ v$ **where** $kp{:}k\|p$ **and** $kq{:}k\|q$ **and** $pu{:}p\|u$ **and** $uv{:}u\|v$ **and** $qv{:}q\|v$ **using** s **by** *blast*
from ‹$(q,r) \in f\widehat{\ }{-1}$› **obtain** $k'\ l\ v'$ **where** $kpq{:}k'\|q$ **and** $kpl{:}k'\|l$ **and** $lr{:}l\|r$ **and** $rvp{:}r\|v'$ **and** $qvp{:}q\|v'$ **using** f **by** *blast*
from *kpq kpl kq* **have** $kl{:}k\|l$ **using** *M1* **by** *blast*
from *qvp qv uv* **have** $uvp{:}u\|v'$ **using** *M1* **by** *blast*
from *pu lr* **have** $p\|r \oplus ((\exists\, t'.\ p\|t' \wedge t'\|r) \oplus (\exists\, t'.\ l\|t' \wedge t'\|u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A{\wedge}\neg?B{\wedge}\neg?C) \vee ((\neg?A{\wedge}?B{\wedge}\neg?C) \vee (\neg?A{\wedge}\neg?B{\wedge}?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov$
proof (*elim disjE*)
{ **assume** $?A{\wedge}\neg?B{\wedge}\neg?C$ **then have** $?A$ **by** *simp*
then have $(p,r) \in m$ **using** m **by** *auto*
thus *?thesis* **using** x **by** *auto*}
next
{ **assume** $\neg?A{\wedge}?B{\wedge}\neg?C$ **then have** $?B$ **by** *simp*
then have $(p,r) \in b$ **using** b **by** *auto*
thus *?thesis* **using** x **by** *auto*}
next
{ **assume** $\neg?A \wedge \neg?B \wedge ?C$ **then have** $?C$ **by** *simp*
then obtain t' **where** $ltp{:}l\|t'$ **and** $t'\|u$ **by** *auto*
with *kp pu uvp kl lr rvp* **have** $(p,r) \in ov$ **using** *ov* **by** *blast*
thus *?thesis* **using** x **by** *auto*}
qed
qed

We prove compositions of the form $r_1 \circ r_2 \subseteq f \cup f^{-1} \cup e$.

lemma *cmmi*:$m\ O\ m\widehat{\ }{-1} \subseteq f \cup f\widehat{\ }{-1} \cup e$
proof
fix $x{::}'a{\times}'a$ **assume** $a{:}x \in m\ O\ m\widehat{\ }{-1}$ **then obtain** $p\ q\ z$ **where** $x{:}x = (p,q)$ **and** $1{:}(p,z) \in m$ **and** $2{:}(z,q) \in m\widehat{\ }{-1}$ **by** *auto*
then have $pz{:}p\|z$ **and** $qz{:}q\|z$ **using** m **by** *auto*
obtain $k\ k'$ **where** $kp{:}k\|p$ **and** $kpq{:}k'\|q$ **using** *M3 meets-wd qz pz* **by** *blast*
from *kp kpq* **have** $k\|q \oplus ((\exists\, t.\ k\|t \wedge t\|q) \oplus (\exists\, t.\ k'\|t \wedge t\|p))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A{\wedge}\neg?B{\wedge}\neg?C){\vee}(\neg?A{\wedge}?B{\wedge}\neg?C){\vee}(\neg?A{\wedge}\neg?B{\wedge}?C)$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in f \cup f\widehat{\ }{-1} \cup e$
proof (*elim disjE*)
{**assume** $(?A{\wedge}\neg?B{\wedge}\neg?C)$ **then have** $?A$ **by** *simp*

then have $p = q$ **using** *M4 kp pz qz* **by** *blast*
then have $(p,q) \in e$ **using** *e* **by** *auto*
thus *?thesis* **using** *x* **by** *simp* **}**
next
{assume $(\neg ?A \wedge ?B \wedge \neg ?C)$ **then have** *?B* **by** *simp*
then obtain *t* **where** $kt{:}k \| t$ **and** $tq{:}t \| q$ **by** *auto*
then have $(p,q) \in f^\frown{-1}$ **using** *f qz pz kp* **by** *blast*
thus *?thesis* **using** *x* **by** *simp***}**
next
{assume $(\neg ?A \wedge \neg ?B \wedge ?C)$ **then have** *?C* **by** *simp*
then obtain *t* **where** $kt{:}k' \| t$ **and** $tp{:}t \| p$ **by** *auto*
with *kpq pz qz* **have** $(p,q) \in f$ **using** *f* **by** *blast*
thus *?thesis* **using** *x* **by** *simp***}**
qed
qed

lemma *cfif*: $f^\frown{-1}\ O\ f \subseteq e \cup f^\frown{-1} \cup f$
proof
fix $x{::}'a \times 'a$ **assume** $a{:}x \in f^\frown{-1}\ O\ f$ **then obtain** *p q z* **where** $x{:}x = (p,q)$ **and** $1{:}(p,z) \in f^\frown{-1}$ **and** $2{:}(z,q) \in f$ **by** *auto*
from *1* **obtain** *k l u* **where** $kp{:}k \| p$ **and** $kl{:}k \| l$ **and** $lz{:}l \| z$ **and** $zu{:}z \| u$ **and** $pu{:}p \| u$ **using** *f* **by** *blast*
from *2* **obtain** $k'\ l'\ u'$ **where** $kpq{:}k' \| q$ **and** $kplp{:}k' \| l'$ **and** $lpz{:}l' \| z$ **and** $zup{:}z \| u'$ **and** $qup{:}q \| u'$ **using** *f* **by** *blast*
from *zu zup qup* **have** $qu{:}q \| u$ **using** *M1* **by** *auto*
from *kp kpq* **have** $k \| q \oplus ((\exists\, t.\ k \| t \wedge t \| q) \oplus (\exists\, t.\ k' \| t \wedge t \| p))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee (\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C)$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*: *elimmeets*)
thus $x \in e \cup f^\frown{-1} \cup f$
proof (*elim disjE*)
{assume $(?A \wedge \neg ?B \wedge \neg ?C)$ **then have** *?A* **by** *simp*
then have $p = q$ **using** *M4 kp pu qu* **by** *blast*
then have $(p,q) \in e$ **using** *e* **by** *auto*
thus *?thesis* **using** *x* **by** *simp* **}**
next
{assume $(\neg ?A \wedge ?B \wedge \neg ?C)$ **then have** *?B* **by** *simp*
then obtain *t* **where** $kt{:}k \| t$ **and** $tq{:}t \| q$ **by** *auto*
then have $(p,q) \in f^\frown{-1}$ **using** *f qu pu kp* **by** *blast*
thus *?thesis* **using** *x* **by** *simp***}**
next
{assume $(\neg ?A \wedge \neg ?B \wedge ?C)$ **then have** *?C* **by** *simp*
then obtain *t* **where** $kt{:}k' \| t$ **and** $tp{:}t \| p$ **by** *auto*
with *kpq pu qu* **have** $(p,q) \in f$ **using** *f* **by** *blast*
thus *?thesis* **using** *x* **by** *simp***}**
qed
qed

lemma *cffi*:$f\ O\ f\hat{}{-1} \subseteq e \cup f \cup f\hat{}{-1}$
proof
fix x::$'a \times 'a$ **assume** $x \in f\ O\ f\hat{}{-1}$ **then obtain** $p\ q\ r$ **where** x:$x = (p,r)$ **and** $(p,q) \in f$ **and** $(q,r) \in f\hat{}{-1}$ **by** *auto*
from ‹$(p,q) \in f$› ‹$(q,r) \in f\hat{}{-1}$› **obtain** $k\ k'$ **where** *kp*:$k \| p$ **and** *kpr*:$k' \| r$ **using** f **by** *blast*
from ‹$(p,q) \in f$› ‹$(q,r) \in f\hat{}{-1}$› **obtain** u **where** *pu*:$p \| u$ **and** $q \| u$ **and** *ru*:$r \| u$ **using** f *M1* **by** *blast*
from *kp kpr* **have** $k \| r \oplus ((\exists\, t.\ k \| t \wedge t \| r) \oplus (\exists\, t.\ k' \| t \wedge t \| p))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in e \cup f \cup f\hat{}{-1}$
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
with *pu ru kp* **have** $p = r$ **using** *M4* **by** *auto*
thus *?thesis* **using** x e **by** *auto*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
then obtain t **where** *kt*:$k \| t$ **and** *tr*:$t \| r$ **by** *auto*
with *ru kp pu* **show** *?thesis* **using** x f **by** *blast*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
then obtain t **where** *rtp*:$k' \| t$ **and** $t \| p$ **by** *auto*
with *kpr ru pu* **show** *?thesis* **using** x f **by** *blast*}
qed
qed

We prove compositions of the form $r_1 \circ r_2 \subseteq e \cup s \cup s^{-1}$.

lemma *cssi*:$s\ O\ s\hat{}{-1} \subseteq e \cup s \cup s\hat{}{-1}$
proof
fix x::$'a \times 'a$ **assume** $x \in s\ O\ s\hat{}{-1}$ **then obtain** $p\ q\ r$ **where** x:$x = (p,r)$ **and** $(p,q) \in s$ **and** $(q,r) \in s\hat{}{-1}$ **by** *auto*
from ‹$(p,q) \in s$› ‹$(q,r) \in s\hat{}{-1}$› **obtain** k **where** *kp*:$k \| p$ **and** *kr*:$k \| r$ **and** *kq*:$k \| q$ **using** s *M1* **by** *blast*
from ‹$(p,q) \in s$› ‹$(q,r) \in s\hat{}{-1}$› **obtain** $u\ u'$ **where** *pu*:$p \| u$ **and** *rup*:$r \| u'$ **using** s **by** *blast*
then have $p \| u' \oplus ((\exists\, t.\ p \| t \wedge t \| u') \oplus (\exists\, t.\ r \| t \wedge t \| u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in e \cup s \cup s\hat{}{-1}$
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
with *rup kp kr* **have** $p = r$ **using** *M4* **by** *auto*
thus *?thesis* **using** x e **by** *auto*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
then obtain t **where** *kt*:$p \| t$ **and** *tr*:$t \| u'$ **by** *auto*

with *rup kp kr* **show** *?thesis* **using** *x s* **by** *blast*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** *?C* **by** *simp*
then obtain *t* **where** $rtp{:}r\|t$ **and** $t\|u$ **by** *auto*
with *pu kp kr* **show** *?thesis* **using** *x s* **by** *blast*}
qed
qed

lemma *csis*: $s\widehat{\ }{-1}\ O\ s \subseteq e \cup s \cup s\widehat{\ }{-1}$
proof
fix $x{::}'a\times'a$ **assume** $x \in s\widehat{\ }{-1}\ O\ s$ **then obtain** *p q r* **where** $x{:}x = (p,r)$ **and** $(p,q)\in s\widehat{\ }{-1}$ **and** $(q,r) \in s$ **by** *auto*
from ‹$(p,q)\in s\widehat{\ }{-1}$› ‹$(q,r) \in s$› **obtain** *k* **where** $kp{:}k\|p$ **and** $kr{:}k\|r$ **and** $kq{:}k\|q$ **using** *s M1* **by** *blast*
from ‹$(p,q)\in s\widehat{\ }{-1}$› ‹$(q,r) \in s$› **obtain** *u u'* **where** $pu{:}p\|u$ **and** $rup{:}r\|u'$ **using** *s* **by** *blast*
then have $p\|u' \oplus ((\exists\, t.\ p\|t \wedge t\|u') \oplus (\exists\, t.\ r\|t \wedge t\|u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A\wedge\neg ?B\wedge\neg ?C) \vee ((\neg ?A\wedge ?B\wedge\neg ?C) \vee (\neg ?A\wedge\neg ?B\wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*: *elimmeets*)
thus $x \in e \cup s \cup s\widehat{\ }{-1}$
proof (*elim disjE*)
{ **assume** $?A\wedge\neg ?B\wedge\neg ?C$ **then have** *?A* **by** *simp*
with *rup kp kr* **have** $p = r$ **using** *M4* **by** *auto*
thus *?thesis* **using** *x e* **by** *auto*}
next
{ **assume** $\neg ?A\wedge ?B\wedge\neg ?C$ **then have** *?B* **by** *simp*
then obtain *t* **where** $kt{:}p\|t$ **and** $tr{:}t\|u'$ **by** *auto*
with *rup kp kr* **show** *?thesis* **using** *x s* **by** *blast*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** *?C* **by** *simp*
then obtain *t* **where** $rtp{:}r\|t$ **and** $t\|u$ **by** *auto*
with *pu kp kr* **show** *?thesis* **using** *x s* **by** *blast*}
qed
qed

lemma *cmim*: $m\widehat{\ }{-1}\ O\ m \subseteq s \cup s\widehat{\ }{-1} \cup e$
proof
fix $x{::}'a\times'a$ **assume** $x \in m\widehat{\ }{-1}\ O\ m$ **then obtain** *p q r* **where** $x{:}x = (p,r)$ **and** $(p,q)\in m\widehat{\ }{-1}$ **and** $(q,r) \in m$ **by** *auto*
from ‹$(p,q)\in m\widehat{\ }{-1}$› ‹$(q,r) \in m$› **have** $qp{:}q\|p$ **and** $qr{:}q\|r$ **using** *m* **by** *auto*
obtain *u u'* **where** $pu{:}p\|u$ **and** $rup{:}r\|u'$ **using** *M3 meets-wd qp qr* **by** *fastforce*
then have $p\|u' \oplus ((\exists\, t.\ p\|t \wedge t\|u') \oplus (\exists\, t.\ r\|t \wedge t\|u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A\wedge\neg ?B\wedge\neg ?C) \vee ((\neg ?A\wedge ?B\wedge\neg ?C) \vee (\neg ?A\wedge\neg ?B\wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*: *elimmeets*)
thus $x \in s \cup s\widehat{\ }{-1} \cup e$
proof (*elim disjE*)
{ **assume** $?A\wedge\neg ?B\wedge\neg ?C$ **then have** *?A* **by** *simp*

with *rup qp qr* **have** $p = r$ **using** *M4* **by** *auto*
thus *?thesis* **using** *x e* **by** *auto*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** *?B* **by** *simp*
then obtain *t* **where** $kt{:}p\|t$ **and** $tr{:}t\|u'$ **by** *auto*
with *rup qp qr* **show** *?thesis* **using** *x s* **by** *blast*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** *?C* **by** *simp*
then obtain *t* **where** $rtp{:}r\|t$ **and** $t\|u$ **by** *auto*
with *pu qp qr* **show** *?thesis* **using** *x s* **by** *blast*}
qed
qed

3.4 β-composition

We prove compositions of the form $r_1 \circ r_2 \subseteq b \cup m \cup ov \cup s \cup d$.

lemma *cbd*:$b\ O\ d \subseteq b \cup m \cup ov \cup s \cup d$
proof
fix $x{::}'a\times'a$ **assume** $x \in b\ O\ d$ **then obtain** *p q z* **where** $x{:}x = (p,q)$ **and** $(p,z) \in b$ **and** $(z,q) \in d$ **by** *auto*
from ‹$(p,z) \in b$› **obtain** *c* **where** $pc{:}p\|c$ **and** $cz{:}c\|z$ **using** *b* **by** *auto*
obtain *a* **where** $ap{:}a\|p$ **using** *M3 meets-wd pc* **by** *blast*
from ‹$(z,q) \in d$› **obtain** *k l u v* **where** $k\|l$ **and** $l\|z$ **and** $kq{:}k\|q$ **and** $zu{:}z\|u$ **and** $uv{:}u\|v$ **and** $qv{:}q\|v$ **using** *d* **by** *blast*
from *pc cz zu* **obtain** *cz* **where** $pcz{:}p\|cz$ **and** $czu{:}cz\|u$ **using** *M5exist-var* **by** *blast*
with *uv* **obtain** *czu* **where** $pczu{:}p\|czu$ **and** $czuv{:}czu\|v$ **using** *M5exist-var* **by** *blast*
from *ap kq* **have** $a\|q \oplus ((\exists\, t.\ a\|t \wedge t\|q) \oplus (\exists\, t.\ k\|t \wedge t\|p))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup s \cup d$
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** *?A* **by** *simp*
with *ap pczu czuv uv qv* **have** $(p,q) \in s$ **using** *s* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** *?B* **by** *simp*
then obtain *t* **where** $at{:}a\|t$ **and** $tq{:}t\|q$ **by** *auto*
from *pc tq* **have** $p\|q \oplus ((\exists\, t'.\ p\|t' \wedge t'\|q) \oplus (\exists\, t'.\ t\|t' \wedge t'\|c))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup s \cup d$
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** *?A* **by** *simp*
thus *?thesis* **using** *x m* **by** *auto*}
next

{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
thus *?thesis* **using** x b **by** *auto*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
then obtain t' **where** $t \| t'$ **and** $t' \| c$ **by** *auto*
with *pc pczu* **have** $t' \| czu$ **using** *M1* **by** *auto*
with *at tq ap pczu czuv qv* ‹$t \| t'$› **have** $(p,q) \in ov$ **using** *ov* **by** *blast*
thus *?thesis* **using** x **by** *auto*}
qed
}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
then obtain t **where** $k \| t$ **and** $t \| p$ **by** *auto*
with *kq pczu czuv uv qv* **have** $(p,q) \in d$ **using** d **by** *blast*
thus *?thesis* **using** x **by** *auto*}
qed
qed

lemma *cbf*: $b\ O\ f \subseteq b \cup m \cup ov \cup s \cup d$
proof
fix $x::'a \times 'a$ **assume** $x \in b\ O\ f$ **then obtain** p q z **where** x:$x = (p,q)$ **and** $(p,z) \in b$ **and** $(z,q) \in f$ **by** *auto*
from ‹$(p,z) \in b$› **obtain** c **where** *pc*:$p \| c$ **and** *cz*:$c \| z$ **using** b **by** *auto*
obtain a **where** *ap*:$a \| p$ **using** *M3 meets-wd pc* **by** *blast*
from ‹$(z,q) \in f$› **obtain** k l u **where** $k \| l$ **and** $l \| z$ **and** *kq*:$k \| q$ **and** *zu*:$z \| u$ **and** *qu*:$q \| u$ **using** f **by** *blast*
from *pc cz zu* **obtain** *cz* **where** *pcz*:$p \| cz$ **and** *czu*:$cz \| u$ **using** *M5exist-var* **by** *blast*
from *ap kq* **have** $a \| q \oplus ((\exists\, t.\ a \| t \wedge t \| q) \oplus (\exists\, t.\ k \| t \wedge t \| p))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup s \cup d$
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
with *ap pcz czu qu* **have** $(p,q) \in s$ **using** s **by** *blast*
thus *?thesis* **using** x **by** *auto*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
then obtain t **where** *at*:$a \| t$ **and** *tq*:$t \| q$ **by** *auto*
from *pc tq* **have** $p \| q \oplus ((\exists\, t'.\ p \| t' \wedge t' \| q) \oplus (\exists\, t'.\ t \| t' \wedge t' \| c))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup s \cup d$
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
thus *?thesis* **using** x m **by** *auto*}
next

{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
thus *?thesis* **using** x b **by** *auto*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
then obtain t' **where** $t \| t'$ **and** $t' \| c$ **by** *auto*
with *pc pcz* **have** $t' \| cz$ **using** *M1* **by** *auto*
with *at tq ap pcz czu qu* ‹$t \| t'$› **have** $(p,q) \in ov$ **using** *ov* **by** *blast*
thus *?thesis* **using** x **by** *auto*}
qed
}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
then obtain t **where** $k \| t$ **and** $t \| p$ **by** *auto*
with *kq pcz czu qu* **have** $(p,q) \in d$ **using** d **by** *blast*
thus *?thesis* **using** x **by** *auto*}
qed
qed

lemma *cbovi*:$b\ O\ ov\hat{}{-1} \subseteq b \cup m \cup ov \cup s \cup d$
proof
fix $x{::}'a \times 'a$ **assume** $x \in b\ O\ ov\hat{}{-1}$ **then obtain** $p\ q\ z$ **where** $x{:}x = (p,q)$ **and** $(p,z) \in b$ **and** $(z,q) \in ov\hat{}{-1}$ **by** *auto*
from ‹$(p,z) \in b$› **obtain** c **where** $pc{:}p \| c$ **and** $cz{:}c \| z$ **using** b **by** *auto*
obtain a **where** $ap{:}a \| p$ **using** *M3 meets-wd pc* **by** *blast*
from ‹$(z,q) \in ov\hat{}{-1}$› **obtain** $k\ l\ u\ v\ w$ **where** $k \| l$ **and** $lz{:}l \| z$ **and** $kq{:}k \| q$ **and** $zv{:}z \| v$ **and** $qu{:}q \| u$ **and** $uv{:}u \| v$ **and** $lw{:}l \| w$ **and** $wu{:}w \| u$ **using** *ov* **by** *blast*
from *cz lz lw* **have** $c \| w$ **using** *M1* **by** *auto*
with *pc wu* **obtain** *cw* **where** $pcw{:}p \| cw$ **and** $cwu{:}cw \| u$ **using** *M5exist-var* **by** *blast*
from *ap kq* **have** $a \| q \oplus ((\exists\, t.\ a \| t \wedge t \| q) \oplus (\exists\, t.\ k \| t \wedge t \| p))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup s \cup d$
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
with *ap qu pcw cwu* **have** $(p,q) \in s$ **using** s **by** *blast*
thus *?thesis* **using** x **by** *auto*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
then obtain t **where** $at{:}a \| t$ **and** $tq{:}t \| q$ **by** *auto*
from *pc tq* **have** $p \| q \oplus ((\exists\, t'.\ p \| t' \wedge t' \| q) \oplus (\exists\, t'.\ t \| t' \wedge t' \| c))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup s \cup d$
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
thus *?thesis* **using** x m **by** *auto*}

next
{ **assume** ¬*?A*∧*?B*∧¬*?C* **then have** *?B* **by** *simp*
thus *?thesis* **using** *x b* **by** *auto*}
next
{ **assume** ¬*?A* ∧ ¬*?B* ∧ *?C* **then have** *?C* **by** *simp*
then obtain t' **where** $t\|t'$ **and** $t'\|c$ **by** *auto*
with *pc pcw* **have** $t'\|cw$ **using** *M1* **by** *auto*
with *at tq ap pcw cwu qu* ‹$t\|t'$› **have** $(p,q)\in ov$ **using** *ov* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
qed
}
next
{ **assume** ¬*?A* ∧ ¬*?B* ∧ *?C* **then have** *?C* **by** *simp*
then obtain t **where** $k\|t$ **and** $t\|p$ **by** *auto*
with *kq pcw cwu qu* **have** $(p,q) \in d$ **using** *d* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
qed
qed

lemma *cbmi*:$b\ O\ m\widehat{\ }{-1} \subseteq b \cup m \cup ov \cup s \cup d$
proof
fix x::$'a\times'a$ **assume** $x \in b\ O\ m\widehat{\ }{-1}$ **then obtain** $p\ q\ z$ **where** x:$x = (p,q)$ **and** $(p,z) \in b$ **and** $(z,q) \in m\widehat{\ }{-1}$ **by** *auto*
from ‹$(p,z) \in b$› **obtain** c **where** *pc*:$p\|c$ **and** *cz*:$c\|z$ **using** *b* **by** *auto*
obtain k **where** *kp*:$k\|p$ **using** *M3 meets-wd pc* **by** *blast*
from ‹$(z,q) \in m\widehat{\ }{-1}$› **have** *qz*:$q\|z$ **using** *m* **by** *auto*
obtain k' **where** *kpq*:$k'\|q$ **using** *M3 meets-wd qz* **by** *blast*
from *kp kpq* **have** $k\|q \oplus ((\exists\, t.\ k\|t \wedge t\|q) \oplus (\exists\, t.\ k'\|t \wedge t\|p))$ (**is** *?A* ⊕ (*?B* ⊕ *?C*)) **using** *M2* **by** *blast*
then have (*?A*∧¬*?B*∧¬*?C*) ∨ ((¬*?A*∧*?B*∧¬*?C*) ∨ (¬*?A*∧¬*?B*∧*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup s \cup d$
proof (*elim disjE*)
{ **assume** *?A*∧¬*?B*∧¬*?C* **then have** *?A* **by** *simp*
with *kp pc cz qz* **have** $(p,q) \in s$ **using** *s* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
next
{ **assume** ¬*?A*∧*?B*∧¬*?C* **then have** *?B* **by** *simp*
then obtain t **where** *kt*:$k\|t$ **and** *tq*:$t\|q$ **by** *auto*
from *pc tq* **have** $p\|q \oplus ((\exists\, t'.\ p\|t' \wedge t'\|q) \oplus (\exists\, t'.\ t\|t' \wedge t'\|c))$ (**is** *?A* ⊕ (*?B* ⊕ *?C*)) **using** *M2* **by** *blast*
then have (*?A*∧¬*?B*∧¬*?C*) ∨ ((¬*?A*∧*?B*∧¬*?C*) ∨ (¬*?A*∧¬*?B*∧*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup s \cup d$
proof (*elim disjE*)
{ **assume** *?A*∧¬*?B*∧¬*?C* **then have** *?A* **by** *simp*
thus *?thesis* **using** *x m* **by** *auto*}
next
{ **assume** ¬*?A*∧*?B*∧¬*?C* **then have** *?B* **by** *simp*

thus *?thesis* **using** x b **by** *auto*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** *?C* **by** *simp*
then obtain t' **where** $t \| t'$ **and** $t' \| c$ **by** *auto*
with *pc cz qz kt tq kp* **have** $(p,q) \in ov$ **using** *ov* **by** *blast*
thus *?thesis* **using** x **by** *auto*}
qed
}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** *?C* **by** *simp*
then obtain t **where** $k' \| t$ **and** $t \| p$ **by** *auto*
with *kpq pc cz qz* **have** $(p,q) \in d$ **using** d **by** *blast*
thus *?thesis* **using** x **by** *auto*}
qed
qed

lemma *cdov*: $d\ O\ ov \subseteq b \cup m \cup ov \cup s \cup d$
proof
fix $x::'a \times 'a$ **assume** $x \in d\ O\ ov$ **then obtain** $p\ q\ z$ **where** $x{:}x = (p,q)$ **and** $(p,z) \in d$ **and** $(z,q) \in ov$ **by** *auto*
from $\langle (p,z) \in d \rangle$ **obtain** $k\ l\ u\ v$ **where** $kl{:}k \| l$ **and** $lp{:}l \| p$ **and** $kz{:}k \| z$ **and** $pu{:}p \| u$ **and** $uv{:}u \| v$ **and** $zv{:}z \| v$ **using** d **by** *blast*
from $\langle (z,q) \in ov \rangle$ **obtain** $k'\ l'\ u'\ v'\ c$ **where** $kplp{:}k' \| l'$ **and** $kpz{:}k' \| z$ **and** $lpq{:}l' \| q$ **and** $zup{:}z \| u'$ **and** $upvp{:}u' \| v'$ **and** $qvp{:}q \| v'$ **and** $l' \| c$ **and** $c \| u'$ **using** *ov* **by** *blast*
from *zup zv uv* **have** $u \| u'$ **using** *M1* **by** *auto*
with *pu upvp* **obtain** *uu* **where** $puu{:}p \| uu$ **and** $uuvp{:}uu \| v'$ **using** *M5exist-var* **by** *blast*
from *lp lpq* **have** $l \| q \oplus ((\exists t.\ l \| t \wedge t \| q) \oplus (\exists t.\ l' \| t \wedge t \| p))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup s \cup d$
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** *?A* **by** *simp*
with *lp puu uuvp qvp* **have** $(p,q) \in s$ **using** s **by** *blast*
thus *?thesis* **using** x **by** *auto*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** *?B* **by** *simp*
then obtain t **where** $lt{:}l \| t$ **and** $tq{:}t \| q$ **by** *auto*
from *pu tq* **have** $p \| q \oplus ((\exists t'.\ p \| t' \wedge t' \| q) \oplus (\exists t'.\ t \| t' \wedge t' \| u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup s \cup d$
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** *?A* **by** *simp*
thus *?thesis* **using** x m **by** *auto*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** *?B* **by** *simp*

thus *?thesis* **using** x b **by** *auto*}
next
{ **assume** $\neg$*?A* $\wedge$ $\neg$*?B* $\wedge$ *?C* **then have** *?C* **by** *simp*
then obtain t' **where** *ttp*:$t \| t'$ **and** $t' \| u$ **by** *auto*
with *pu puu* **have** $t' \| uu$ **using** *M1* **by** *auto*
with *lp puu qvp uuvp lt tq ttp* **have** $(p,q) \in ov$ **using** *ov* **by** *blast*
thus *?thesis* **using** x **by** *auto*}
qed
}
next
{ **assume** $\neg$*?A* $\wedge$ $\neg$*?B* $\wedge$ *?C* **then have** *?C* **by** *simp*
then obtain t **where** $l' \| t$ **and** $t \| p$ **by** *auto*
with *lpq puu uuvp qvp* **have** $(p,q) \in d$ **using** d **by** *blast*
thus *?thesis* **using** x **by** *auto*}
qed
qed

lemma *cdfi*:$d\ O\ f\hat{}{-1} \subseteq b \cup m \cup ov \cup s \cup d$
proof
fix $x{::}'a \times 'a$ **assume** $x \in d\ O\ f\hat{}{-1}$ **then obtain** p q z **where** x:$x = (p,q)$ **and** $(p,z) \in d$ **and** $(z,q) \in f\hat{}{-1}$ **by** *auto*
from ‹$(p,z) \in d$› **obtain** k l u v **where** *kl*:$k \| l$ **and** *lp*:$l \| p$ **and** *kz*:$k \| z$ **and** *pu*:$p \| u$ **and** *uv*:$u \| v$ **and** *zv*:$z \| v$ **using** d **by** *blast*
from ‹$(z,q) \in f\hat{}{-1}$› **obtain** k' l' u' **where** *kpz*:$k' \| z$ **and** *kplp*:$k' \| l'$ **and** *lpq*:$l' \| q$ **and** *zup*:$z \| u'$ **and** *qup*:$q \| u'$ **using** f **by** *blast*
from *zup zv uv* **have** *uup*:$u \| u'$ **using** *M1* **by** *auto*
from *lp lpq* **have** $l \| q \oplus ((\exists\, t.\ l \| t \wedge t \| q) \oplus (\exists\, t.\ l' \| t \wedge t \| p))$ (**is** *?A* $\oplus$ (*?B* $\oplus$ *?C*)) **using** *M2* **by** *blast*
then have (*?A*$\wedge\neg$*?B*$\wedge\neg$*?C*) $\vee$ (($\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C*) $\vee$ ($\neg$*?A*$\wedge\neg$*?B*$\wedge$*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup s \cup d$
proof (*elim disjE*)
{ **assume** *?A*$\wedge\neg$*?B*$\wedge\neg$*?C* **then have** *?A* **by** *simp*
with *lp pu uup qup* **have** $(p,q) \in s$ **using** s **by** *blast*
thus *?thesis* **using** x **by** *auto*}
next
{ **assume** $\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C* **then have** *?B* **by** *simp*
then obtain t **where** *lt*:$l \| t$ **and** *tq*:$t \| q$ **by** *auto*
from *pu tq* **have** $p \| q \oplus ((\exists\, t'.\ p \| t' \wedge t' \| q) \oplus (\exists\, t'.\ t \| t' \wedge t' \| u))$ (**is** *?A* $\oplus$ (*?B* $\oplus$ *?C*)) **using** *M2* **by** *blast*
then have (*?A*$\wedge\neg$*?B*$\wedge\neg$*?C*) $\vee$ (($\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C*) $\vee$ ($\neg$*?A*$\wedge\neg$*?B*$\wedge$*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup s \cup d$
proof (*elim disjE*)
{ **assume** *?A*$\wedge\neg$*?B*$\wedge\neg$*?C* **then have** *?A* **by** *simp*
thus *?thesis* **using** x m **by** *auto*}
next
{ **assume** $\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C* **then have** *?B* **by** *simp*
thus *?thesis* **using** x b **by** *auto*}

next
{ **assume** $\neg$*?A* $\wedge$ $\neg$*?B* $\wedge$ *?C* **then have** *?C* **by** *simp*
then obtain t' **where** *ttp*:$t \| t'$ **and** *tpu*:$t' \| u$ **by** *auto*
with *lt tq lp pu uup qup* **have** $(p,q) \in ov$ **using** *ov* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
qed
}
next
{ **assume** $\neg$*?A* $\wedge$ $\neg$*?B* $\wedge$ *?C* **then have** *?C* **by** *simp*
then obtain t **where** $l' \| t$ **and** $t \| p$ **by** *auto*
with *lpq pu uup qup* **have** $(p,q) \in d$ **using** *d* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
qed
qed

We prove compositions of the form $r_1 \circ r_2 \subseteq b \cup m \cup ov \cup f^{-1} \cup d^{-1}$.

lemma *covdi*:*ov O d^−1* $\subseteq$ *b* $\cup$ *m* $\cup$ *ov* $\cup$ *f^−1* $\cup$ *d^−1*
proof
fix $x{::}'a \times 'a$ **assume** $x \in ov\ O\ d\hat{}{-1}$ **then obtain** $p\ q\ z$ **where** $(p,z) : ov$ **and** $(z,q) : d\hat{}{-1}$ **and** $x{:}x = (p,q)$ **by** *auto*
from ‹$(p,z) : ov$› **obtain** $k\ l\ u\ v\ c$ **where** *kp*:$k \| p$ **and** *kl*:$k \| l$ **and** *lz*:$l \| z$ **and** *pu*:$p \| u$ **and** *uv*:$u \| v$ **and** *zv*:$z \| v$ **and** *lc*:$l \| c$ **and** *cu*:$c \| u$ **using** *ov* **by** *blast*
from ‹$(z,q) : d\hat{}{-1}$› **obtain** $l'\ k'\ u'\ v'$ **where** *lpq*:$l' \| q$ **and** *kplp*:$k' \| l'$ **and** *kpz*:$k' \| z$ **and** *qup*:$q \| u'$ **and** *upvp*:$u' \| v'$ **and** *zvp*:$z \| v'$ **using** *d* **by** *blast*
from *lz kpz kplp* **have** $l \| l'$ **using** *M1* **by** *auto*
with *kl lpq* **obtain** *ll* **where** *kll*:$k \| ll$ **and** *llq*:$ll \| q$ **using** *M5exist-var* **by** *blast*
from *pu qup* **have** $p \| u' \oplus ((\exists\, t.\ p \| t \wedge t \| u') \oplus (\exists\, t.\ q \| t \wedge t \| u))$ (**is** *?A* $\oplus$ (*?B* $\oplus$ *?C*)) **using** *M2* **by** *blast*
then have (*?A*$\wedge\neg$*?B*$\wedge\neg$*?C*) $\vee$ (($\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C*) $\vee$ ($\neg$*?A*$\wedge\neg$*?B*$\wedge$*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
proof (*elim disjE*)
{ **assume** *?A*$\wedge\neg$*?B*$\wedge\neg$*?C* **then have** *?A* **by** *simp*
with *qup kll llq kp* **have** $(p,q) \in f\hat{}{-1}$ **using** *f* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
next
{ **assume** $\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C* **then have** *?B* **by** *simp*
then obtain t **where** *pt*:$p \| t$ **and** *tup*:$t \| u'$ **by** *auto*
from *pt lpq* **have** $p \| q \oplus ((\exists\, t'.\ p \| t' \wedge t' \| q) \oplus (\exists\, t'.\ l' \| t' \wedge t' \| t))$ (**is** *?A* $\oplus$ (*?B* $\oplus$ *?C*)) **using** *M2* **by** *blast*
then have (*?A*$\wedge\neg$*?B*$\wedge\neg$*?C*) $\vee$ (($\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C*) $\vee$ ($\neg$*?A*$\wedge\neg$*?B*$\wedge$*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
proof (*elim disjE*)
{ **assume** *?A*$\wedge\neg$*?B*$\wedge\neg$*?C* **then have** *?A* **by** *simp*
thus *?thesis* **using** *x m* **by** *auto*}
next
{ **assume** $\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C* **then have** *?B* **by** *simp*
thus *?thesis* **using** *x b* **by** *auto*}

next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
then obtain t' **where** $lptp{:}l' \| t'$ **and** $tpt{:}t' \| t$ **by** *auto*
from *lpq lptp llq* **have** $ll \| t'$ **using** *M1* **by** *auto*
with *kp kll llq pt tup qup tpt* **have** $(p,q) \in ov$ **using** *ov* **by** *blast*
thus *?thesis* **using** x **by** *auto*}
qed
}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
then obtain t **where** $q \| t$ **and** $t \| u$ **by** *auto*
with *pu kll llq kp* **have** $(p,q) \in d\hat{\ }{-1}$ **using** d **by** *blast*
thus *?thesis* **using** x **by** *auto*}
qed
qed

lemma $cdib{:}d\hat{\ }{-1}\ O\ b \subseteq b \cup m \cup ov \cup f\hat{\ }{-1} \cup d\hat{\ }{-1}$
proof
fix $x{::}'a \times 'a$ **assume** $x \in d\hat{\ }{-1}\ O\ b$ **then obtain** $p\ q\ z$ **where** $(p,z) : d\hat{\ }{-1}$ **and** $(z,q) : b$ **and** $x{:}x = (p,q)$ **by** *auto*
from $\langle (p,z) : d\hat{\ }{-1} \rangle$ **obtain** $k\ l\ u\ v$ **where** $kp{:}k \| p$ **and** $kl{:}k \| l$ **and** $lz{:}l \| z$ **and** $pv{:}p \| v$ **and** $uv{:}u \| v$ **and** $zu{:}z \| u$ **using** d **by** *blast*
from $\langle (z,q) : b \rangle$ **obtain** c **where** $zc{:}z \| c$ **and** $cq{:}c \| q$ **using** b **by** *blast*
with *kl lz* **obtain** *lzc* **where** $klzc{:}k \| lzc$ **and** $lzcq{:}lzc \| q$ **using** *M5exist-var* **by** *blast*
obtain v' **where** $qvp{:}q \| v'$ **using** *M3 meets-wd cq* **by** *blast*
from *pv qvp* **have** $p \| v' \oplus ((\exists\, t.\ p \| t \wedge t \| v') \oplus (\exists\, t.\ q \| t \wedge t \| v))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup f\hat{\ }{-1} \cup d\hat{\ }{-1}$
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
with *qvp kp klzc lzcq* **have** $(p,q) \in f\hat{\ }{-1}$ **using** f **by** *blast*
thus *?thesis* **using** x **by** *auto*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
then obtain t **where** $pt{:}p \| t$ **and** $tvp{:}t \| v'$ **by** *auto*
from *pt cq* **have** $p \| q \oplus ((\exists\, t'.\ p \| t' \wedge t' \| q) \oplus (\exists\, t'.\ c \| t' \wedge t' \| t))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup f\hat{\ }{-1} \cup d\hat{\ }{-1}$
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
thus *?thesis* **using** $x\ m$ **by** *auto*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
thus *?thesis* **using** $x\ b$ **by** *auto*}

next
{ **assume** $\neg$*?A* $\wedge$ $\neg$*?B* $\wedge$ *?C* **then have** *?C* **by** *simp*
then obtain t' **where** $ctp{:}c\|t'$ **and** $tpt{:}t'\|t$ **by** *auto*
from *lzcq cq ctp* **have** $lzc\|t'$ **using** *M1* **by** *auto*
with *pt tvp qvp kp klzc lzcq tpt* **have** $(p,q) \in ov$ **using** *ov* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
qed
}
next
{ **assume** $\neg$*?A* $\wedge$ $\neg$*?B* $\wedge$ *?C* **then have** *?C* **by** *simp*
then obtain t **where** $q\|t$ **and** $t\|v$ **by** *auto*
with *pv kp klzc lzcq* **have** $(p,q) \in d\hat{}{-1}$ **using** *d* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
qed
qed

lemma *csdi*:$s\ O\ d\hat{}{-1} \subseteq b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
proof
fix $x{::}'a\times'a$ **assume** $x \in s\ O\ d\hat{}{-1}$ **then obtain** *p q z* **where** $(p,z) : s$ **and** $(z,q) : d\hat{}{-1}$ **and** $x{:}x = (p,q)$ **by** *auto*
from $\langle(p,z) : s\rangle$ **obtain** *k u v* **where** $kp{:}k\|p$ **and** $kz{:}k\|z$ **and** $pu{:}p\|u$ **and** $uv{:}u\|v$ **and** $zv{:}z\|v$ **using** *s* **by** *blast*
from $\langle(z,q) : d\hat{}{-1}\rangle$ **obtain** $l'\ k'\ u'\ v'$ **where** $lpq{:}l'\|q$ **and** $kplp{:}k'\|l'$ **and** $kpz{:}k'\|z$ **and** $qup{:}q\|u'$ **and** $upvp{:}u'\|v'$ **and** $zvp{:}z\|v'$ **using** *d* **by** *blast*
from *kp kz kpz* **have** $kpp{:}k'\|p$ **using** *M1* **by** *auto*
from *pu qup* **have** $p\|u' \oplus ((\exists t.\ p\|t \wedge t\|u') \oplus (\exists t.\ q\|t \wedge t\|u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A\wedge\neg?B\wedge\neg?C) \vee ((\neg?A\wedge?B\wedge\neg?C) \vee (\neg?A\wedge\neg?B\wedge?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
proof (*elim disjE*)
{ **assume** *?A*$\wedge\neg$*?B*$\wedge\neg$*?C* **then have** *?A* **by** *simp*
with *qup kpp kplp lpq* **have** $(p,q) \in f\hat{}{-1}$ **using** *f* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
next
{ **assume** $\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C* **then have** *?B* **by** *simp*
then obtain t **where** $pt{:}p\|t$ **and** $tup{:}t\|u'$ **by** *auto*
from *pt lpq* **have** $p\|q \oplus ((\exists t'.\ p\|t' \wedge t'\|q) \oplus (\exists t'.\ l'\|t' \wedge t'\|t))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A\wedge\neg?B\wedge\neg?C) \vee ((\neg?A\wedge?B\wedge\neg?C) \vee (\neg?A\wedge\neg?B\wedge?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
proof (*elim disjE*)
{ **assume** *?A*$\wedge\neg$*?B*$\wedge\neg$*?C* **then have** *?A* **by** *simp*
thus *?thesis* **using** *x m* **by** *auto*}
next
{ **assume** $\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C* **then have** *?B* **by** *simp*
thus *?thesis* **using** *x b* **by** *auto*}
next

{ **assume** $\neg$ *?A* $\wedge$ $\neg$ *?B* $\wedge$ *?C* **then have** *?C* **by** *simp*
then obtain t' **where** *lptp*:$l'\|t'$ **and** *tpt*:$t'\|t$ **by** *auto*
with *pt tup qup kpp kplp lpq* **have** $(p,q) \in ov$ **using** *ov* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
qed
}
next
{ **assume** $\neg$ *?A* $\wedge$ $\neg$ *?B* $\wedge$ *?C* **then have** *?C* **by** *simp*
then obtain t **where** $q\|t$ **and** $t\|u$ **by** *auto*
with *pu kpp kplp lpq* **have** $(p,q) \in d\hat{\ }{-1}$ **using** *d* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
qed
qed

lemma *csib*:$s\hat{\ }{-1}$ O $b \subseteq b \cup m \cup ov \cup f\hat{\ }{-1} \cup d\hat{\ }{-1}$
proof
fix x::$'a\times'a$ **assume** $x \in s\hat{\ }{-1}$ O b **then obtain** p q z **where** (p,z) : $s\hat{\ }{-1}$ **and** (z,q) : b **and** x:$x = (p,q)$ **by** *auto*
from ‹(p,z) : $s\hat{\ }{-1}$› **obtain** k u v **where** *kp*:$k\|p$ **and** *kz*:$k\|z$ **and** *zu*:$z\|u$ **and** *uv*:$u\|v$ **and** *pv*:$p\|v$ **using** *s* **by** *blast*
from ‹(z,q) : b› **obtain** c **where** *zc*:$z\|c$ **and** *cq*:$c\|q$ **using** *b* **by** *blast*
from *kz zc cq* **obtain** *zc* **where** *kzc*:$k\|zc$ **and** *zcq*:$zc\|q$ **using** *M5exist-var* **by** *blast*
obtain v' **where** *qvp*:$q\|v'$ **using** *M3 meets-wd cq* **by** *blast*
from *pv qvp* **have** $p\|v' \oplus ((\exists\, t.\ p\|t \wedge t\|v') \oplus (\exists\, t.\ q\|t \wedge t\|v))$ (**is** *?A* $\oplus$ (*?B* $\oplus$ *?C*)) **using** *M2* **by** *blast*
then have (*?A*$\wedge\neg$*?B*$\wedge\neg$*?C*) $\vee$ (($\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C*) $\vee$ ($\neg$*?A*$\wedge\neg$*?B*$\wedge$*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup f\hat{\ }{-1} \cup d\hat{\ }{-1}$
proof (*elim disjE*)
{ **assume** *?A*$\wedge\neg$*?B*$\wedge\neg$*?C* **then have** *?A* **by** *simp*
with *qvp kp kzc zcq* **have** $(p,q) \in f\hat{\ }{-1}$ **using** *f* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
next
{ **assume** $\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C* **then have** *?B* **by** *simp*
then obtain t **where** *pt*:$p\|t$ **and** *tvp*:$t\|v'$ **by** *auto*
from *pt cq* **have** $p\|q \oplus ((\exists\, t'.\ p\|t' \wedge t'\|q) \oplus (\exists\, t'.\ c\|t' \wedge t'\|t))$ (**is** *?A* $\oplus$ (*?B* $\oplus$ *?C*)) **using** *M2* **by** *blast*
then have (*?A*$\wedge\neg$*?B*$\wedge\neg$*?C*) $\vee$ (($\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C*) $\vee$ ($\neg$*?A*$\wedge\neg$*?B*$\wedge$*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup f\hat{\ }{-1} \cup d\hat{\ }{-1}$
proof (*elim disjE*)
{ **assume** *?A*$\wedge\neg$*?B*$\wedge\neg$*?C* **then have** *?A* **by** *simp*
thus *?thesis* **using** *x m* **by** *auto*}
next
{ **assume** $\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C* **then have** *?B* **by** *simp*
thus *?thesis* **using** *x b* **by** *auto*}
next
{ **assume** $\neg$ *?A* $\wedge$ $\neg$ *?B* $\wedge$ *?C* **then have** *?C* **by** *simp*

then obtain t' **where** *ctp*:$c\|t'$ **and** *tpt*:$t'\|t$ **by** *auto*
from *zcq cq ctp* **have** $zc\|t'$ **using** *M1* **by** *auto*
with *zcq pt tvp qvp kzc kp ctp tpt* **have** $(p,q) \in ov$ **using** *ov* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
qed
}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** *?C* **by** *simp*
then obtain t **where** $q\|t$ **and** $t\|v$ **by** *auto*
with *pv kp kzc zcq* **have** $(p,q) \in d\hat{}{-1}$ **using** *d* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
qed
qed

lemma *covib*:$ov\hat{}{-1}\ O\ b \subseteq b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
proof
fix $x{::}'a{\times}'a$ **assume** $x \in ov\hat{}{-1}\ O\ b$ **then obtain** $p\ q\ z$ **where** $(p,z) : ov\hat{}{-1}$ **and** $(z,q) : b$ **and** $x{:}x = (p,q)$ **by** *auto*
from ‹$(p,z) : ov\hat{}{-1}$› **obtain** $k\ l\ u\ v\ c$ **where** *kz*:$k\|z$ **and** *kl*:$k\|l$ **and** *lp*:$l\|p$ **and** *zu*:$z\|u$ **and** *uv*:$u\|v$ **and** *pv*:$p\|v$ **and** *lc*:$l\|c$ **and** *cu*:$c\|u$ **using** *ov* **by** *blast*
from ‹$(z,q) : b$› **obtain** w **where** *zw*:$z\|w$ **and** *wq*:$w\|q$ **using** *b* **by** *blast*
from *cu zu zw* **have** *cw*:$c\|w$ **using** *M1* **by** *auto*
with *lc wq* **obtain** *cw* **where** *lcw*:$l\|cw$ **and** *cwq*:$cw\|q$ **using** *M5exist-var* **by** *blast*
obtain v' **where** *qvp*:$q\|v'$ **using** *M3 meets-wd wq* **by** *blast*
from *pv qvp* **have** $p\|v' \oplus ((\exists\, t.\ p\|t \wedge t\|v') \oplus (\exists\, t.\ q\|t \wedge t\|v))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A\wedge\neg ?B\wedge\neg ?C) \vee ((\neg ?A\wedge ?B\wedge\neg ?C) \vee (\neg ?A\wedge\neg ?B\wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
proof (*elim disjE*)
{ **assume** $?A\wedge\neg ?B\wedge\neg ?C$ **then have** *?A* **by** *simp*
with *qvp lp lcw cwq* **have** $(p,q) \in f\hat{}{-1}$ **using** *f* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
next
{ **assume** $\neg ?A\wedge ?B\wedge\neg ?C$ **then have** *?B* **by** *simp*
then obtain t **where** *pt*:$p\|t$ **and** *tvp*:$t\|v'$ **by** *auto*
from *pt wq* **have** $p\|q \oplus ((\exists\, t'.\ p\|t' \wedge t'\|q) \oplus (\exists\, t'.\ w\|t' \wedge t'\|t))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A\wedge\neg ?B\wedge\neg ?C) \vee ((\neg ?A\wedge ?B\wedge\neg ?C) \vee (\neg ?A\wedge\neg ?B\wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
proof (*elim disjE*)
{ **assume** $?A\wedge\neg ?B\wedge\neg ?C$ **then have** *?A* **by** *simp*
thus *?thesis* **using** *x m* **by** *auto*}
next
{ **assume** $\neg ?A\wedge ?B\wedge\neg ?C$ **then have** *?B* **by** *simp*
thus *?thesis* **using** *x b* **by** *auto*}
next

{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
then obtain t' **where** $wtp{:}w \| t'$ **and** $tpt{:}t' \| t$ **by** *auto*
moreover with *wq cwq* **have** $cw \| t'$ **using** *M1* **by** *auto*
ultimately have $(p,q) \in ov$ **using** *ov cwq lp lcw pt tvp qvp* **by** *blast*
thus *?thesis* **using** x **by** *auto*}
qed
}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
then obtain t **where** $q \| t$ **and** $t \| v$ **by** *auto*
with *pv lp lcw cwq* **have** $(p,q) \in d\hat{}{-1}$ **using** d **by** *blast*
thus *?thesis* **using** x **by** *auto*}
qed
qed

lemma $cmib{:}m\hat{}{-1}\ O\ b \subseteq b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
proof
fix $x{::}'a \times 'a$ **assume** $x \in m\hat{}{-1}\ O\ b$ **then obtain** $p\ q\ z$ **where** $(p,z) : m\hat{}{-1}$ **and** $(z,q) : b$ **and** $x{:}x = (p,q)$ **by** *auto*
from $\langle (p,z) : m\hat{}{-1} \rangle$ **have** $zp{:}z \| p$ **using** m **by** *auto*
from $\langle (z,q) : b \rangle$ **obtain** w **where** $zw{:}z \| w$ **and** $wq{:}w \| q$ **using** b **by** *blast*
obtain v **where** $pv{:}p \| v$ **using** *M3 meets-wd zp* **by** *blast*
obtain v' **where** $qvp{:}q \| v'$ **using** *M3 meets-wd wq* **by** *blast*

from *pv qvp* **have** $p \| v' \oplus ((\exists t.\ p \| t \wedge t \| v') \oplus (\exists t.\ q \| t \wedge t \| v))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
with *zp zw wq qvp* **have** $(p,q) \in f\hat{}{-1}$ **using** f **by** *blast*
thus *?thesis* **using** x **by** *auto*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
then obtain t **where** $pt{:}p \| t$ **and** $tvp{:}t \| v'$ **by** *auto*
from *pt wq* **have** $p \| q \oplus ((\exists t'.\ p \| t' \wedge t' \| q) \oplus (\exists t'.\ w \| t' \wedge t' \| t))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
thus *?thesis* **using** $x\ m$ **by** *auto*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
thus *?thesis* **using** $x\ b$ **by** *auto*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*

then obtain t' **where** *wtp*:$w\|t'$ **and** *tpt*:$t'\|t$ **by** *auto*
with *zp zw wq pt tvp qvp* **have** $(p,q) \in ov$ **using** *ov* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
qed
}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** *?C* **by** *simp*
then obtain t **where** $q\|t$ **and** $t\|v$ **by** *auto*
with *zp zw wq pv* **have** $(p,q) \in d\hat{}{-1}$ **using** *d* **by** *blast*
thus *?thesis* **using** *x* **by** *auto*}
qed
qed

3.5 γ-composition

We prove compositions of the form $r_1 \circ r_2 \subseteq ov \cup s \cup d \cup f \cup e \cup f^{-1} \cup d^{-1} \cup s^{-1} \cup ov^{-1}$.

lemma *covovi*:$ov\ O\ ov\hat{}{-1} \subseteq e \cup ov \cup ov\hat{}{-1} \cup d \cup d\hat{}{-1} \cup s \cup s\hat{}{-1} \cup f \cup f\hat{}{-1}$
proof
fix x::$'a\times'a$ **assume** $x \in ov\ O\ ov\hat{}{-1}$ **then obtain** $p\ q\ z$ **where** x:$x = (p,q)$ **and** $(p,z) \in ov$ **and** $(z, q) \in ov\hat{}{-1}$ **by** *auto*
from ‹$(p,z) \in ov$› **obtain** $k\ l\ c\ u$ **where** *kp*:$k\|p$ **and** *kl*:$k\|l$ **and** *lz*:$l\|z$ **and** *lc*:$l\|c$ **and** *pu*:$p\|u$ **and** *cu*:$c\|u$ **using** *ov* **by** *blast*
from ‹$(z,q) \in ov\hat{}{-1}$› **obtain** $k'\ l'\ c'\ u'$ **where** *kpq*:$k'\|q$ **and** *kplp*:$k'\|l'$ **and** *lpz*:$l'\|z$ **and** *lpcp*:$l'\|c'$ **and** *qup*:$q\|u'$ **and** *cpup*:$c'\|u'$ **using** *ov* **by** *blast*

from *kp kpq* **have** $k\|q \oplus ((\exists t.\ k\|t \wedge t\|q) \oplus (\exists t.\ k'\|t \wedge t\|p))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A\wedge\neg ?B\wedge\neg ?C) \vee ((\neg ?A\wedge ?B\wedge\neg ?C) \vee (\neg ?A\wedge\neg ?B\wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in e \cup ov \cup ov\hat{}{-1} \cup d \cup d\hat{}{-1} \cup s \cup s\hat{}{-1} \cup f \cup f\hat{}{-1}$
proof (*elim disjE*)
{ **assume** $?A\wedge\neg ?B\wedge\neg ?C$ **then have** *kq*:*?A* **by** *simp*
from *pu qup* **have** $p\|u' \oplus ((\exists t'.\ p\|t' \wedge t'\|u') \oplus (\exists t'.\ q\|t' \wedge t'\|u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A\wedge\neg ?B\wedge\neg ?C) \vee ((\neg ?A\wedge ?B\wedge\neg ?C) \vee (\neg ?A\wedge\neg ?B\wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{ **assume** $?A\wedge\neg ?B\wedge\neg ?C$ **then have** *?A* **by** *simp*
with *kq kp qup* **have** $p = q$ **using** *M4* **by** *auto*
thus *?thesis* **using** *x e* **by** *auto*}
next
{ **assume** $\neg ?A\wedge ?B\wedge\neg ?C$ **then have** *?B* **by** *simp*
with *kq kp qup* **show** *?thesis* **using** *x s* **by** *blast*}
next
{ **assume** $\neg ?A\wedge\neg ?B\wedge ?C$ **then have** *?C* **by** *simp*
with *kq kp pu* **show** *?thesis* **using** *x s* **by** *blast*}

qed}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** *?B* **by** *simp*
then obtain t **where** $kt{:}k \| t$ **and** $tq{:}t \| q$ **by** *auto*
from *pu qup* **have** $p \| u' \oplus ((\exists\, t'.\ p \| t' \wedge t' \| u') \oplus (\exists\, t'.\ q \| t' \wedge t' \| u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** *?A* **by** *simp*
with *qup kp kt tq* **show** *?thesis* **using** *x f* **by** *blast*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** *?B* **by** *simp*
then obtain t' **where** $ptp{:}p \| t'$ **and** $tpup{:}t' \| u'$ **by** *auto*
from *tq kpq kplp* **have** $t \| l'$ **using** *M1* **by** *auto*
moreover with *lpz lz lc* **have** $l' \| c$ **using** *M1* **by** *auto*
moreover with *cu pu ptp* **have** $c \| t'$ **using** *M1* **by** *auto*
ultimately obtain *lc* **where** $t \| lc$ **and** $lc \| t'$ **using** *M5exist-var* **by** *blast*
with *ptp tpup kp kt tq qup* **show** *?thesis* **using** *x ov* **by** *blast*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** *?C* **by** *simp*
with *pu kp kt tq* **show** *?thesis* **using** *x d* **by** *blast*}

qed}
next
{**assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** *?C* **by** *auto*
then obtain t **where** $kpt{:}k' \| t$ **and** $tp{:}t \| p$ **by** *auto*
from *pu qup* **have** $p \| u' \oplus ((\exists\, t'.\ p \| t' \wedge t' \| u') \oplus (\exists\, t'.\ q \| t' \wedge t' \| u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** *?A* **by** *simp*
with *kpq kpt tp qup* **show** *?thesis* **using** *x f* **by** *blast*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** *?B* **by** *simp*
then obtain t' **where** $p \| t'$ **and** $t' \| u'$ **by** *auto*
with *kpq kpt tp qup* **show** *?thesis* **using** *x d* **by** *blast*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** *?C* **by** *simp*
then obtain t' **where** $qtp{:}q \| t'$ **and** $tpu{:}t' \| u$ **by** *auto*
from *tp kp kl* **have** $t \| l$ **using** *M1* **by** *auto*
moreover with *lpcp lpz lz* **have** $l \| c'$ **using** *M1* **by** *auto*
moreover with *cpup qup qtp* **have** $c' \| t'$ **using** *M1* **by** *auto*
ultimately obtain *lc* **where** $t \| lc$ **and** $lc \| t'$ **using** *M5exist-var* **by** *blast*
with *kpt tp kpq qtp tpu pu* **show** *?thesis* **using** *x ov* **by** *blast*}
qed}

qed
qed

lemma *cdid*:$d\widehat{\ }{-1}\ O\ d \subseteq e \cup ov \cup ov\widehat{\ }{-1} \cup d \cup d\widehat{\ }{-1} \cup s \cup s\widehat{\ }{-1} \cup f \cup f\widehat{\ }{-1}$
proof
fix $x::'a\times'a$ **assume** $x \in d\widehat{\ }{-1}\ O\ d$ **then obtain** $p\ q\ z$ **where** $x{:}x = (p,q)$ **and** $(p,z) \in d\widehat{\ }{-1}$ **and** $(z,\ q) \in d$ **by** *auto*
from ‹$(p,z) \in d\widehat{\ }{-1}$› **obtain** $k\ l\ u\ v$ **where** $kp{:}k\|p$ **and** $kl{:}k\|l$ **and** $lz{:}l\|z$ **and** $pv{:}p\|v$ **and** $zu{:}z\|u$ **and** $uv{:}u\|v$ **using** d **by** *blast*
from ‹$(z,q) \in d$› **obtain** $k'\ l'\ u'\ v'$ **where** $kpq{:}k'\|q$ **and** $kplp{:}k'\|l'$ **and** $lpz{:}l'\|z$ **and** $qvp{:}q\|v'$ **and** $zup{:}z\|u'$ **and** $upvp{:}u'\|v'$ **using** d **by** *blast*

from *kp kpq* **have** $k\|q \oplus ((\exists\, t.\ k\|t \wedge t\|q) \oplus (\exists\, t.\ k'\|t \wedge t\|p))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A\wedge\neg ?B\wedge\neg ?C) \vee ((\neg ?A\wedge ?B\wedge\neg ?C) \vee (\neg ?A\wedge\neg ?B\wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in e \cup ov \cup ov\widehat{\ }{-1} \cup d \cup d\widehat{\ }{-1} \cup s \cup s\widehat{\ }{-1} \cup f \cup f\widehat{\ }{-1}$
proof (*elim disjE*)
{ assume $?A\wedge\neg ?B\wedge\neg ?C$ **then have** $kq{:}?A$ **by** *simp*
from *pv qvp* **have** $p\|v' \oplus ((\exists\, t'.\ p\|t' \wedge t'\|v') \oplus (\exists\, t'.\ q\|t' \wedge t'\|v))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A\wedge\neg ?B\wedge\neg ?C) \vee ((\neg ?A\wedge ?B\wedge\neg ?C) \vee (\neg ?A\wedge\neg ?B\wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{ assume $?A\wedge\neg ?B\wedge\neg ?C$ **then have** $?A$ **by** *simp*
with *kq kp qvp* **have** $p = q$ **using** *M4* **by** *auto*
thus *?thesis* **using** *x e* **by** *auto***}**
next
{ assume $\neg ?A\wedge ?B\wedge\neg ?C$ **then have** $?B$ **by** *simp*
with *kq kp qvp* **show** *?thesis* **using** *x s* **by** *blast***}**
next
{ assume $\neg ?A\wedge\neg ?B\wedge ?C$ **then have** $?C$ **by** *simp*
with *kq kp pv* **show** *?thesis* **using** *x s* **by** *blast***}**
qed}
next
{ assume $\neg ?A\wedge ?B\wedge\neg ?C$ **then have** $?B$ **by** *simp*
then obtain t **where** $kt{:}k\|t$ **and** $tq{:}t\|q$ **by** *auto*
from *pv qvp* **have** $p\|v' \oplus ((\exists\, t'.\ p\|t' \wedge t'\|v') \oplus (\exists\, t'.\ q\|t' \wedge t'\|v))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A\wedge\neg ?B\wedge\neg ?C) \vee ((\neg ?A\wedge ?B\wedge\neg ?C) \vee (\neg ?A\wedge\neg ?B\wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{ assume $?A\wedge\neg ?B\wedge\neg ?C$ **then have** $?A$ **by** *simp*
with *qvp kp kt tq* **show** *?thesis* **using** *x f* **by** *blast***}**
next
{ assume $\neg ?A\wedge ?B\wedge\neg ?C$ **then have** $?B$ **by** *simp*

then obtain t' **where** *ptp*:$p\|t'$ **and** *tpvp*:$t'\|v'$ **by** *auto*
from *tq kpq kplp* **have** $t\|l'$ **using** *M1* **by** *auto*
moreover with *ptp pv uv* **have** $u\|t'$ **using** *M1* **by** *auto*
moreover with *lpz zu* ‹$t\|l'$› **obtain** *lzu* **where** $t\|lzu$ **and** $lzu\|t'$ **using** *M5exist-var* **by** *blast*
ultimately show *?thesis* **using** *x ov kt tq kp ptp tpvp qvp* **by** *blast*}
next
{ **assume** $\neg$*?A*$\wedge\neg$*?B*$\wedge$*?C* **then have** *?C* **by** *simp*
with *pv kp kt tq* **show** *?thesis* **using** *x d* **by** *blast*}

qed}
next
{**assume** $\neg$*?A*$\wedge\neg$*?B*$\wedge$*?C* **then have** *?C* **by** *auto*
then obtain t **where** *kpt*:$k'\|t$ **and** *tp*:$t\|p$ **by** *auto*
from *pv qvp* **have** $p\|v' \oplus ((\exists\, t'.\ p\|t' \wedge t'\|v') \oplus (\exists\, t'.\ q\|t' \wedge t'\|v))$ (**is** *?A* $\oplus$ (*?B* $\oplus$ *?C*)) **using** *M2* **by** *blast*
then have (*?A*$\wedge\neg$*?B*$\wedge\neg$*?C*) $\vee$ (($\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C*) $\vee$ ($\neg$*?A*$\wedge\neg$*?B*$\wedge$*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{ **assume** *?A*$\wedge\neg$*?B*$\wedge\neg$*?C* **then have** *?A* **by** *simp*
with *kpq kpt tp qvp* **show** *?thesis* **using** *x f* **by** *blast*}
next
{ **assume** $\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C* **then have** *?B* **by** *simp*
then obtain t' **where** $p\|t'$ **and** $t'\|v'$ **by** *auto*
with *kpq kpt tp qvp* **show** *?thesis* **using** *x d* **by** *blast*}
next
{ **assume** $\neg$*?A*$\wedge\neg$*?B*$\wedge$*?C* **then have** *?C* **by** *simp*
then obtain t' **where** *qtp*:$q\|t'$ **and** *tpv*:$t'\|v$ **by** *auto*
from *tp kp kl* **have** $t\|l$ **using** *M1* **by** *auto*
moreover with *qtp qvp upvp* **have** $u'\|t'$ **using** *M1* **by** *auto*
moreover with *lz zup* ‹$t\|l$› **obtain** *lzu* **where** $t\|lzu$ **and** $lzu\|t'$ **using** *M5exist-var* **by** *blast*
ultimately show *?thesis* **using** *x ov kpt tp kpq qtp tpv pv* **by** *blast*}
qed}
qed
qed

lemma *coviov*:$ov\widehat{\ }{-1}\ O\ ov \subseteq e \cup ov \cup ov\widehat{\ }{-1} \cup d \cup d\widehat{\ }{-1} \cup s \cup s\widehat{\ }{-1} \cup f \cup f\widehat{\ }{-1}$
proof
fix x::$'a\times'a$ **assume** $x \in ov\widehat{\ }{-1}\ O\ ov$ **then obtain** $p\ q\ z$ **where** *x*:$x = (p,q)$ **and** $(p,z) \in ov\widehat{\ }{-1}$ **and** $(z,\ q) \in ov$ **by** *auto*
from ‹$(p,z) \in ov\widehat{\ }{-1}$› **obtain** $k\ l\ c\ u\ v$ **where** *kz*:$k\|z$ **and** *kl*:$k\|l$ **and** *lp*:$l\|p$ **and** *lc*:$l\|c$ **and** *zu*:$z\|u$ **and** *pv*:$p\|v$ **and** *cu*:$c\|u$ **and** *uv*:$u\|v$ **using** *ov* **by** *blast*
from ‹$(z,q) \in ov$› **obtain** $k'\ l'\ c'\ u'\ v'$ **where** *kpz*:$k'\|z$ **and** *kplp*:$k'\|l'$ **and** *lpq*:$l'\|q$ **and** *lpcp*:$l'\|c'$ **and** *qvp*:$q\|v'$ **and** *zup*:$z\|u'$ **and** *cpup*:$c'\|u'$ **and** *upvp*:$u'\|v'$ **using** *ov* **by** *blast*

from *lp lpq* **have** $l \| q \oplus ((\exists t.\ l \| t \wedge t \| q) \oplus (\exists t.\ l' \| t \wedge t \| p))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in e \cup ov \cup ov\hat{}{-1} \cup d \cup d\hat{}{-1} \cup s \cup s\hat{}{-1} \cup f \cup f\hat{}{-1}$
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** *lq*:$?A$ **by** *simp*
from *pv qvp* **have** $p \| v' \oplus ((\exists t'.\ p \| t' \wedge t' \| v') \oplus (\exists t'.\ q \| t' \wedge t' \| v))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
with *lq lp qvp* **have** $p = q$ **using** *M4* **by** *auto*
thus *?thesis* **using** *x e* **by** *auto*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
with *lq lp qvp* **show** *?thesis* **using** *x s* **by** *blast*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
with *lq lp pv* **show** *?thesis* **using** *x s* **by** *blast*}
qed}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
then obtain t **where** *lt*:$l \| t$ **and** *tq*:$t \| q$ **by** *auto*
from *pv qvp* **have** $p \| v' \oplus ((\exists t'.\ p \| t' \wedge t' \| v') \oplus (\exists t'.\ q \| t' \wedge t' \| v))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
with *qvp lp lt tq* **show** *?thesis* **using** *x f* **by** *blast*}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
then obtain t' **where** *ptp*:$p \| t'$ **and** *tpvp*:$t' \| v'$ **by** *auto*
from *tq lpq lpcp* **have** $t \| c'$ **using** *M1* **by** *auto*
moreover with *cpup zup zu* **have** $c' \| u$ **using** *M1* **by** *auto*
moreover with *ptp pv uv* **have** $u \| t'$ **using** *M1* **by** *auto*
ultimately obtain *cu* **where** $t \| cu$ **and** $cu \| t'$ **using** *M5exist-var* **by** *blast*
with *lt tq lp ptp tpvp qvp* **show** *?thesis* **using** *x ov* **by** *blast*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
with *pv lp lt tq* **show** *?thesis* **using** *x d* **by** *blast*}

qed}
next

{**assume** $\neg$*?A*$\wedge\neg$*?B*$\wedge$*?C* **then have** *?C* **by** *auto*
then obtain t **where** *lpt*:$l'\|t$ **and** *tp*:$t\|p$ **by** *auto*
from *pv qvp* **have** $p\|v' \oplus ((\exists\, t'.\ p\|t' \wedge t'\|v') \oplus (\exists\, t'.\ q\|t' \wedge t'\|v))$ (**is** *?A* $\oplus$ (*?B* $\oplus$ *?C*)) **using** *M2* **by** *blast*
then have (*?A*$\wedge\neg$*?B*$\wedge\neg$*?C*) $\vee$ (($\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C*) $\vee$ ($\neg$*?A*$\wedge\neg$*?B*$\wedge$*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{ **assume** *?A*$\wedge\neg$*?B*$\wedge\neg$*?C* **then have** *?A* **by** *simp*
with *qvp lpq lpt tp* **show** *?thesis* **using** *x f* **by** *blast*}
next
{ **assume** $\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C* **then have** *?B* **by** *simp*
then obtain t' **where** $p\|t'$ **and** $t'\|v'$ **by** *auto*
with *qvp lpq lpt tp* **show** *?thesis* **using** *x d* **by** *blast*}
next
{ **assume** $\neg$*?A*$\wedge\neg$*?B*$\wedge$*?C* **then have** *?C* **by** *simp*
then obtain t' **where** *qtp*:$q\|t'$ **and** *tpv*:$t'\|v$ **by** *auto*
from *tp lp lc* **have** $t\|c$ **using** *M1* **by** *auto*
moreover with *cu zu zup* **have** $c\|u'$ **using** *M1* **by** *auto*
moreover with *qtp qvp upvp* **have** $u'\|t'$ **using** *M1* **by** *auto*
ultimately obtain *cu* **where** $t\|cu$ **and** $cu\|t'$ **using** *M5exist-var* **by** *blast*
with *lpt tp lpq pv qtp tpv* **show** *?thesis* **using** *x ov* **by** *blast*}
qed}
qed
qed

3.6 γ-composition

We prove compositions of the form $r_1 \circ r_2 \subseteq b \cup m \cup ov \cup s \cup d \cup f \cup e \cup f^{-1} \cup d^{-1} \cup s^{-1} \cup ov^{-1} \cup b^{-1} \cup m^{-1}$.

lemma *cbbi*:*b O b^−1* $\subseteq$ *b* $\cup$ *b^−1* $\cup$ *m* $\cup$ *m^−1* $\cup$ *e* $\cup$ *ov* $\cup$ *ov^−1* $\cup$ *s* $\cup$ *s^−1* $\cup$ *d* $\cup$ *d^−1* $\cup$ *f* $\cup$ *f^−1* (**is** *b O b^−1* $\subseteq$ *?R*)
proof
fix *x*::$'a\times'a$ **assume** $x \in$ *b O b^−1* **then obtain** *p q z*::$'a$ **where** *x*:$x = (p,q)$ **and** $(p,z) \in b$ **and** $(z,q) \in$ *b^−1* **by** *auto*
from ‹$(p,z)\in b$› **obtain** c **where** *pc*:$p\|c$ **and** $c\|z$ **using** *b* **by** *blast*
from ‹$(z,q) \in$ *b^−1*› **obtain** c' **where** *qcp*:$q\|c'$ **and** $c'\|z$ **using** *b* **by** *blast*
obtain $k\ k'$ **where** *kp*:$k\|p$ **and** *kpq*:$k'\|q$ **using** *M3 meets-wd pc qcp* **by** *fastforce*
then have $k\|q \oplus ((\exists\, t.\ k\|t \wedge t\|q) \oplus (\exists\, t.\ k'\|t \wedge t\|p))$ (**is** *?A* $\oplus$ (*?B* $\oplus$ *?C*)) **using** *M2* **by** *blast*
then have (*?A*$\wedge\neg$*?B*$\wedge\neg$*?C*) $\vee$ (($\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C*) $\vee$ ($\neg$*?A*$\wedge\neg$*?B*$\wedge$*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in$*?R*
proof (*elim disjE*)
{ **assume** *?A*$\wedge\neg$*?B*$\wedge\neg$*?C* **then have** *kq*:*?A* **by** *simp*
from *pc qcp* **have** $p\|c' \oplus ((\exists\, t'.\ p\|t' \wedge t'\|c') \oplus (\exists\, t'.\ q\|t' \wedge t'\|c))$ (**is** *?A* $\oplus$ (*?B* $\oplus$ *?C*)) **using** *M2* **by** *blast*
then have (*?A*$\wedge\neg$*?B*$\wedge\neg$*?C*) $\vee$ (($\neg$*?A*$\wedge$*?B*$\wedge\neg$*?C*) $\vee$ ($\neg$*?A*$\wedge\neg$*?B*$\wedge$*?C*)) **by**

(*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{**assume** (*?A*∧¬*?B*∧¬*?C*) **then have** *?A* **by** *simp*
with *kp kq qcp* **have** $p = q$ **using** *M4* **by** *auto*
thus *?thesis* **using** *x e* **by** *auto*}
next
{**assume** ¬*?A*∧*?B*∧¬*?C* **then have** *?B* **by** *simp*
with *kq kp qcp* **show** *?thesis* **using** *x s* **by** *blast*}
next
{**assume** (¬*?A*∧¬*?B*∧*?C*) **then have** *?C* **by** *simp*
with *kq kp pc* **show** *?thesis* **using** *x s* **by** *blast*}
qed}
next
{ **assume** ¬*?A*∧*?B*∧¬*?C* **then have** *?B* **by** *simp*
then obtain t **where** *kt*:$k \| t$ **and** *tq*:$t \| q$ **by** *auto*
from *pc qcp* **have** $p \| c' \oplus ((\exists\, t'.\ p \| t' \wedge t' \| c') \oplus (\exists\, t'.\ q \| t' \wedge t' \| c))$ (**is** *?A* ⊕ (*?B* ⊕ *?C*)) **using** *M2* **by** *blast*
then have (*?A*∧¬*?B*∧¬*?C*) ∨ ((¬*?A*∧*?B*∧¬*?C*) ∨ (¬*?A*∧¬*?B*∧*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{**assume** *?A*∧¬*?B*∧¬*?C* **then have** *?A* **by** *simp*
with *kp qcp kt tq* **show** *?thesis* **using** *f x* **by** *blast*}
next
{**assume** ¬*?A*∧*?B*∧¬*?C* **then have** *?B* **by** *simp*
then obtain t' **where** *ptp*:$p \| t'$ **and** *tpcp*:$t' \| c'$ **by** *auto*
from *pc tq* **have** $p \| q \oplus ((\exists\, t''.\ p \| t'' \wedge t'' \| q) \oplus (\exists\, t''.\ t \| t'' \wedge t'' \| c))$ (**is** *?A* ⊕ (*?B* ⊕ *?C*)) **using** *M2* **by** *blast*
then have (*?A*∧¬*?B*∧¬*?C*) ∨ ((¬*?A*∧*?B*∧¬*?C*) ∨ (¬*?A*∧¬*?B*∧*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{**assume** *?A*∧¬*?B*∧¬*?C* **then have** *?A* **by** *simp*
thus *?thesis* **using** *x m* **by** *auto*}
next
{**assume** ¬*?A*∧*?B*∧¬*?C* **then have** *?B* **by** *simp*
thus *?thesis* **using** *x b* **by** *auto*}
next
{ **assume** ¬*?A*∧¬*?B*∧*?C* **then have** *?C* **by** *simp*
then obtain g **where** $t \| g$ **and** $g \| c$ **by** *auto*
moreover with *pc ptp* **have** $g \| t'$ **using** *M1* **by** *blast*
ultimately show *?thesis* **using** *x ov kt tq kp ptp tpcp qcp* **by** *blast*}
qed}
next
{**assume** ¬*?A*∧¬*?B*∧*?C* **then have** *?C* **by** *simp*
then obtain t' **where** $q \| t'$ **and** $t' \| c$ **by** *auto*
with *kp kt tq pc* **show** *?thesis* **using** *d x* **by** *blast*}
qed}

next
{ assume $\neg$*?A*$\wedge\neg$*?B*$\wedge$*?C* **then have** *?C* **by** *simp*
then obtain t **where** *kpt*:$k'\|t$ **and** *tp*:$t\|p$ **by** *auto*
from *pc qcp* **have** $p\|c' \oplus ((\exists t'.\ p\|t' \wedge t'\|c') \oplus (\exists t'.\ q\|t' \wedge t'\|c))$ (**is** *?A* $\oplus$ (*?B* $\oplus$ *?C*)) **using** *M2* **by** *blast*
then have (*?A*$\wedge\neg$*?B*$\wedge\neg$*?C*) $\vee$ ((¬*?A*$\wedge$*?B*$\wedge\neg$*?C*) $\vee$ (¬*?A*$\wedge\neg$*?B*$\wedge$*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{assume *?A*$\wedge\neg$*?B*$\wedge\neg$*?C* **then have** *?A* **by** *simp*
with *qcp kpt tp kpq* **show** *?thesis* **using** *x f* **by** *blast***}**
next
{assume ¬*?A*$\wedge$*?B*$\wedge\neg$*?C* **then have** *?B* **by** *simp*
with *qcp kpt tp kpq* **show** *?thesis* **using** *x d* **by** *blast***}**
next
{assume ¬*?A*$\wedge\neg$*?B*$\wedge$*?C* **then obtain** t' **where** *qt'*:$q\|t'$ **and** *tpc*:$t'\|c$ **by** *auto*
from *qcp tp* **have** $q\|p \oplus ((\exists t''.\ q\|t'' \wedge t''\|p) \oplus (\exists t''.\ t\|t'' \wedge t''\|c'))$ (**is** *?A* $\oplus$ (*?B* $\oplus$ *?C*)) **using** *M2* **by** *blast*
then have (*?A*$\wedge\neg$*?B*$\wedge\neg$*?C*) $\vee$ ((¬*?A*$\wedge$*?B*$\wedge\neg$*?C*) $\vee$ (¬*?A*$\wedge\neg$*?B*$\wedge$*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{assume *?A*$\wedge\neg$*?B*$\wedge\neg$*?C* **then have** *?A* **by** *simp*
thus *?thesis* **using** *x m* **by** *auto***}**
next
{assume ¬*?A*$\wedge$*?B*$\wedge\neg$*?C* **then have** *?B* **by** *simp*
thus *?thesis* **using** *x b* **by** *auto***}**
next
{ assume ¬*?A*$\wedge\neg$*?B*$\wedge$*?C* **then obtain** g **where** *tg*:$t\|g$ **and** $g\|c'$ **by** *auto*
with *qcp qt'* **have** $g\|t'$ **using** *M1* **by** *blast*
with *qt' tpc pc kpq kpt tp tg* **show** *?thesis* **using** *x ov* **by** *blast***}**
qed}
qed}
qed
qed

lemma *cbib*:$b^{\frown}{-1}\ O\ b \subseteq b \cup b^{\frown}{-1} \cup m \cup m^{\frown}{-1} \cup e \cup ov \cup ov^{\frown}{-1} \cup s \cup s^{\frown}{-1} \cup d \cup d^{\frown}{-1} \cup f \cup f^{\frown}{-1}$ (**is** $b^{\frown}{-1}\ O\ b \subseteq$ *?R*)
proof
fix x::$'a\times'a$ **assume** $x \in b^{\frown}{-1}\ O\ b$ **then obtain** $p\ q\ z$::$'a$ **where** x:$x = (p,q)$ **and** $(p,z) \in b^{\frown}{-1}$ **and** $(z,q) \in b$ **by** *auto*
from ‹$(p,z) \in b^{\frown}{-1}$› **obtain** c **where** *zc*:$z\|c$ **and** *cp*:$c\|p$ **using** *b* **by** *blast*
from ‹$(z,q) \in b$› **obtain** c' **where** *zcp*:$z\|c'$ **and** *cpq*:$c'\|q$ **using** *b* **by** *blast*
obtain $u\ u'$ **where** *pu*:$p\|u$ **and** *qup*:$q\|u'$ **using** *M3 meets-wd cp cpq* **by** *fastforce*
from *cp cpq* **have** $c\|q \oplus ((\exists t.\ c\|t \wedge t\|q) \oplus (\exists t.\ c'\|t \wedge t\|p))$ (**is** *?A* $\oplus$ (*?B* $\oplus$

?C)) **using** *M2* **by** *blast*
then have (*?A*∧¬*?B*∧¬*?C*) ∨ ((¬*?A*∧*?B*∧¬*?C*) ∨ (¬*?A*∧¬*?B*∧*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in$*?R*
proof (*elim disjE*)
{ **assume** *?A*∧¬*?B*∧¬*?C* **then have** *cq*:*?A* **by** *simp*
from *pu qup* **have** $p\|u' \oplus ((\exists\, t'.\ p\|t' \wedge t'\|u') \oplus (\exists\, t'.\ q\|t' \wedge t'\|u))$ (**is** *?A* ⊕ (*?B* ⊕ *?C*)) **using** *M2* **by** *blast*
then have (*?A*∧¬*?B*∧¬*?C*) ∨ ((¬*?A*∧*?B*∧¬*?C*) ∨ (¬*?A*∧¬*?B*∧*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{**assume** (*?A*∧¬*?B*∧¬*?C*) **then have** *?A* **by** *simp*
with *cq cp qup* **have** $p = q$ **using** *M4* **by** *auto*
thus *?thesis* **using** *x e* **by** *auto*}
next
{**assume** ¬*?A*∧*?B*∧¬*?C* **then have** *?B* **by** *simp*
with *cq cp qup* **show** *?thesis* **using** *x s* **by** *blast*}
next
{**assume** (¬*?A*∧¬*?B*∧*?C*) **then have** *?C* **by** *simp*
with *pu cq cp* **show** *?thesis* **using** *x s* **by** *blast*}
qed}
next
{ **assume** ¬*?A*∧*?B*∧¬*?C* **then have** *?B* **by** *simp*
then obtain *t* **where** $ct{:}c\|t$ **and** $tq{:}t\|q$ **by** *auto*
from *pu qup* **have** $p\|u' \oplus ((\exists\, t'.\ p\|t' \wedge t'\|u') \oplus (\exists\, t'.\ q\|t' \wedge t'\|u))$ (**is** *?A* ⊕ (*?B* ⊕ *?C*)) **using** *M2* **by** *blast*
then have (*?A*∧¬*?B*∧¬*?C*) ∨ ((¬*?A*∧*?B*∧¬*?C*) ∨ (¬*?A*∧¬*?B*∧*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{**assume** *?A*∧¬*?B*∧¬*?C* **then have** *?A* **by** *simp*
with *qup ct tq cp* **show** *?thesis* **using** *f x* **by** *blast*}
next
{**assume** ¬*?A*∧*?B*∧¬*?C* **then have** *?B* **by** *simp*
then obtain t' **where** $ptp{:}p\|t'$ **and** $tpup{:}t'\|u'$ **by** *auto*
from *pu tq* **have** $p\|q \oplus ((\exists\, t''.\ p\|t'' \wedge t''\|q) \oplus (\exists\, t''.\ t\|t'' \wedge t''\|u))$ (**is** *?A* ⊕ (*?B* ⊕ *?C*)) **using** *M2* **by** *blast*
then have (*?A*∧¬*?B*∧¬*?C*) ∨ ((¬*?A*∧*?B*∧¬*?C*) ∨ (¬*?A*∧¬*?B*∧*?C*)) **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{**assume** *?A*∧¬*?B*∧¬*?C* **then have** *?A* **by** *simp*
thus *?thesis* **using** *x m* **by** *auto*}
next
{**assume** ¬*?A*∧*?B*∧¬*?C* **then have** *?B* **by** *simp*
thus *?thesis* **using** *x b* **by** *auto*}
next
{ **assume** ¬*?A*∧¬*?B*∧*?C* **then have** *?C* **by** *simp*

then obtain g **where** $t\|g$ **and** $g\|u$ **by** *auto*
moreover with *pu ptp* **have** $g\|t'$ **using** *M1* **by** *blast*
ultimately show *?thesis* **using** *x ov ct tq cp ptp tpup qup* **by** *blast*}
qed}
next
{**assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** *?C* **by** *simp*
then obtain t' **where** $q\|t'$ **and** $t'\|u$ **by** *auto*
with *cp ct tq pu* **show** *?thesis* **using** *d x* **by** *blast*}
qed}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** *?C* **by** *simp*
then obtain t **where** $cpt{:}c'\|t$ **and** $tp{:}t\|p$ **by** *auto*
from *pu qup* **have** $p\|u' \oplus ((\exists\, t'.\ p\|t' \wedge t'\|u') \oplus (\exists\, t'.\ q\|t' \wedge t'\|u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{**assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** *?A* **by** *simp*
with *qup cpt tp cpq* **show** *?thesis* **using** *x f* **by** *blast*}
next
{**assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** *?B* **by** *simp*
with *qup cpt tp cpq* **show** *?thesis* **using** *x d* **by** *blast*}
next
{**assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then obtain** t' **where** $qt'{:}q\|t'$ **and** $tpc{:}t'\|u$ **by** *auto*
from *qup tp* **have** $q\|p \oplus ((\exists\, t''.\ q\|t'' \wedge t''\|p) \oplus (\exists\, t''.\ t\|t'' \wedge t''\|u'))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{**assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** *?A* **by** *simp*
thus *?thesis* **using** *x m* **by** *auto*}
next
{**assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** *?B* **by** *simp*
thus *?thesis* **using** *x b* **by** *auto*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then obtain** g **where** $tg{:}t\|g$ **and** $g\|u'$ **by** *auto*
with *qup qt'* **have** $g\|t'$ **using** *M1* **by** *blast*
with *qt' tpc pu cpq cpt tp tg* **show** *?thesis* **using** *x ov* **by** *blast*}
qed}
qed}
qed
qed

lemma *cddi*:$d\ O\ d\hat{}{-1} \subseteq b \cup b\hat{}{-1} \cup m \cup m\hat{}{-1} \cup e \cup ov \cup ov\hat{}{-1} \cup s \cup s\hat{}{-1} \cup d \cup d\hat{}{-1} \cup f \cup f\hat{}{-1}$ (**is** $d\ O\ d\hat{}{-1} \subseteq ?R$)

proof
fix x::$'a \times 'a$ **assume** $x \in d$ O $d\widehat{\ }{-1}$ **then obtain** p q z::$'a$ **where** x:$x = (p,q)$ **and** $(p,z) \in d$ **and** $(z,q) \in d\widehat{\ }{-1}$ **by** *auto*
from ‹$(p,z) \in d$› **obtain** k l u v **where** lp:$l\|p$ **and** kl:$k\|l$ **and** kz:$k\|z$ **and** pu:$p\|u$ **and** uv:$u\|v$ **and** zv:$z\|v$ **using** d **by** *blast*
from ‹$(z,q) \in d\widehat{\ }{-1}$› **obtain** k' l' u' v' **where** lpq:$l'\|q$ **and** $kplp$:$k'\|l'$ **and** kpz:$k'\|z$ **and** qup:$q\|u'$ **and** $upvp$:$u'\|v'$ **and** zv':$z\|v'$ **using** d **by** *blast*
from lp lpq **have** $l\|q \oplus ((\exists\, t.\ l\|t \wedge t\|q) \oplus (\exists\, t.\ l'\|t \wedge t\|p))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** $M2$ **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus $x \in ?R$
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** lq:$?A$ **by** *simp*
from pu qup **have** $p\|u' \oplus ((\exists\, t'.\ p\|t' \wedge t'\|u') \oplus (\exists\, t'.\ q\|t' \wedge t'\|u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** $M2$ **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{**assume** $(?A \wedge \neg ?B \wedge \neg ?C)$ **then have** $?A$ **by** *simp*
with lq lp qup **have** $p = q$ **using** $M4$ **by** *auto*
thus *?thesis* **using** x e **by** *auto*}
next
{**assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
with lq lp qup **show** *?thesis* **using** x s **by** *blast*}
next
{**assume** $(\neg ?A \wedge \neg ?B \wedge ?C)$ **then have** $?C$ **by** *simp*
with pu lq lp **show** *?thesis* **using** x s **by** *blast*}
qed}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
then obtain t **where** lt:$l\|t$ **and** tq:$t\|q$ **by** *auto*
from pu qup **have** $p\|u' \oplus ((\exists\, t'.\ p\|t' \wedge t'\|u') \oplus (\exists\, t'.\ q\|t' \wedge t'\|u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** $M2$ **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{**assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
with qup lt tq lp **show** *?thesis* **using** f x **by** *blast*}
next
{**assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
then obtain t' **where** ptp:$p\|t'$ **and** $tpup$:$t'\|u'$ **by** *auto*
from pu tq **have** $p\|q \oplus ((\exists\, t''.\ p\|t'' \wedge t''\|q) \oplus (\exists\, t''.\ t\|t'' \wedge t''\|u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** $M2$ **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*

proof (*elim disjE*)
{**assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
thus *?thesis* **using** *x m* **by** *auto*}
next
{**assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
thus *?thesis* **using** *x b* **by** *auto*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
then obtain g **where** $t \| g$ **and** $g \| u$ **by** *auto*
moreover with *pu ptp* **have** $g \| t'$ **using** *M1* **by** *blast*
ultimately show *?thesis* **using** *x ov lt tq lp ptp tpup qup* **by** *blast*}
qed}
next
{**assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
then obtain t' **where** $q \| t'$ **and** $t' \| u$ **by** *auto*
with *lp lt tq pu* **show** *?thesis* **using** *d x* **by** *blast*}
qed}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then have** $?C$ **by** *simp*
then obtain t **where** $lpt{:}l' \| t$ **and** $tp{:}t \| p$ **by** *auto*
from *pu qup* **have** $p \| u' \oplus ((\exists t'.\ p \| t' \wedge t' \| u') \oplus (\exists t'.\ q \| t' \wedge t' \| u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{**assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
with *qup lpt tp lpq* **show** *?thesis* **using** *x f* **by** *blast*}
next
{**assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
with *qup lpt tp lpq* **show** *?thesis* **using** *x d* **by** *blast*}
next
{**assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then obtain** t' **where** $qt'{:}q \| t'$ **and** $tpc{:}t' \| u$ **by** *auto*
from *qup tp* **have** $q \| p \oplus ((\exists t''.\ q \| t'' \wedge t'' \| p) \oplus (\exists t''.\ t \| t'' \wedge t'' \| u'))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{**assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $?A$ **by** *simp*
thus *?thesis* **using** *x m* **by** *auto*}
next
{**assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** $?B$ **by** *simp*
thus *?thesis* **using** *x b* **by** *auto*}
next
{ **assume** $\neg ?A \wedge \neg ?B \wedge ?C$ **then obtain** g **where** $tg{:}t \| g$ **and** $g \| u'$ **by** *auto*
with *qup qt'* **have** $g \| t'$ **using** *M1* **by** *blast*

with *qt′ tpc pu lpq lpt tp tg* **show** *?thesis* **using** *x ov* **by** *blast*}
qed}
qed}
qed
qed

3.7 The rest of the composition table

Because of the symmetry $(r_1 \circ r_2)^{-1} = r_2^{-1} \circ r_1^{-1}$, the rest of the compositions is easily deduced.

lemma *cmbi*:$m\ O\ b\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup s\hat{}{-1} \cup ov\hat{}{-1} \cup d\hat{}{-1}$
using *cbmi* **by** *auto*

lemma *covmi*:$ov\ O\ m\hat{}{-1} \subseteq ov\hat{}{-1} \cup d\hat{}{-1} \cup s\hat{}{-1}$
using *cmovi* **by** *auto*

lemma *covbi*:$ov\ O\ b\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup s\hat{}{-1} \cup ov\hat{}{-1} \cup d\hat{}{-1}$
using *cbovi* **by** *auto*

lemma *cfiovi*:$f\hat{}{-1}\ O\ ov\hat{}{-1} \subseteq ov\hat{}{-1} \cup s\hat{}{-1} \cup d\hat{}{-1}$
using *covf* **by** *auto*

lemma *cfimi*:$(f\hat{}{-1}\ O\ m\hat{}{-1}) \subseteq s\hat{}{-1} \cup ov\hat{}{-1} \cup d\hat{}{-1}$
using *cmf* **by** *auto*

lemma *cfibi*:$f\hat{}{-1}\ O\ b\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1} \cup s\hat{}{-1} \cup d\hat{}{-1}$
using *cbf* **by** *auto*

lemma *cdif*:$d\hat{}{-1}\ O\ f \subseteq ov\hat{}{-1} \cup s\hat{}{-1} \cup d\hat{}{-1}$
using *cfid* **by** *auto*

lemma *cdiovi*:$d\hat{}{-1}\ O\ ov\ \hat{}{-1} \subseteq ov\hat{}{-1} \cup s\hat{}{-1} \cup d\hat{}{-1}$
using *covd* **by** *auto*

lemma *cdimi*:$d\hat{}{-1}\ O\ m\hat{}{-1} \subseteq s\hat{}{-1} \cup ov\hat{}{-1} \cup d\hat{}{-1}$
using *cmd* **by** *auto*

lemma *cdibi*:$d\hat{}{-1}\ O\ b\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1} \cup s\hat{}{-1} \cup d\hat{}{-1}$
using *cbd* **by** *auto*

lemma *csd*:$s\ O\ d \subseteq d$
using *cdisi* **by** *auto*

lemma *csf*:$s\ O\ f \subseteq d$
using *cfisi* **by** *auto*

lemma *csovi*:$s\ O\ ov\hat{}{-1} \subseteq ov\hat{}{-1} \cup f \cup d$
using *covsi* **by** *auto*

lemma *csmi*:$s\ O\ m\hat{}{-1} \subseteq m\hat{}{-1}$
using *cmsi* **by** *auto*

lemma *csbi*:$s\ O\ b\hat{}{-1} \subseteq b\hat{}{-1}$
using *cbsi* **by** *auto*

lemma *csisi*:$s\hat{}{-1}\ O\ s\hat{}{-1} \subseteq s\hat{}{-1}$
using *css* **by** *auto*

lemma *csid*:$s\hat{}{-1}\ O\ d \subseteq ov\hat{}{-1} \cup f \cup d$
using *cdis* **by** *auto*

lemma *csif*:$s\hat{}{-1}\ O\ f \subseteq ov\hat{}{-1}$
using *cfis* **by** *auto*

lemma *csiovi*:$s\hat{}{-1}\ O\ ov\hat{}{-1} \subseteq ov\hat{}{-1}$
using *covs* **by** *auto*

lemma *csimi*:$s\hat{}{-1}\ O\ m\hat{}{-1} \subseteq m\hat{}{-1}$
using *cms* **by** *auto*

lemma *csibi*:$s\hat{}{-1}\ O\ b\hat{}{-1} \subseteq b\hat{}{-1}$
using *cbs* **by** *auto*

lemma *cds*:$d\ O\ s \subseteq d$
using *csidi* **by** *auto*

lemma *cdsi*:$d\ O\ s\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1} \cup f \cup d$
using *csdi* **by** *auto*

lemma *cdd*:$d\ O\ d \subseteq d$
using *cdidi* **by** *auto*

lemma *cdf*:$d\ O\ f \subseteq d$
using *cfidi* **by** *auto*

lemma *cdovi*:$d\ O\ ov\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1} \cup f \cup d$
using *covdi* **by** *auto*

lemma *cdmi*:$d\ O\ m\hat{}{-1} \subseteq b\hat{}{-1}$
using *cmdi* **by** *auto*

lemma *cdbi*:$d\ O\ b\hat{}{-1} \subseteq b\hat{}{-1}$
using *cbdi* **by** *auto*

lemma *cfdi*:$f\ O\ d\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1} \cup s\hat{}{-1} \cup d\hat{}{-1}$
using *cdfi* **by** *auto*

lemma *cfs*:$f\ O\ s \subseteq d$
using *csifi* **by** *auto*

lemma *cfsi*:$f\ O\ s\hat{}{-1} \subseteq b\ \hat{}{-1} \cup m\hat{}{-1} \cup ov\ \hat{}{-1}$
using *csfi* **by** *auto*

lemma *cfd*:$f\ O\ d \subseteq d$
using *cdifi* **by** *auto*

lemma *cff*:$f\ O\ f \subseteq f$
using *cfifi* **by** *auto*

lemma *cfovi*:$f\ O\ ov\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1}$
using *covfi* **by** *auto*

lemma *cfmi*:$f\ O\ m\hat{}{-1} \subseteq b\hat{}{-1}$
using *cmfi* **by** *auto*

lemma *cfbi*:$f\ O\ b\hat{}{-1} \subseteq b\hat{}{-1}$
using *cbfi* **by** *auto*

lemma *covifi*:$ov\hat{}{-1}\ O\ f\hat{}{-1} \subseteq ov\hat{}{-1} \cup s\hat{}{-1} \cup d\hat{}{-1}$
using *cfov* **by** *auto*

lemma *covidi*:$ov\hat{}{-1}\ O\ d\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup s\hat{}{-1} \cup ov\hat{}{-1} \cup d\hat{}{-1}$
using *cdov* **by** *auto*

lemma *covis*:$ov\hat{}{-1}\ O\ s \subseteq ov\hat{}{-1} \cup f \cup d$
using *csiov* **by** *auto*

lemma *covisi*:$ov\hat{}{-1}\ O\ s\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1}$
using *csov* **by** *auto*

lemma *covid*:$ov\hat{}{-1}\ O\ d \subseteq ov\hat{}{-1} \cup f \cup d$
using *cdiov* **by** *auto*

lemma *covif*:$ov\hat{}{-1}\ O\ f \subseteq ov\hat{}{-1}$
using *cfiov* **by** *auto*

lemma *coviovi*:$ov\hat{}{-1}\ O\ ov\hat{}{-1} \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1}$
using *covov* **by** *auto*

lemma *covimi*:$ov\hat{}{-1}\ O\ m\hat{}{-1} \subseteq b\hat{}{-1}$
using *cmov* **by** *auto*

lemma *covibi*:$ov\hat{}{-1}\ O\ b\hat{}{-1} \subseteq b\hat{}{-1}$
using *cbov* **by** *auto*

lemma *cmiov*:$m\hat{}-1\ O\ ov \subseteq ov\hat{}-1 \cup d \cup f$
using *covim* **by** *auto*

lemma *cmifi*:$m\hat{}-1\ O\ f\hat{}-1 \subseteq m\hat{}-1$
using *cfm* **by** *auto*

lemma *cmidi*:$m\hat{}-1\ O\ d\hat{}-1 \subseteq b\hat{}-1$
using *cdm* **by** *auto*

lemma *cmis*:$m\hat{}-1\ O\ s \subseteq ov\hat{}-1 \cup d \cup f$
using *csim* **by** *auto*

lemma *cmisi*:$m\hat{}-1\ O\ s\hat{}-1 \subseteq b\hat{}-1$
using *csm* **by** *auto*

lemma *cmid*:$m\hat{}-1\ O\ d \subseteq ov\hat{}-1 \cup d \cup f$
using *cdim* **by** *auto*

lemma *cmif*:$m\hat{}-1\ O\ f \subseteq m\hat{}-1$
using *cfim* **by** *auto*

lemma *cmiovi*:$m\hat{}-1\ O\ ov\hat{}-1 \subseteq b\hat{}-1$
using *covm* **by** *auto*

lemma *cmimi*:$m\hat{}-1\ O\ m\hat{}-1 \subseteq b\hat{}-1$
using *cmm* **by** *auto*

lemma *cmibi*:$m\hat{}-1\ O\ b\hat{}-1 \subseteq b\hat{}-1$
using *cbm* **by** *auto*

lemma *cbim*:$b\hat{}-1\ O\ m \subseteq b\hat{}-1 \cup m\hat{}-1 \cup ov\hat{}-1 \cup f \cup d$
using *cmib* **by** *auto*

lemma *cbiov*:$b\hat{}-1\ O\ ov \subseteq b\hat{}-1 \cup m\hat{}-1 \cup ov\hat{}-1 \cup f \cup d$
using *covib* **by** *auto*

lemma *cbifi*:$b\hat{}-1\ O\ f\hat{}-1 \subseteq b\hat{}-1$
using *cfb* **by** *auto*

lemma *cbidi*:$b\hat{}-1\ O\ d\hat{}-1 \subseteq b\hat{}-1$
using *cdb* **by** *auto*

lemma *cbis*:$b\hat{}-1\ O\ s \subseteq b\hat{}-1 \cup m\hat{}-1 \cup ov\hat{}-1 \cup f \cup d$
using *csib* **by** *auto*

lemma *cbisi*:$b\hat{}-1\ O\ s\hat{}-1 \subseteq b\hat{}-1$
using *csb* **by** *auto*

lemma *cbid*:$b\hat{}-1\ O\ d \subseteq b\hat{}-1 \cup m\hat{}-1 \cup ov\hat{}-1 \cup f \cup d$

using *cdib* **by** *auto*

lemma *cbif*:$b^\frown{-1}$ *O f* $\subseteq$ $b^\frown{-1}$
using *cfib* **by** *auto*

lemma *cbiovi*:$b^\frown{-1}$ *O* $ov^\frown{-1}$ $\subseteq$ $b^\frown{-1}$
using *covb* **by** *auto*

lemma *cbimi*:$b^\frown{-1}$ *O* $m^\frown{-1}$ $\subseteq$ $b^\frown{-1}$
using *cmb* **by** *auto*

lemma *cbibi*:$b^\frown{-1}$ *O* $b^\frown{-1}$ $\subseteq$ $b^\frown{-1}$
using *cbb* **by** *auto*

3.8 Composition rules

named-theorems *ce-rules* **declare** *cem*[*ce-rules*] **and** *ceb*[*ce-rules*] **and** *ceov*[*ce-rules*] **and** *ces*[*ce-rules*] **and** *cef*[*ce-rules*] **and** *ced*[*ce-rules*] **and** *cemi*[*ce-rules*] **and** *cebi*[*ce-rules*] **and** *ceovi*[*ce-rules*] **and** *cesi*[*ce-rules*] **and** *cefi*[*ce-rules*] **and** *cedi*[*ce-rules*]

named-theorems *cm-rules* **declare** *cme*[*cm-rules*] **and** *cmb*[*cm-rules*] **and** *cmm*[*cm-rules*] **and** *cmov*[*cm-rules*] **and** *cms* [*cm-rules*] **and** *cmd*[*cm-rules*] **and** *cmf*[*cm-rules*] **and** *cmbi*[*cm-rules*] **and** *cmmi*[*cm-rules*] **and** *cmovi*[*cm-rules*] **and** *cmsi*[*cm-rules*] **and** *cmdi*[*cm-rules*] **and** *cmfi*[*cm-rules*]

named-theorems *cb-rules* **declare** *cbe*[*cb-rules*] **and** *cbm*[*cb-rules*] **and** *cbb*[*cb-rules*] **and** *cbov*[*cb-rules*] **and** *cbs* [*cb-rules*] **and** *cbd*[*cb-rules*] **and** *cbf*[*cb-rules*] **and** *cbbi*[*cb-rules*] **and** *cbbi*[*cb-rules*] **and** *cbovi*[*cb-rules*] **and** *cbsi*[*cb-rules*] **and** *cbdi*[*cb-rules*] **and** *cbfi*[*cb-rules*]

named-theorems *cov-rules* **declare** *cove*[*cov-rules*] **and** *covb*[*cov-rules*] **and** *covb*[*cov-rules*] **and** *covov*[*cov-rules*] **and** *covs* [*cov-rules*] **and** *covd*[*cov-rules*] **and** *covf*[*cov-rules*] **and** *covbi*[*cov-rules*] **and** *covbi*[*cov-rules*] **and** *covovi*[*cov-rules*] **and** *covsi*[*cov-rules*] **and** *covdi*[*cov-rules*] **and** *covfi*[*cov-rules*]

named-theorems *cs-rules* **declare** *cse*[*cs-rules*] **and** *csb*[*cs-rules*] **and** *csb*[*cs-rules*] **and** *csov*[*cs-rules*] **and** *css* [*cs-rules*] **and** *csd*[*cs-rules*] **and** *csf*[*cs-rules*] **and** *csbi*[*cs-rules*] **and** *csbi*[*cs-rules*] **and** *csovi*[*cs-rules*] **and** *cssi*[*cs-rules*] **and** *csdi*[*cs-rules*] **and** *csfi*[*cs-rules*]

named-theorems *cf-rules* **declare** *cfe*[*cf-rules*] **and** *cfb*[*cf-rules*] **and** *cfb*[*cf-rules*] **and** *cfov*[*cf-rules*] **and** *cfs* [*cf-rules*] **and** *cfd*[*cf-rules*] **and** *cff*[*cf-rules*] **and** *cfbi*[*cf-rules*] **and** *cfbi*[*cf-rules*] **and** *cfovi*[*cf-rules*] **and** *cfsi*[*cf-rules*] **and** *cfdi*[*cf-rules*] **and** *cffi*[*cf-rules*]

named-theorems *cd-rules* **declare** *cde*[*cd-rules*] **and** *cdb*[*cd-rules*] **and** *cdb*[*cd-rules*]

and *cdov*[*cd-rules*] **and** *cds* [*cd-rules*] **and** *cdd*[*cd-rules*] **and** *cdf*[*cd-rules*] **and** *cdbi*[*cd-rules*] **and** *cdbi*[*cd-rules*] **and** *cdovi*[*cd-rules*] **and** *cdsi*[*cd-rules*] **and** *cddi*[*cd-rules*] **and** *cdfi*[*cd-rules*]

named-theorems *cmi-rules* **declare** *cmie*[*cmi-rules*] **and** *cmib*[*cmi-rules*] **and** *cmib*[*cmi-rules*] **and** *cmiov*[*cmi-rules*] **and** *cmis* [*cmi-rules*] **and** *cmid*[*cmi-rules*] **and** *cmif*[*cmi-rules*] **and**
cmibi[*cmi-rules*] **and** *cmibi*[*cmi-rules*] **and** *cmiovi*[*cmi-rules*] **and** *cmisi*[*cmi-rules*] **and** *cmidi*[*cmi-rules*] **and** *cmifi*[*cmi-rules*]

named-theorems *cbi-rules* **declare** *cbie*[*cbi-rules*] **and** *cbim*[*cbi-rules*] **and** *cbib*[*cbi-rules*] **and** *cbiov*[*cbi-rules*] **and** *cbis* [*cbi-rules*] **and** *cbid*[*cbi-rules*] **and** *cbif*[*cbi-rules*] **and** *cbimi*[*cbi-rules*] **and** *cbibi*[*cbi-rules*] **and** *cbiovi*[*cbi-rules*] **and** *cbisi*[*cbi-rules*] **and** *cbidi*[*cbi-rules*] **and** *cbifi*[*cbi-rules*]

named-theorems *covi-rules* **declare** *covie*[*covi-rules*] **and** *covib*[*covi-rules*] **and** *covib*[*covi-rules*] **and** *coviov*[*covi-rules*] **and** *covis* [*covi-rules*] **and** *covid*[*covi-rules*] **and** *covif*[*covi-rules*] **and**
covibi[*covi-rules*] **and** *covibi*[*covi-rules*] **and** *coviovi*[*covi-rules*] **and** *covisi*[*covi-rules*] **and** *covidi*[*covi-rules*] **and** *covifi*[*covi-rules*]

named-theorems *csi-rules* **declare** *csie*[*csi-rules*] **and** *csib*[*csi-rules*] **and** *csib*[*csi-rules*] **and** *csiov*[*csi-rules*] **and** *csis* [*csi-rules*] **and** *csid*[*csi-rules*] **and** *csif*[*csi-rules*] **and** *csibi*[*csi-rules*] **and** *csibi*[*csi-rules*] **and** *csiovi*[*csi-rules*] **and** *csisi*[*csi-rules*] **and** *csidi*[*csi-rules*] **and** *csifi*[*csi-rules*]

named-theorems *cfi-rules* **declare** *cfie*[*cfi-rules*] **and** *cfib*[*cfi-rules*] **and** *cfib*[*cfi-rules*] **and** *cfiov*[*cfi-rules*] **and** *cfis* [*cfi-rules*] **and** *cfid*[*cfi-rules*] **and** *cfif*[*cfi-rules*] **and** *cfibi*[*cfi-rules*] **and** *cfibi*[*cfi-rules*] **and** *cfiovi*[*cfi-rules*] **and** *cfisi*[*cfi-rules*] **and** *cfidi*[*cfi-rules*] **and** *cfifi*[*cfi-rules*]

named-theorems *cdi-rules* **declare** *cdie*[*cdi-rules*] **and** *cdib*[*cdi-rules*] **and** *cdib*[*cdi-rules*] **and** *cdiov*[*cdi-rules*] **and** *cdis* [*cdi-rules*] **and** *cdid*[*cdi-rules*] **and** *cdif*[*cdi-rules*] **and** *cdibi*[*cdi-rules*] **and** *cdibi*[*cdi-rules*] **and** *cdiovi*[*cdi-rules*] **and** *cdisi*[*cdi-rules*] **and** *cdidi*[*cdi-rules*] **and** *cdifi*[*cdi-rules*]

named-theorems *cre-rules* **declare** *cee*[*cre-rules*] **and** *cme*[*cre-rules*] **and** *cbe*[*cre-rules*] **and** *cove*[*cre-rules*] **and** *cse*[*cre-rules*] **and** *cfe*[*cre-rules*] **and** *cde*[*cre-rules*] **and** *cmie*[*cre-rules*] **and** *cbie*[*cre-rules*] **and** *covie*[*cre-rules*] **and** *csie*[*cre-rules*] **and** *cfie*[*cre-rules*] **and** *cdie*[*cre-rules*]

named-theorems *crm-rules* **declare** *cem*[*crm-rules*] **and** *cbm*[*crm-rules*] **and** *cmm*[*crm-rules*] **and** *covm*[*crm-rules*] **and** *csm*[*crm-rules*] **and** *cfm*[*crm-rules*] **and** *cdm*[*crm-rules*] **and**
cmim[*crm-rules*] **and** *cbim*[*crm-rules*] **and** *covim*[*crm-rules*] **and** *csim*[*crm-rules*] **and** *cfim*[*crm-rules*] **and** *cdim*[*crm-rules*]

named-theorems *crmi-rules* **declare** *cemi*[*crmi-rules*] **and** *cbmi*[*crmi-rules*] **and** *cmmi*[*crmi-rules*] **and** *covmi*[*crmi-rules*] **and** *csmi*[*crmi-rules*] **and** *cfmi*[*crmi-rules*]

and *cdmi*[*crmi-rules*] **and**
cmimi[*crmi-rules*] **and** *cbimi*[*crmi-rules*] **and** *covimi*[*crmi-rules*] **and** *csimi*[*crmi-rules*] **and** *cfimi*[*crmi-rules*] **and** *cdimi*[*crmi-rules*]

named-theorems *crs-rules* **declare** *ces*[*crs-rules*] **and** *cbs*[*crs-rules*] **and** *cms*[*crs-rules*] **and** *covs*[*crs-rules*] **and** *css*[*crs-rules*] **and** *cfs*[*crs-rules*] **and** *cds*[*crs-rules*] **and** *cmis*[*crs-rules*] **and** *cbis*[*crs-rules*] **and** *covis*[*crs-rules*] **and** *csis*[*crs-rules*] **and** *cfis*[*crs-rules*] **and** *cdis*[*crs-rules*]

named-theorems *crsi-rules* **declare** *cesi*[*crsi-rules*] **and** *cbsi*[*crsi-rules*] **and** *cmsi*[*crsi-rules*] **and** *covsi*[*crsi-rules*] **and** *cssi*[*crsi-rules*] **and** *cfsi*[*crsi-rules*] **and** *cdsi*[*crsi-rules*]
and
cmisi[*crsi-rules*] **and** *cbisi*[*crsi-rules*] **and** *covisi*[*crsi-rules*] **and** *csisi*[*crsi-rules*] **and** *cfisi*[*crsi-rules*] **and** *cdisi*[*crsi-rules*]

named-theorems *crb-rules* **declare** *ceb*[*crb-rules*] **and** *cbb*[*crb-rules*] **and** *cmb*[*crb-rules*] **and** *covb*[*crb-rules*] **and** *csb*[*crb-rules*] **and** *cfb*[*crb-rules*] **and** *cdb*[*crb-rules*] **and** *cmib*[*crb-rules*] **and** *cbib*[*crb-rules*] **and** *covib*[*crb-rules*] **and** *csib*[*crb-rules*] **and** *cfib*[*crb-rules*] **and** *cdib*[*crb-rules*]

named-theorems *crbi-rules* **declare** *cebi*[*crbi-rules*] **and** *cbbi*[*crbi-rules*] **and** *cmbi*[*crbi-rules*] **and** *covbi*[*crbi-rules*] **and** *csbi*[*crbi-rules*] **and** *cfbi*[*crbi-rules*] **and** *cdbi*[*crbi-rules*]
and
cmibi[*crbi-rules*] **and** *cbibi*[*crbi-rules*] **and** *covibi*[*crbi-rules*] **and** *csibi*[*crbi-rules*] **and** *cfibi*[*crbi-rules*] **and** *cdibi*[*crbi-rules*]

named-theorems *crov-rules* **declare** *ceov*[*crov-rules*] **and** *cbov*[*crov-rules*] **and** *cmov*[*crov-rules*] **and** *covov*[*crov-rules*] **and** *csov*[*crov-rules*] **and** *cfov*[*crov-rules*] **and** *cdov*[*crov-rules*] **and**
cmiov[*crov-rules*] **and** *cbiov*[*crov-rules*] **and** *coviov*[*crov-rules*] **and** *csiov*[*crov-rules*] **and** *cfiov*[*crov-rules*] **and** *cdiov*[*crov-rules*]

named-theorems *crovi-rules* **declare** *ceovi*[*crovi-rules*] **and** *cbovi*[*crovi-rules*] **and** *cmovi*[*crovi-rules*] **and** *covovi*[*crovi-rules*] **and** *csovi*[*crovi-rules*] **and** *cfovi*[*crovi-rules*] **and** *cdovi*[*crovi-rules*] **and**
cmiovi[*crovi-rules*] **and** *cbiovi*[*crovi-rules*] **and** *coviovi*[*crovi-rules*] **and** *csiovi*[*crovi-rules*] **and** *cfiovi*[*crovi-rules*] **and** *cdiovi*[*crovi-rules*]

named-theorems *crf-rules* **declare** *cef*[*crf-rules*] **and** *cbf*[*crf-rules*] **and** *cmf*[*crf-rules*] **and** *covf*[*crf-rules*] **and** *csf*[*crf-rules*] **and** *cff*[*crf-rules*] **and** *cdf*[*crf-rules*] **and** *cmif*[*crf-rules*] **and** *cbif*[*crf-rules*] **and** *covif*[*crf-rules*] **and** *csif*[*crf-rules*] **and** *cfif*[*crf-rules*] **and** *cdif*[*crf-rules*]

named-theorems *crfi-rules* **declare** *cefi*[*crfi-rules*] **and** *cbfi*[*crfi-rules*] **and** *cmfi*[*crfi-rules*] **and** *covfi*[*crfi-rules*] **and** *csfi*[*crfi-rules*] **and** *cffi*[*crfi-rules*] **and** *cdfi*[*crfi-rules*] **and**

cmifi[*crfi-rules*] **and** *cbifi*[*crfi-rules*] **and** *covifi*[*crfi-rules*] **and** *csifi*[*crfi-rules*] **and** *cfifi*[*crfi-rules*] **and** *cdifi*[*crfi-rules*]

named-theorems *crd-rules* **declare** *ced*[*crd-rules*] **and** *cbd*[*crd-rules*] **and** *cmd*[*crd-rules*] **and** *covd*[*crd-rules*] **and** *csd*[*crd-rules*] **and** *cfd*[*crd-rules*] **and** *cdd*[*crd-rules*] **and** *cmid*[*crd-rules*] **and** *cbid*[*crd-rules*] **and** *covid*[*crd-rules*] **and** *csid*[*crd-rules*] **and** *cfid*[*crd-rules*] **and** *cdid*[*crd-rules*]

named-theorems *crdi-rules* **declare** *cedi*[*crdi-rules*] **and** *cbdi*[*crdi-rules*] **and** *cmdi*[*crdi-rules*] **and** *covdi*[*crdi-rules*] **and** *csdi*[*crdi-rules*] **and** *cfdi*[*crdi-rules*] **and** *cddi*[*crdi-rules*] **and**
cmidi[*crdi-rules*] **and** *cbidi*[*crdi-rules*] **and** *covidi*[*crdi-rules*] **and** *csidi*[*crdi-rules*] **and** *cfidi*[*crdi-rules*] **and** *cdidi*[*crdi-rules*]

end

theory *disjoint-relations*

imports
allen

begin

4 PD property

The 13 time interval relations (i.e. e, b, m, s, f, d, ov and their inverse relations) are pairwise disjoint.

lemma *em* :$e \cap m = \{\}$
using *e m meets-irrefl*
by (*metis ComplI disjoint-eq-subset-Compl meets-wd subrelI*)

lemma *eb* :$e \cap b = \{\}$
using *b e meets-asym*
by (*metis ComplI disjoint-eq-subset-Compl subrelI*)

lemma *eov* :$e \cap ov = \{\}$
apply (*auto simp*: *e ov*)
using *elimmeets* **by** *blast*

lemma *es* :$e \cap s = \{\}$
apply (*auto simp*:*e s*)
using *elimmeets* **by** *blast*

lemma *ef* :$e \cap f = \{\}$
using *e f* **by** (*metis* (*no-types*, *lifting*) *ComplI disjoint-eq-subset-Compl meets-atrans subrelI*)

lemma *ed* :$e \cap d = \{\}$
using *e d* **by** (*metis* (*no-types*, *lifting*) *ComplI disjoint-eq-subset-Compl meets-atrans subrelI*)

lemma *emi* :$e \cap m\hat{}{-1} = \{\}$
using *converseE em e*
by (*metis disjoint-iff-not-equal*)

lemma *ebi* :$e \cap b\hat{}{-1} = \{\}$
using *converseE eb e*
by (*metis disjoint-iff-not-equal*)

lemma *eovi* :$e \cap ov\hat{}{-1} = \{\}$
using *converseE eov e*
by (*metis disjoint-iff-not-equal*)

lemma *esi* :$e \cap s\hat{}{-1} = \{\}$
using *converseE es e*
by (*metis disjoint-iff-not-equal*)

lemma *efi* :$e \cap f\hat{}{-1} = \{\}$
using *converseE ef e*
by (*metis disjoint-iff-not-equal*)

lemma *edi* :$e \cap d\hat{}{-1} = \{\}$
using *converseE ed e*
by (*metis disjoint-iff-not-equal*)

lemma *mb* :$m \cap b = \{\}$
using *m b*
apply *auto*
using *elimmeets* **by** *blast*

lemma *mov* : $m \cap ov = \{\}$
apply (*auto simp*:*m ov*)
by (*meson M1 elimmeets*)

lemma *ms* :$m \cap s = \{\}$
apply (*auto simp*:*m s*)
by (*meson M1 elimmeets*)

lemma *mf* :$m \cap f = \{\}$
apply (*auto simp*:*m f*)
using *elimmeets* **by** *blast*

lemma *md* :$m \cap d = \{\}$
apply (*auto simp*: *m d*)

using *trans2* **by** *blast*

lemma *mi* :$m \cap m\hat{}{-}1 = \{\}$
apply (*auto simp*:*m*)
using *converseE m meets-asym* **by** *blast*

lemma *mbi* :$m \cap b\hat{}{-}1 = \{\}$
apply (*auto simp*:*mb*)
apply (*auto simp*: *m b*)
using *nontrans2* **by** *blast*

lemma *movi* :$m \cap ov\hat{}{-}1 = \{\}$
using *m ov*
apply *auto*
using *trans2* **by** *blast*

lemma *msi* :$m \cap s\hat{}{-}1 = \{\}$
apply (*auto simp*:*m s*)
by (*meson M1 elimmeets*)

lemma *mfi* :$m \cap f\hat{}{-}1 = \{\}$
apply (*auto simp*:*m f*)
by (*meson M1 elimmeets*)

lemma *mdi* :$m \cap d\hat{}{-}1 = \{\}$
apply (*auto simp*:*m d*)
using *trans2* **by** *blast*

lemma *bov* :$b \cap ov = \{\}$
apply (*auto simp*:*b ov*)
by (*meson M1 trans2*)

lemma *bs* :$b \cap s = \{\}$
apply (*auto simp*:*b s*)
by (*meson M1 trans2*)

lemma *bf* :$b \cap f = \{\}$
apply (*auto simp*: *b f*)
by (*meson M1 trans2*)

lemma *bd* :$b \cap d = \{\}$
apply (*auto simp*:*b d*)
by (*meson M1 nonmeets4*)

lemma *bmi* :$b \cap m\hat{}{-}1 = \{\}$
using *mbi* **by** *auto*

lemma *bi* :$b \cap b\hat{}{-}1 = \{\}$

apply (*auto simp*:*b*)
using *M5exist-var3 trans2* **by** *blast*

lemma *bovi* :*b* ∩ *ov*^−*1* = {}
apply (*auto simp*:*bov*)
apply (*auto simp*:*b ov*)
by (*meson M1 nontrans2*)

lemma *bsi* :*b* ∩ *s*^−*1* = {}
using *bs* **apply** *auto* **using** *b s* **apply** *auto*
using *trans2* **by** *blast*

lemma *bfi* :*b* ∩ *f*^−*1* = {}
using *bf* **apply** *auto* **using** *b f* **apply** *auto*
using *trans2* **by** *blast*

lemma *bdi* :*b* ∩ *d*^−*1* = {}
apply (*auto simp*:*bd*)
apply (*auto simp*:*b d*)
using *trans2*
using *M1 nonmeets4* **by** *blast*

lemma *ovs* :*ov* ∩ *s* = {}
apply (*auto simp*:*ov s*)
by (*meson M1 meets-atrans*)

lemma *ovf* :*ov* ∩ *f* = {}
apply (*auto simp*:*ov f*)
by (*meson M1 meets-atrans*)

lemma *ovd* :*ov*∩ *d* = {}
apply (*auto simp*:*ov d*)
by (*meson M1 trans2*)

lemma *ovmi* :*ov* ∩ *m*^−*1* = {}
using *movi* **by** *auto*

lemma *ovbi* :*ov* ∩ *b*^−*1* = {}
using *bovi* **by** *blast*

lemma *ovi* :*ov* ∩ *ov*^−*1* = {}
apply (*auto simp*:*ov*)
by (*meson M1 trans2*)

lemma *ovsi* :*ov* ∩ *s*^−*1* = {}
apply (*auto simp*:*ov s*)
by (*meson M1 elimmeets*)

lemma *ovfi* :$ov \cap f$^−*1* = {}
apply (*auto simp*:*ov f*)
by (*meson M1 elimmeets*)

lemma *ovdi* :$ov \cap d$^−*1* = {}
apply (*auto simp*:*ov d*)
by (*meson M1 trans2*)

lemma *sf* :$s \cap f$ = {}
apply (*auto simp*:*s f*)
by (*metis M4 elimmeets*)

lemma *sd* :$s \cap d$ = {}
apply (*auto simp*:*s d*)
by (*metis M1 meets-atrans*)

lemma *smi* :$s \cap m$^−*1* = {}
using *msi* **by** *auto*

lemma *sbi* :$s \cap b$^−*1* = {}
using *bsi* **by** *blast*

lemma *sovi* :$s \cap ov$^−*1* = {}
using *ovsi* **by** *auto*

lemma *si* :$s \cap s$^−*1* = {}
apply (*auto simp*:*s*)
by (*meson M1 trans2*)

lemma *sfi* :$s \cap f$^−*1* = {}
apply (*auto simp*:*s f*)
by (*metis M4 elimmeets*)

lemma *sdi* :$s \cap d$^−*1* = {}
apply (*auto simp*:*s d*)
by (*meson M1 meets-atrans*)

lemma *fd* :$f \cap d$ = {}
apply (*auto simp*:*f d*)
by (*meson M1 meets-atrans*)

lemma *fmi* :$f \cap m$^−*1* = {}
using *mfi* **by** *auto*

lemma *fbi* :$f \cap b$^−*1* = {}

using *bfi converse-Int* **by** *auto*

lemma *fovi* :$f \cap ov\widehat{\ }{-1} = \{\}$
  using *ovfi* **by** *auto*

lemma *fsi* :$f \cap s\widehat{\ }{-1} = \{\}$
using *sfi* **by** *auto*

lemma *fi* :$f \cap f\widehat{\ }{-1} = \{\}$
apply (*auto simp*:*f*)
by (*meson M1 trans2*)

lemma *fdi* :$f \cap d\widehat{\ }{-1} = \{\}$
apply (*auto simp*:*f d*)
by (*meson M1 trans2*)

lemma *dmi* :$d \cap m\widehat{\ }{-1} = \{\}$
using *mdi* **by** *auto*

lemma *dbi* :$d \cap b\widehat{\ }{-1} = \{\}$
using *bdi* **by** *blast*

lemma *dovi* :$d \cap ov\widehat{\ }{-1} = \{\}$
using *ovdi* **by** *auto*

lemma *dsi* :$d \cap s\widehat{\ }{-1} = \{\}$
using *sdi* **by** *auto*

lemma *dfi* :$d \cap f\widehat{\ }{-1} = \{\}$
apply (*auto simp*:*d f*)
by (*meson M1 trans2*)

lemma *di* :$d \cap d\widehat{\ }{-1} = \{\}$
apply (*auto simp*:*d*)
by (*meson M1 trans2*)

lemma *mibi* :$m\widehat{\ }{-1} \cap b\widehat{\ }{-1} = \{\}$
using *mb* **by** *auto*

lemma *miovi* :$m\widehat{\ }{-1} \cap ov\widehat{\ }{-1} = \{\}$
using *mov* **by** *auto*

lemma *misi* :$m\widehat{\ }{-1} \cap s\widehat{\ }{-1} = \{\}$
using *ms* **by** *auto*

lemma *mifi* :$m\widehat{\ }{-1} \cap f\widehat{\ }{-1} = \{\}$

using *mf* **by** *auto*

lemma *midi* :$m\widehat{~}{-1} \cap d\widehat{~}{-1} = \{\}$
using *md* **by** *auto*

lemma *bid* :$b\widehat{~}{-1} \cap d = \{\}$
by (*simp add*: *dbi inf-sup-aci*(*1*))

lemma *bimi* :$b\widehat{~}{-1} \cap m\widehat{~}{-1} = \{\}$
using *mibi* **by** *auto*

lemma *biovi* :$b\widehat{~}{-1} \cap ov\widehat{~}{-1} = \{\}$
using *bov* **by** *blast*

lemma *bisi* :$b\widehat{~}{-1} \cap s\widehat{~}{-1} = \{\}$
using *bs* **by** *blast*

lemma *bifi* :$b\widehat{~}{-1} \cap f\widehat{~}{-1} = \{\}$
using *bf* **by** *blast*

lemma *bidi* :$b\widehat{~}{-1} \cap d\widehat{~}{-1} = \{\}$
using *bd* **by** *blast*

lemma *ovisi* :$ov\widehat{~}{-1} \cap s\widehat{~}{-1} = \{\}$
using *ovs* **by** *blast*

lemma *ovifi* :$ov\widehat{~}{-1} \cap f\widehat{~}{-1} = \{\}$
using *ovf* **by** *blast*

lemma *ovidi* :$ov\widehat{~}{-1} \cap d\widehat{~}{-1} = \{\}$
using *ovd* **by** *blast*

lemma *sifi* :$s\widehat{~}{-1} \cap f\widehat{~}{-1} = \{\}$
using *sf* **by** *blast*

lemma *sidi* :$s\widehat{~}{-1} \cap d\widehat{~}{-1} = \{\}$
using *sd* **by** *blast*

lemma *fidi* :$f\widehat{~}{-1} \cap d\widehat{~}{-1} = \{\}$
using *fd* **by** *blast*

lemma *eei*[*simp*]:$e\widehat{~}{-1} = e$
using *e*

by (*metis converse-iff subrelI subset-antisym*)

lemma *rdisj-sym*:$A \cap B = \{\} \Longrightarrow B \cap A = \{\}$
by *auto*

4.1 Intersection rules

named-theorems *e-rules* **declare** *em*[*e-rules*] **and** *eb*[*e-rules*] **and** *eov*[*e-rules*] **and** *es*[*e-rules*] **and** *ef*[*e-rules*] **and** *ed*[*e-rules*] **and** *emi*[*e-rules*] **and** *ebi*[*e-rules*] **and** *eovi*[*e-rules*]
and *esi*[*e-rules*] **and** *efi*[*e-rules*] **and** *edi*[*e-rules*]

named-theorems *m-rules* **declare** *em*[*THEN rdisj-sym*, *m-rules*] **and** *mb* [*m-rules*] **and** *ms* [*m-rules*] **and** *mov* [*m-rules*] **and** *mf*[*m-rules*] **and**
md[*m-rules*] **and** *mi* [*m-rules*] **and** *mbi* [*m-rules*] **and** *movi* [*m-rules*] **and** *msi* [*m-rules*] **and** *mfi* [*m-rules*] **and** *mdi* [*m-rules*] **and** *emi*[*m-rules*]

named-theorems *b-rules* **declare** *eb*[*THEN rdisj-sym*, *b-rules*] **and** *mb* [*THEN rdisj-sym*, *b-rules*] **and** *bs* [*b-rules*] **and** *bov* [*b-rules*] **and** *bf*[*b-rules*] **and**
bd[*b-rules*] **and** *bmi* [*b-rules*] **and** *bi* [*b-rules*] **and** *bovi* [*b-rules*] **and** *bsi* [*b-rules*] **and** *bfi* [*b-rules*] **and** *bdi* [*b-rules*] **and** *ebi*[*b-rules*]

named-theorems *ov-rules* **declare** *eov*[*THEN rdisj-sym*, *ov-rules*] **and** *mov* [*THEN rdisj-sym*, *ov-rules*] **and** *ovs* [*ov-rules*] **and** *bov* [*THEN rdisj-sym*,*ov-rules*] **and** *ovf*[*ov-rules*] **and**
ovd[*ov-rules*] **and** *ovmi* [*ov-rules*] **and** *ovi* [*ov-rules*] **and** *ovsi* [*ov-rules*] **and** *ovfi* [*ov-rules*] **and** *ovdi* [*ov-rules*] **and** *eovi*[*ov-rules*]

named-theorems *s-rules* **declare** *es*[*THEN rdisj-sym*, *s-rules*] **and** *ms* [*THEN rdisj-sym*, *s-rules*] **and** *ovs* [*THEN rdisj-sym*, *s-rules*] **and** *bs* [*THEN rdisj-sym*,*s-rules*] **and** *sf*[*s-rules*] **and**
sd[*s-rules*] **and** *smi* [*s-rules*] **and** *sovi* [*s-rules*] **and** *si* [*s-rules*] **and** *sfi* [*s-rules*] **and** *sdi* [*s-rules*]

named-theorems *d-rules* **declare** *ed*[*THEN rdisj-sym*, *d-rules*] **and** *md* [*THEN rdisj-sym*, *d-rules*] **and** *sd* [*THEN rdisj-sym*, *d-rules*] **and** *fd*[*THEN rdisj-sym*, *d-rules*] **and**
ovd[*THEN rdisj-sym*,*d-rules*] **and** *dmi* [*d-rules*] **and** *dovi* [*d-rules*] **and** *dsi* [*d-rules*] **and** *dfi* [*d-rules*] **and** *di* [*d-rules*]

named-theorems *f-rules* **declare** *ef*[*THEN rdisj-sym*, *f-rules*] **and** *mf* [*THEN rdisj-sym*, *f-rules*] **and** *sf* [*THEN rdisj-sym*, *f-rules*] **and** *ovf* [*THEN rdisj-sym*,*f-rules*] **and** *fd*[*f-rules*] **and**
fmi [*f-rules*] **and** *fovi* [*f-rules*] **and** *fsi* [*f-rules*] **and** *fi* [*f-rules*] **and** *fdi* [*f-rules*]

end

theory *jointly-exhaustive*

imports
allen

begin

5 JE property

The 13 time interval relations are jointly exhaustive. For any two intervals x and y, we can find a basic relation r such that $(x, y) \in r$.

lemma (**in** *arelations*) *jointly-exhaustive*:
assumes $\mathcal{I}\ p\ \mathcal{I}\ q$
shows $(p{::}'a, q{::}'a) \in b \vee (p,q) \in m \vee (p,q) \in ov \vee (p,q) \in s \vee (p,q) \in d \vee (p,q) \in f\hat{\ }{-1} \vee (p,q) \in e \vee$
$(p,q) \in f \vee (p,q) \in s\hat{\ }{-1} \vee (p,q) \in d\hat{\ }{-1} \vee (p,q) \in ov\hat{\ }{-1} \vee (p,q) \in m\hat{\ }{-1} \vee (p,q) \in b\hat{\ }{-1}$ (**is** *?R*)
proof −
obtain $k\ k'\ u\ u'{::}'a$ **where** $kp{:}k\|p$ **and** $kpq{:}k'\|q$ **and** $pu{:}p\|u$ **and** $qup{:}q\|u'$ **using** *M3 meets-wd assms*(*1*,*2*) **by** *fastforce*
from *kp kpq* **have** $k\|q \oplus ((\exists\, t.\ k\|t \wedge t\|q) \oplus (\exists\, t.\ k'\|t \wedge t\|p))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{ **assume** $?A \wedge \neg ?B \wedge \neg ?C$ **then have** $kq{:}?A$ **by** *simp*
from *pu qup* **have** $p\|u' \oplus ((\exists\, t'{::}'a.\ p\|t' \wedge t'\|u') \oplus (\exists\, t'.\ q\|t' \wedge t'\|u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{**assume** $(?A \wedge \neg ?B \wedge \neg ?C)$ **then have** *?A* **by** *simp*
with *kp kq qup* **have** $p = q$ **using** *M4* **by** *auto*
thus *?thesis* **using** *e* **by** *auto*}
next
{**assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** *?B* **by** *simp*
with *kq kp qup* **show** *?thesis* **using** *s* **by** *blast*}
next
{**assume** $(\neg ?A \wedge \neg ?B \wedge ?C)$ **then have** *?C* **by** *simp*
then obtain t' **where** $q\|t'$ **and** $t'\|u$ **by** *blast*
with *kq kp pu* **show** *?thesis* **using** *s* **by** *blast* }
qed}
next
{ **assume** $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** *?B* **by** *simp*

then obtain t **where** $kt{:}k\|t$ **and** $tq{:}t\|q$ **by** *auto*
from *pu qup* **have** $p\|u' \oplus ((\exists\, t'.\ p\|t' \wedge t'\|u') \oplus (\exists\, t'.\ q\|t' \wedge t'\|u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A\wedge\neg ?B\wedge\neg ?C) \vee ((\neg ?A\wedge ?B\wedge\neg ?C) \vee (\neg ?A\wedge\neg ?B\wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{**assume** $?A\wedge\neg ?B\wedge\neg ?C$ **then have** $?A$ **by** *simp*
with *kp qup kt tq* **show** *?thesis* **using** f **by** *blast*}
next
{**assume** $\neg ?A\wedge ?B\wedge\neg ?C$ **then have** $?B$ **by** *simp*
then obtain t' **where** $ptp{:}p\|t'$ **and** $tpup{:}t'\|u'$ **by** *auto*
from *pu tq* **have** $p\|q \oplus ((\exists\, t''.\ p\|t'' \wedge t''\|q) \oplus (\exists\, t''.\ t\|t'' \wedge t''\|u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A\wedge\neg ?B\wedge\neg ?C) \vee ((\neg ?A\wedge ?B\wedge\neg ?C) \vee (\neg ?A\wedge\neg ?B\wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{**assume** $?A\wedge\neg ?B\wedge\neg ?C$ **then have** $?A$ **by** *simp*
thus *?thesis* **using** m **by** *auto*}
next
{**assume** $\neg ?A\wedge ?B\wedge\neg ?C$ **then have** $?B$ **by** *simp*
thus *?thesis* **using** b **by** *auto*}
next
{ **assume** $\neg ?A\wedge\neg ?B\wedge ?C$ **then have** $?C$ **by** *simp*
then obtain g **where** $t\|g$ **and** $g\|u$ **by** *auto*
moreover with *pu ptp* **have** $g\|t'$ **using** *M1* **by** *blast*
ultimately show *?thesis* **using** *ov kt tq kp ptp tpup qup* **by** *blast*}
qed}
next
{**assume** $\neg ?A\wedge\neg ?B\wedge ?C$ **then have** $?C$ **by** *simp*
then obtain t' **where** $q\|t'$ **and** $t'\|u$ **by** *auto*
with *kp kt tq pu* **show** *?thesis* **using** d **by** *blast*}
qed}
next
{ **assume** $\neg ?A\wedge\neg ?B\wedge ?C$ **then have** $?C$ **by** *simp*
then obtain t **where** $kpt{:}k'\|t$ **and** $tp{:}t\|p$ **by** *auto*
from *pu qup* **have** $p\|u' \oplus ((\exists\, t'.\ p\|t' \wedge t'\|u') \oplus (\exists\, t'.\ q\|t' \wedge t'\|u))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A\wedge\neg ?B\wedge\neg ?C) \vee ((\neg ?A\wedge ?B\wedge\neg ?C) \vee (\neg ?A\wedge\neg ?B\wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{**assume** $?A\wedge\neg ?B\wedge\neg ?C$ **then have** $?A$ **by** *simp*
with *qup kpt tp kpq* **show** *?thesis* **using** f **by** *blast*}
next
{**assume** $\neg ?A\wedge ?B\wedge\neg ?C$ **then have** $?B$ **by** *simp*
with *qup kpt tp kpq* **show** *?thesis* **using** d **by** *blast*}
next

{assume $\neg ?A \wedge \neg ?B \wedge ?C$ **then obtain** t' **where** qt':$q \| t'$ **and** tpc:$t' \| u$ **by** *auto*
from *qup tp* **have** $q \| p \oplus ((\exists\, t''.\ q \| t'' \wedge t'' \| p) \oplus (\exists\, t''.\ t \| t'' \wedge t'' \| u'))$ (**is** $?A \oplus (?B \oplus ?C)$) **using** *M2* **by** *blast*
then have $(?A \wedge \neg ?B \wedge \neg ?C) \vee ((\neg ?A \wedge ?B \wedge \neg ?C) \vee (\neg ?A \wedge \neg ?B \wedge ?C))$ **by** (*insert xor-distr-L*[*of ?A ?B ?C*], *auto simp*:*elimmeets*)
thus *?thesis*
proof (*elim disjE*)
{assume $?A \wedge \neg ?B \wedge \neg ?C$ **then have** *?A* **by** *simp*
thus *?thesis* **using** *m* **by** *auto***}**
next
{assume $\neg ?A \wedge ?B \wedge \neg ?C$ **then have** *?B* **by** *simp*
thus *?thesis* **using** *b* **by** *auto***}**
next
{ assume $\neg ?A \wedge \neg ?B \wedge ?C$ **then obtain** g **where** tg:$t \| g$ **and** $g \| u'$ **by** *auto*
with *qup* qt' **have** $g \| t'$ **using** *M1* **by** *blast*
with qt' *tpc pu kpq kpt tp tg* **show** *?thesis* **using** *ov* **by** *blast***}**
qed}
qed}
qed
qed

lemma (**in** *arelations*) *JE*:
assumes $\mathcal{I}\ p\ \mathcal{I}\ q$
shows $(p::'a,q::'a) \in b \cup m \cup ov \cup s \cup d \cup f\hat{}{-1} \cup e \cup f \cup s\hat{}{-1} \cup d\hat{}{-1} \cup ov\hat{}{-1} \cup m\hat{}{-1} \cup b\hat{}{-1}$
using *jointly-exhaustive UnCI assms*(*1*,*2*) **by** *blast*

end

theory *examples*

imports

disjoint-relations

begin

6 Examples

6.1 Compositions of non-basic relations

Basic relations are the 13 time interval relations. The unions of basic relations are also relations and their compositions is the union of compositions.

We prove few of these compositions that are required in theory nest.thy.

method (**in** *arelations*) *e-compose* = (*match* **conclusion in** $e\ O\ b \subseteq$ - $\Rightarrow$ ‹*insert ceb*, *blast*›
| - $\Rightarrow$ ‹*match conclusion in* $e\ O\ m \subseteq$ - $\Rightarrow$ ‹*insert cem*, *blast*› | - $\Rightarrow$ ‹*fail*››)

declare [[*simp-trace-depth-limit=4*]]

lemma *eovisidifmifiOm*:$(e \cup\ ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup m^{-1} \cup f\hat{}{-1})\ O\ m \subseteq m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1} \cup s \cup s^{-1} \cup e$
apply (*simp*, *intro conjI*)
  using *cem* **apply** *blast*
    using *crm-rules* **by** *auto*

lemma *ovsmfidiesiOmi*:$(ov \cup s \cup m\ \cup f\hat{}{-1} \cup d\hat{}{-1} \cup e \cup s\hat{}{-1})\ O\ m\hat{}{-1} \subseteq d\hat{}{-1} \cup s\hat{}{-1} \cup\ ov\hat{}{-1} \cup m\hat{}{-1} \cup f\hat{}{-1} \cup f \cup e$
apply (*simp*, *intro conjI*)
  using *crmi-rules* **by** *auto*

lemma *ovsmfidiesiOm*:$(ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1})\ O\ m \subseteq b \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1} \cup m$
apply (*simp*, *intro conjI*)
  using *crm-rules* **by** *auto*

lemma *ovsmfidiesiOssie*:$(ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1})\ O\ (s \cup s\hat{}{-1} \cup e) \subseteq ov \cup f\hat{}{-1} \cup d\hat{}{-1} \cup s \cup e \cup s\hat{}{-1} \cup m$
apply (*simp*, *intro conjI*)
  using *crs-rules* **apply** *auto*[*7*]
    using *crsi-rules* **apply** *auto*[*7*]
      using *cre-rules* **by** *auto*[*7*]

lemma $(b \cup m \cup ov \cup s \cup d)\ O\ (b \cup m \cup ov \cup s \cup d) \subseteq\ b \cup m \cup ov \cup s \cup d$
apply (*simp*, *intro conjI*)
 using *crb-rules* **apply** *auto*[*5*]
  using *crm-rules* **apply** *auto*[*5*]
    using *crov-rules* **apply** *auto*[*5*]
      using *crs-rules* **apply** *auto*[*5*]
        using *crd-rules* **by** *auto*[*5*]

lemma *ebmovovissifsiddib*:$(e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1})\ O\ b \subseteq b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
apply (*simp*, *intro conjI*)
   using *crb-rules* **by** *auto*

lemma *ebmovovissiffiddibmovsd*:$(e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup$

$d \cup d^{-1}$) O ($b \cup m \cup ov \cup s \cup d$) $\subseteq$ ($b \cup m \cup ov \cup s \cup d \cup f\hat{}{-1} \cup d\hat{}{-1} \cup ov\hat{}{-1} \cup s^{-1} \cup f \cup e$)
apply (*simp*, *intro conjI*)
  using *crb-rules* **apply** *auto*[*11*]
    using *crm-rules* **apply** *auto*[*11*]
      using *crov-rules* **apply** *auto*[*11*]
        using *crs-rules* **apply** *auto*[*11*]
          using *crd-rules* **by** *auto*

lemma *difimov*:($d\hat{}{-1} \cup f\hat{}{-1} \cup ov \cup e \cup f \cup m \cup b \cup s\hat{}{-1} \cup s$) O ($m \cup ov \cup s \cup d \cup b \cup f\hat{}{-1} \cup f \cup e$) $\subseteq$ ($e \cup b \cup m \cup ov \cup ov\hat{}{-1} \cup s \cup s\hat{}{-1} \cup f \cup f^{-1} \cup d \cup d^{-1}$)
apply (*simp*, *intro conjI*)
  using *crm-rules* **apply** *auto*[*9*]
    using *crov-rules* **apply** *auto*[*9*]
      using *crs-rules* **apply** *auto*[*9*]
        using *crd-rules* **apply** *auto*[*9*]
          using *crb-rules* **apply** *auto*[*9*]
            using *crfi-rules* **apply** *auto*[*9*]
              using *crf-rules* **apply** *auto*[*9*]
                using *cre-rules* **by** *auto*

lemma *difibs*:($d^{-1} \cup f^{-1} \cup ov \cup e \cup f \cup m \cup b \cup s^{-1} \cup s$) O ($b \cup s \cup m$) $\subseteq$ ($b \cup m \cup ov \cup f^{-1} \cup d^{-1} \cup d \cup e \cup s \cup s^{-1}$)
apply (*simp*, *intro conjI*)
  using *crb-rules* **apply** *auto*[*9*]
    using *crs-rules* **apply** *auto*[*9*]
      using *crm-rules* **by** *auto*

lemma *bebmovovissiffiddi*:b O ($e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1}$) $\subseteq$ ($b \cup m \cup ov \cup s \cup d$)
apply (*simp*, *intro conjI*)
  using *cb-rules* **by** *auto*[*11*]

lemma *ovsmfidiesi*:((($ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}$) O ($ov\hat{}{-1} \cup s\hat{}{-1} \cup m\hat{}{-1} \cup f \cup d \cup e \cup s$)) $\subseteq$ ($s \cup s\hat{}{-1} \cup f \cup f\hat{}{-1} \cup d \cup d\hat{}{-1} \cup e \cup ov \cup ov\hat{}{-1} \cup m \cup m\hat{}{-1}$))
apply (*simp*, *intro conjI*)
  using *crovi-rules* **apply** *auto*[*7*]
    using *crsi-rules* **apply** *auto*[*7*]
      using *crmi-rules* **apply** *auto*[*7*]
        using *crf-rules* **apply** *auto*[*7*]
          using *crd-rules* **apply** *auto*[*7*]
            using *cre-rules* **apply** *auto*[*7*]
              using *crs-rules* **by** *auto*

lemma *piiq*:$(p,i) \in ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1} \Longrightarrow (i,q) \in ov\hat{}{-1} \cup s\hat{}{-1} \cup m\hat{}{-1} \cup f \cup d \cup e \cup s \Longrightarrow (p,q) \in s \cup s\hat{}{-1} \cup f \cup f\hat{}{-1} \cup d \cup d\hat{}{-1} \cup e \cup ov \cup ov\hat{}{-1} \cup m \cup m\hat{}{-1}$

using *ovsmfidiesi relcomp.relcompI subsetCE* **by** *blast*

lemma *ceovisidiffimi-ffie*:$(e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1})$ O $(f \cup f^{-1} \cup e) \subseteq$ $e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$
apply (*simp*, *intro conjI*)
  using *crf-rules* **apply** *auto*[*7*]
    using *crfi-rules* **apply** *auto*[*7*]
      using *cre-rules* **by** *auto*

lemma *ceovisidiffimi-ffie-simp*:$(p,i) \in (e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1})$ $\Longrightarrow (i, q) \in (f \cup f^{-1} \cup e) \Longrightarrow (p,q) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$

using *ceovisidiffimi-ffie relcomp.relcompI subsetCE* **by** *blast*

lemma *ceovisidiffimi-fife*: $(e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1})$ O $(f^{-1} \cup f \cup e) \subseteq$ $e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$
apply (*simp*, *intro conjI*)
 using *cefi covifi csifi cdifi cffi cfifi cmifi covifi csifi cdifi* **apply** *auto*[*7*]
   using *cef covif csif cdif cff cfif cmif* **apply** *auto*[*7*]
     using *cee covie csie cdie cfe cfie cmie* **by** *auto*[*7*]

lemma $(x, j) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1} \Longrightarrow (j, i) \in f^{-1} \cup f \cup e \Longrightarrow (x, i) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$
using *ceovisidiffimi-ffie-simp* **by** *blast*

lemma *m-ovsmfidiesi*:m O $(ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}) \subseteq b \cup s \cup m$
apply (*simp*, *intro conjI*)
 using *cm-rules* **by** *auto*

lemma *ovsmfidiesi-d*:$(ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1})$ O $d \subseteq$ $e \cup$ $s \cup d \cup ov \cup ov\hat{}{-1} \cup s\hat{}{-1} \cup f \cup f\hat{}{-1}$ $\cup d\hat{}{-1}$
apply (*simp*, *intro conjI*)
  using *crd-rules* **by** *auto*[*7*]

lemma *cbi-esdovovisiffidi*:$b\hat{}{-1}$ O $(e \cup s \cup d \cup ov \cup ov^{-1} \cup s^{-1} \cup f \cup f^{-1} \cup d^{-1}) \subseteq b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1} \cup f \cup d$
apply (*simp*, *intro conjI*)
  using *cbi-rules* **by** *auto*[*9*]

lemma *cm-alpha1ialpha4mi*:m O $(e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}) \subseteq$ $m \cup ov \cup s \cup d \cup b \cup f\hat{}{-1} \cup f \cup e$
apply (*simp*, *intro conjI*)
using *cm-rules* **by** *auto*

lemma *cbi-alpha1ialpha4mi*:$b\hat{}{-1}$ O $($ $e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1})$ $\subseteq$ $b\hat{}{-1}$

apply (*simp*, *intro conjI*)
using *cbi-rules* **by** *auto*

lemma *cbeta2-beta2*:$(b \cup m \cup ov \cup f^{-1} \cup d^{-1})\ O\ (b \cup m \cup ov \cup f^{-1} \cup d^{-1}) \subseteq b \cup m \cup ov \cup f^{-1} \cup d^{-1}$
apply (*simp*, *intro conjI*)
using *crb-rules* **apply** *auto*[*5*]
using *crm-rules* **apply** *auto*[*5*]
using *crov-rules* **apply** *auto*[*5*]
using *crfi-rules* **apply** *auto*[*5*]
using *crdi-rules* **by** *auto*

lemma *cbeta2-gammabm*: $(b \cup m \cup ov \cup f^{-1} \cup d^{-1})\ O\ (e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1}) \subseteq (e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1})$
apply (*simp*, *intro conjI*)
using *cre-rules* **apply** *auto*[*5*]
using *crb-rules* **apply** *auto*[*5*]
using *crm-rules* **apply** *auto*[*5*]
using *crov-rules* **apply** *auto*[*5*]
using *crovi-rules* **apply** *auto*[*5*]
using *crs-rules* **apply** *auto*[*5*]
using *crsi-rules* **apply** *auto*[*5*]
using *crf-rules* **apply** *auto*[*5*]
using *crfi-rules* **apply** *auto*[*5*]
using *crd-rules* **apply** *auto*[*5*]
using *crdi-rules* **by** *auto*

lemma *calpha1-alpha1*:$(b \cup m \cup ov \cup s \cup d)\ O\ (b \cup m \cup ov \cup s \cup d) \subseteq (b \cup m \cup ov \cup s \cup d)$
apply (*simp*, *intro conjI*)
using *crb-rules* **apply** *auto*[*5*]
using *crm-rules* **apply** *auto*[*5*]
using *crov-rules* **apply** *auto*[*5*]
using *crs-rules* **apply** *auto*[*5*]
using *crd-rules* **by** *auto*

6.2 Intersection of non-basic relations

lemma *inter-ov*:
assumes $(i,j) \in (b \cup m \cup ov \cup f^{-1} \cup d^{-1}) \cap (e \cup b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1} \cup ov \cup s\hat{}{-1} \cup s \cup f\hat{}{-1} \cup f \cup d\hat{}{-1} \cup d) \cap (b \cup m \cup ov \cup s \cup d)$
shows $(i,j) \in ov$
using *assms* **apply** *auto*
using *b-rules* **apply** *auto*[*43*]
using *e-rules* **apply** *auto*[*9*]
using *b-rules* **apply** *auto*[*30*]
using *m-rules* **apply** *auto*[*24*]
using *b-rules* **apply** *auto*[*6*]

using *m-rules* **apply** *auto*[*20*]
using *f-rules* **apply** *auto*[*14*]
using *d-rules* **by** *auto*

lemma *neq-beta2i-alpha2alpha5m*:
assumes $(q, j) \in b^{-1} \cup d \cup f \cup ov^{-1} \cup m^{-1}$ **and** $(q, j) \in ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}$
shows *False*
using *assms* **apply** *auto*
using *b-rules* **apply** *auto*[*7*]
using *ov-rules* **apply** *auto*[*4*]
using *d-rules* **apply** *auto*[*6*]
using *s-rules* **apply** *auto*[*3*]
using *f-rules* **apply** *auto*[*5*]
using *m-rules* **apply** *auto*[*2*]
using *ov-rules* **apply** *auto*[*4*]
using *m-rules* **by** *auto*

lemma *neq-bi-alpha1ialpha4mi*:
assumes $(q,i) \in b\hat{}{-1}$ **and** $(q, i) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$
shows *False*
using *assms* **apply** *auto*
using *b-rules* **by** *auto*

end

theory *nest*

imports
Main jointly-exhaustive examples
HOL−Eisbach.Eisbach-Tools

begin

7 Nests

Nests are sets of intervals that share a meeting point. We define relation before between nests that give the ordering properties of points.

7.1 Definitions

type-synonym *'a nest = 'a set*

definition (**in** *arelations*) *BEGIN* :: $'a \Rightarrow 'a$ *nest*
where *BEGIN i* $= \{j \mid j.\ (j,i) \in ov \cup s \cup m \cup f\hat{}{-1} \cup d\hat{}{-1} \cup e \cup s\hat{}{-1}\}$

definition (**in** *arelations*) *END* :: *'a* ⇒ *'a nest*
where $END\ i\ = \{j \mid j.\ (j,i) \in e \cup ov^{\frown}{-1} \cup s^{\frown}{-1} \cup d^{\frown}{-1} \cup f \cup f^{\frown}{-1} \cup m^{\frown}{-1}\}$

definition (**in** *arelations*) *NEST* ::*'a nest* ⇒ *bool*
where $NEST\ S \equiv \exists\, i.\ \mathcal{I}\ i \land (S = BEGIN\ i \lor S = END\ i)$

definition (**in** *arelations*) *before* :: *'a nest* ⇒ *'a nest* ⇒ *bool* (**infix** ≪ *100*)
where $before\ N\ M \equiv NEST\ N \land NEST\ M \land (\exists\, n\ m.\ \cancel{\mathcal{I}\ m \land \mathcal{I}\ n \land}\ n \in N \land m \in M \land (n,m) \in b)$

7.2 Properties of Nests

lemma *intv1*:
assumes $\mathcal{I}\ i$
shows $i \in BEGIN\ i$
unfolding *BEGIN-def*
by (*simp add*:*e assms*)

lemma *intv2*:
assumes $\mathcal{I}\ i$
shows $i \in END\ i$
unfolding *END-def*
by (*simp add*: *e assms*)

lemma *NEST-nonempty*:
assumes *NEST S*
shows $S \neq \{\}$
using *assms* **unfolding** *NEST-def*
by (*insert intv1 intv2*, *auto*)

lemma *NEST-BEGIN*:
assumes $\mathcal{I}\ i$
shows *NEST* (*BEGIN i*)
using *NEST-def assms* **by** *auto*

lemma *NEST-END*:
assumes $\mathcal{I}\ i$
shows *NEST* (*END i*)
using *NEST-def assms* **by** *auto*

lemma *before*:
assumes $a{:}\mathcal{I}\ i$
shows $BEGIN\ i \ll END\ i$
proof −

obtain *p* **where** $pi{:}(p,i) \in m$
using *a M3 m* **by** *blast*
then have $p{:}p \in BEGIN\ i$ **using** *BEGIN-def* **by** *auto*

```
    obtain q where qi:(q,i) ∈ m^−1
    using a M3 m by blast
    then have q:q ∈ END i using END-def by auto

    from pi qi have c1:(p,q) ∈ b using b  m
    by blast

    with  c1 p q assms show ?thesis  by (auto simp:NEST-def  before-def)

qed

lemma  meets:
fixes i j
 assumes ℐ i and ℐ j
shows (i,j) ∈ m = ((END i) = (BEGIN j))
proof
  assume ij:(i,j) ∈ m then have ibj:i ∈ (BEGIN j) unfolding BEGIN-def by
auto
  from ij have ji:(j,i) ∈ m^−1 by simp
  then have jeo:j ∈ (END i) unfolding END-def by simp
  show ((END i) = (BEGIN j))
  proof
      {fix x::'a assume a:x ∈ (END i)
      then have asimp:(x,i) ∈ e ∪  ov⁻¹ ∪ s⁻¹ ∪ d⁻¹ ∪ f ∪ m⁻¹ ∪ f^−1
      unfolding END-def by auto
      then have x ∈ (BEGIN j) using ij eovisidifmifiOm
      by (auto simp:BEGIN-def)
      }
      thus conc1:END i ⊆ BEGIN j by auto
   next
     {fix x assume b:x ∈ (BEGIN j)
     then have bsimp:(x,j) ∈ ov ∪ s∪ m  ∪ f^−1 ∪ d^−1 ∪ e ∪ s^−1
     unfolding BEGIN-def by auto
     then have x ∈ (END i) using ij ovsmfidiesiOmi
     by (auto simp:END-def)
     }thus conc2:BEGIN j ⊆ END i by auto
  qed
next
  assume a0:END (i::'a) = BEGIN (j::'a) show (i,j) ∈ m
  proof (rule ccontr)
     assume a:(i,j) ∉ m then have ¬i∥j using m by auto
      from a have (i,j) ∈ b  ∪ ov  ∪ s ∪ d ∪ f^−1 ∪ e ∪ f  ∪ s^−1  ∪ d^−1 ∪
ov^−1 ∪ m^−1  ∪ b^−1 using assms JE by auto
     thus False
     proof (auto)
       {assume ij:(i,j) ∈ e
        obtain p where ip:i∥p using M3 assms(1)  by auto
        then  have pi:(p,i)∈ m^−1 using m by auto
```

```
then have p ∈ END i using END-def by auto
with a0 have pj:p ∈ (BEGIN j)  by auto
from ij pi have (p,j) ∈ m^−1  by (simp add: e)
with pj show  ?thesis
apply (auto simp:BEGIN-def)
using m-rules by auto[7] }
next
{assume ij: (j,i) ∈ m
 obtain p where ip:i∥p using M3  assms(1) by auto
 then  have pi:(p,i)∈ m^−1 using m by auto
 then have p ∈ END i using END-def by auto
 with a0 have pj:p ∈ (BEGIN j)  by auto
 from ij have (i,j) ∈ m^−1 by simp
 with  pi have (p,j) ∈ b^−1 using cmimi  by auto
 with pj show  ?thesis
 apply (auto simp:BEGIN-def)
   using b-rules by auto
 }

next

{assume ij:(i,j)∈ b
 have ii:(i,i)∈e and i ∈ END i using assms  intv2 e by auto
 with a0 have j:i ∈ BEGIN j by simp
 with ij  show  ?thesis
 apply (auto simp:BEGIN-def)
   using b-rules by auto
  }

 next

 { assume ji:(j,i) ∈ b then have ij:(i,j) ∈ b^−1 by simp
   have ii:(i,i)∈e and i ∈ END i using assms intv2 e by auto
   with a0 have j:i ∈ BEGIN j by simp
   with ij  show  ?thesis
   apply (auto simp:BEGIN-def)
     using b-rules by auto}

 next

 {assume ij:(i,j)∈ov
  then obtain u v::'a where iu:i∥u and uv:u∥v and uv:u∥v using ov by
blast
  from iu have u ∈ END i using m END-def by auto
  with a0 have u:u ∈ BEGIN j by simp
  from iu have (u,i) ∈ m^−1 using m by auto
  with ij have uj:(u,j) ∈ ov^−1 ∪ d ∪ f using covim by auto
  show ?thesis using u uj
  apply (auto simp:BEGIN-def)
```

```
  using ov-rules eovi apply auto[9]
   using s-rules apply auto[2]
     using d-rules apply auto[5]
       using f-rules by auto[5]
}

next

{assume (j,i) ∈ ov then have  ij:(i,j)∈ov^-1 by simp let ?p = i
from ij have pi:(?p, i) ∈ e by (simp add:e)
from ij have pj:(?p,j) ∈ ov^-1 by simp
from pi have ?p ∈ END i using END-def by auto
with a0 have ?p ∈ (BEGIN j)  by auto
with pj show ?thesis
apply (auto simp:BEGIN-def)
  using ov-rules by auto
}
next
{assume ij:(i,j) ∈ s
 then obtain p q t where ip:i‖p and pq:p‖q and  jq:j‖q and ti:t‖i and
tj:t‖j using s by blast
 from ip have (p,i) ∈ m^-1 using m by auto
 then have p ∈ END i using END-def by auto
 with a0 have p:p ∈ BEGIN j by simp
 from ti ip pq tj jq have (p,j) ∈ f using f by blast
 with p show ?thesis
 apply (auto simp:BEGIN-def)
   using f-rules by auto

 }
 next
 {assume (j,i) ∈ s then have  ij:(i,j) ∈ s^-1 by simp
  then obtain u v where ju:j‖u and uv:u‖v and iv:i‖v using s by blast
  from iv have (v,i) ∈ m^-1 using m by blast
  then have v ∈ END i using END-def by auto
  with a0 have v:v ∈ BEGIN j by simp
  from ju uv have (v,j) ∈ b^-1 using b by auto
  with v show ?thesis
  apply (auto simp:BEGIN-def)
    using b-rules by auto}
 next
 {assume ij:(i,j) ∈ f
  have (i,i) ∈ e and i ∈ END i
  by (simp add: e)  (auto simp: assms intv2 )
  with a0 have i ∈ BEGIN j by simp
  with ij show ?thesis
  apply (auto simp:BEGIN-def)
   using f-rules by auto
 }
```

next
{assume $(j,i) \in f$ **then have** $(i,j) \in f\widehat{\ }{-1}$ **by** *simp*
then obtain u **where** $ju{:}j\|u$ **and** $iu{:}i\|u$ **using** f **by** *auto*
then have $ui{:}(u,i) \in m\widehat{\ }{-1}$ **and** $u \in END\ i$
apply (*simp add*: *converse.intros m*)
using *END-def iu m* **by** *auto*
with *a0* **have** $ubj{:}u \in BEGIN\ j$ **by** *simp*
from *ju* **have** $(u,j) \in m\widehat{\ }{-1}$ **by** (*simp add*: *converse.intros m*)
with *ubj* **show** *?thesis*
apply (*auto simp*:*BEGIN-def*)
using *m-rules* **by** *auto*
}
next
{assume $ij{:}(i,j) \in d$ **then**
have $(i,i) \in e$ **and** $i \in END\ i$ **using** *assms e* **by** (*blast*, *simp add*: *intv2*)
with *a0* **have** $i \in BEGIN\ j$ **by** *simp*
with *ij* **show** *?thesis*
apply (*auto simp*:*BEGIN-def*)
using *d-rules* **by** *auto***}**
next
{assume $ji{:}(j,i) \in d$ **then have** $(i,j) \in d\widehat{\ }{-1}$ **using** d **by** *simp*
then obtain $u\ v$ **where** $ju{:}j\|u$ **and** $uv{:}u\|v$ **and** $iv{:}i\|v$ **using** d **using** *ji* **by** *blast*
then have $(v,i) \in m\widehat{\ }{-1}$ **and** $v \in END\ i$ **using** *m END-def* **by** *auto*
with *a0 ju uv* **have** $vj{:}(v,j) \in b\widehat{\ }{-1}$ **and** $v \in BEGIN\ j$ **using** b **by** *auto*
with *vj* **show** *?thesis*
apply (*auto simp*:*BEGIN-def*)
using *b-rules* **by** *auto***}**

qed
qed
qed

lemma *starts*:
fixes $i\ j$
assumes $\mathcal{I}\ i$ **and** $\mathcal{I}\ j$
shows $((i,j) \in s \cup s\widehat{\ }{-1} \cup e) = (BEGIN\ i = BEGIN\ j)$
proof
assume $a3{:}(i,j) \in s \cup s\widehat{\ }{-1} \cup e$ **show** $BEGIN\ i = BEGIN\ j$
proof $-$
{ fix x **assume** $x \in BEGIN\ i$ **then have** $(x,i) \in ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}$ **unfolding** *BEGIN-def* **by** *auto*
hence $x \in BEGIN\ j$ **using** *a3 ovsmfidiesiOssie*
by (*auto simp*:*BEGIN-def*)
} note $c1 = this$

{ fix x **assume** $x \in BEGIN\ j$ **then have** $xj{:}(x,j) \in ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}$ **unfolding** *BEGIN-def* **by** *auto*

then have $x \in$ *BEGIN i*
apply (*insert converseI*[*OF a3*] *xj*)
apply (*subst* (*asm*) *converse-Un*)+
apply (*subst* (*asm*) *converse-converse*)
using *ovsmfidiesiOssie*
by (*auto simp*:*BEGIN-def*)
} **note** *c2* = *this*

from *c1* **have** *BEGIN i* $\subseteq$ *BEGIN j* **by** *auto*
moreover with *c2* **have** *BEGIN j* $\subseteq$ *BEGIN i* **by** *auto*
ultimately show *?thesis* **by** *auto*
qed
next
assume *a4*:*BEGIN i* = *BEGIN j*
with *assms* **have** $i \in$ *BEGIN j* **and** $j \in$ *BEGIN i* **using** *intv1* **by** *auto*
then have *ij*:$(i,j) \in ov \cup s \cup m \cup f\hat{}{-1} \cup d\hat{}{-1} \cup e \cup s\hat{}{-1}$ **and** *ji*:$(j,i) \in ov \cup s \cup m \cup f\hat{}{-1} \cup d\hat{}{-1} \cup e \cup s\hat{}{-1}$
unfolding *BEGIN-def* **by** *auto*
then have *ijov*:$(i,j) \notin ov$
apply *auto*
using *ov-rules* **by** *auto*

from *ij ji* **have** *ijm*:$(i,j) \notin m$
apply (*simp-all*, *elim disjE*, *simp-all*)
using *ov-rules* **apply** *auto*[*13*]
using *s-rules* **apply** *auto*[*11*]
using *m-rules* **apply** *auto*[*9*]
using *f-rules* **apply** *auto*[*7*]
using *d-rules* **apply** *auto*[*5*]
using *m-rules* **by** *auto*[*4*]

from *ij ji* **have** *ijfi*:$(i,j) \notin f\hat{}{-1}$
apply (*simp-all*, *elim disjE*, *simp-all*)
using *ov-rules* **apply** *auto*[*13*]
using *s-rules* **apply** *auto*[*11*]
using *m-rules* **apply** *auto*[*9*]
using *f-rules* **apply** *auto*[*7*]
using *d-rules* **apply** *auto*[*5*]
using *f-rules* **by** *auto*[*4*]

from *ij ji* **have** *ijdi*:$(i,j) \notin d\hat{}{-1}$
apply (*simp-all*, *elim disjE*, *simp-all*)
using *ov-rules* **apply** *auto*[*13*]
using *s-rules* **apply** *auto*[*11*]
using *m-rules* **apply** *auto*[*9*]
using *f-rules* **apply** *auto*[*7*]
using *d-rules* **apply** *auto*[*5*]
using *d-rules* **by** *auto*[*4*]

from *ij ijm ijov ijfi ijdi* **show** $(i, j) \in s \cup s^{-1} \cup e$ **by** *auto*

qed

lemma *xj-set*:$x \in \{a \mid a.\ (a, j) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}\} = ((x,j) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1})$
by *blast*

lemma *ends*:
fixes *i j*
assumes $\mathcal{I}$ *i* **and** $\mathcal{I}$ *j*
shows $((i,j) \in f \cup f\widehat{\ }{-1} \cup e) = (END\ i = END\ j)$
proof
assume *a3*:$(i,j) \in f \cup f\widehat{\ }{-1} \cup e$ **show** $END\ i = END\ j$
proof $-$
{ fix *x* **assume** $x \in END\ i$ **then have** $(x,i) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$ **unfolding** *END-def* **by** *auto*
then have $x \in END\ j$ **using** *a3* **unfolding** *END-def*
apply (*subst xj-set*)
using *ceovisidiffimi-ffie-simp* **by** *simp*
} **note** *c1* =*this*

{ fix *x* **assume** $x \in END\ j$ **then have** $(x,j) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$ **unfolding** *END-def* **by** *auto*
then have $x \in END\ i$ **using** *a3* **unfolding** *END-def*
by (*metis Un-iff ceovisidiffimi-ffie-simp converse-iff eei mem-Collect-eq*)
} **note** *c2* = *this*

from *c1* **have** $END\ i \subseteq END\ j$ **by** *auto*
moreover with *c2* **have** $END\ j \subseteq END\ i$ **by** *auto*
ultimately show *?thesis* **by** *auto*
qed
next
assume *a4*:$END\ i = END\ j$
with *assms* **have** $i \in END\ j$ **and** $j \in END\ i$ **using** *intv2* **by** *auto*
then have *ij*:$(i,j) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$ **and** *ji*:$(j,i) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$
unfolding *END-def* **by** *auto*
then have *ijov*:$(i,j) \notin ov\widehat{\ }{-1}$
apply (*simp-all*, *elim disjE*, *simp-all*)
using *eov es ed efi ef em eov* **apply** *auto*[*13*]
using *ov-rules* **apply** *auto*[*11*]
using *s-rules* **apply** *auto*[*9*]
using *d-rules* **apply** *auto*[*7*]
using *f-rules* **apply** *auto*[*8*]
using *movi* **by** *auto*

from *ij ji* **have** *ijm*:$(i,j) \notin m\widehat{\ }{-1}$
apply (*simp-all*, *elim disjE*, *simp-all*)

using *m-rules* **by** *auto*

from *ij ji* **have** *ijfi*:$(i,j) \notin s\hat{}{-1}$
apply (*simp-all*, *elim disjE*, *simp-all*)
using *s-rules* **by** *auto*

from *ij ji* **have** *ijdi*:$(i,j) \notin d\hat{}{-1}$
apply (*simp-all*, *elim disjE*, *simp-all*)
using *d-rules* **by** *auto*

from *ij ijm ijov ijfi ijdi* **show** $(i, j) \in f \cup f^{-1} \cup e$ **by** *auto*
qed

lemma *before-irrefl*:
fixes *a*
shows $\neg\ a \ll a$
proof (*rule ccontr*, *auto*)
assume *a0*:$a \ll a$
then have *NEST a* **unfolding** *before-def* **by** *auto*
then obtain *i* **where** *i*:$a = BEGIN\ i \vee a = END\ i$ **unfolding** *NEST-def* **by** *auto*
from *i* **show** *False*
proof
assume $a = BEGIN\ i$
with *a0* **have** $BEGIN\ i \ll BEGIN\ i$ **by** *simp*
then obtain *p q* **where** $p\in BEGIN\ i$ **and** $q \in BEGIN\ i$ **and** *b*:$(p,q) \in b$ **unfolding** *before-def* **by** *auto*
then have *a1*:$(p,i) \in ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}$ **and** *a2*:$(i,q) \in ov\hat{}{-1} \cup s\hat{}{-1} \cup m\hat{}{-1} \cup f \cup d \cup e \cup s$ **unfolding** *BEGIN-def*
apply *auto*
using *eei* **apply** *fastforce*
by (*simp add*: *e*)+
with *b* **show** *False*
using *piiq*[*of p i q*]
using *b-rules* **by** *safe fast+*
next
assume $a = END\ i$
with *a0* **have** $END\ i \ll END\ i$ **by** *simp*
then obtain *p q* **where** $p\in END\ i$ **and** $q \in END\ i$ **and** *b*:$(p,q) \in b$ **unfolding** *before-def* **by** *auto*
then have *a1*:$(p,i) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$ **and** *a2*:$(q,i) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$ **unfolding** *END-def*
by *auto*
with *b* **show** *False*
apply (*subst* (*asm*) *converse-iff*[*THEN sym*])
using *cbi-alpha1ialpha4mi neq-bi-alpha1ialpha4mi relcomp.relcompI subsetCE* **by** *blast*
qed
qed

lemma *BEGIN-before*:
fixes *i j*
assumes $\mathcal{I}$ *i* **and** $\mathcal{I}$ *j*
shows *BEGIN i* $\ll$ *BEGIN j* $= ((i,j) \in b \cup m \cup ov \cup f^{-1} \cup d^{-1})$
proof

assume *a3*:*BEGIN i* $\ll$ *BEGIN j*
from *a3* **obtain** *p q* **where** *pa*:$p \in$ *BEGIN i* **and** *qc*:$q \in$ *BEGIN j* **and** *pq*:$(p,q) \in b$ **unfolding** *before-def* **by** *auto*
then obtain *r* **where** $p\|r$ **and** $r\|q$ **using** *b* **by** *auto*
then have *pr*:$(p,r) \in m$ **and** *rq*:$(r,q) \in m$ **using** *m* **by** *auto*
from *pa* **have** *pi*:$(p,i) \in ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}$ **unfolding** *BEGIN-def* **by** *auto*
moreover with *pr* **have** $(r,p) \in m\hat{}{-1}$ **by** *simp*
ultimately have $(r,i) \in d \cup f \cup ov\hat{}{-1} \cup e \cup f\hat{}{-1} \cup m\hat{}{-1} \cup b\hat{}{-1} \cup s \cup s\hat{}{-1}$
using *cmiov cmis cmim cmifi cmidi cmisi*
apply (*simp-all*,*elim disjE*, *auto*)
by (*simp add*: *e*)

then have *ir*:$(i,r) \in d\hat{}{-1} \cup f\hat{}{-1} \cup ov \cup e \cup f \cup m \cup b \cup s\hat{}{-1} \cup s$
by (*metis* (*mono-tags*, *lifting*) *converseD converse-Un converse-converse eei*)

from *qc* **have** $(q,j) \in ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}$ **unfolding** *BEGIN-def* **by** *auto*
with *rq* **have** *rj*:$(r,j) \in b \cup s \cup m$
using *m-ovsmfidiesi* **using** *contra-subsetD relcomp.relcompI* **by** *blast*

with *ir* **have** *c1*:$(i,j) \in b \cup m \cup ov \cup f^{-1} \cup d^{-1} \cup d \cup e \cup s \cup s^{-1}$
using *difibs* **by** *blast*
{assume $(i,j) \in s \vee (i,j) \in s\hat{}{-1} \vee (i,j) \in e$ **then have** *BEGIN i* = *BEGIN j*
using *starts Un-iff assms*(*1*) *assms*(*2*) **by** *blast*
with *a3* **have** *False* **by** (*simp add*: *before-irrefl*)**}**

from *c1* **have** *c1′*:$(i,j) \in b \cup m \cup ov \cup f^{-1} \cup d^{-1} \cup d$
using ‹$(i, j) \in s \vee (i, j) \in s^{-1} \vee (i, j) \in e \Longrightarrow$ *False*› **by** *blast*

{assume $(i,j) \in d$ **with** *pi* **have** $(p,j) \in e \cup s \cup d \cup ov \cup ov\hat{}{-1} \cup s\hat{}{-1} \cup f \cup f\hat{}{-1} \cup d\hat{}{-1}$
using *ovsmfidiesi-d* **using** *relcomp.relcompI subsetCE* **by** *blast*
with *pq* **have** $(q,j) \in b\hat{}{-1} \cup d \cup f \cup ov\hat{}{-1} \cup m\hat{}{-1}$
apply (*subst* (*asm*) *converse-iff*[*THEN sym*])
using *cbi-esdovovisiffidi* **by** *blast*
with *qc* **have** *False* **unfolding** *BEGIN-def*
apply (*subgoal-tac* $(q, j) \in ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}$)
prefer *2*
apply *simp*

using *neq-beta2i-alpha2alpha5m* **by** *auto*
}

with *c1'* **show** $((i, j) \in b \cup m \cup ov \cup f^{-1} \cup d^{-1})$ **by** *auto*
next
assume $(i, j) \in b \cup m \cup ov \cup f^{-1} \cup d^{-1}$
then show *BEGIN i* $\ll$ *BEGIN j*
proof (*simp-all*,*elim disjE*, *simp-all*)
assume $(i,j) \in b$ **thus** *?thesis* **using** *intv1* **using** *before-def NEST-BEGIN assms* **by** *metis*
next
assume *iu*:$(i,j) \in m$
obtain *l* **where** *li*:$(l,i) \in m$ **using** *M3 m meets-wd assms* **by** *blast*
with *iu* **have** $(l,j) \in b$ **using** *cmm* **by** *auto*
moreover from *li* **have** $l \in$ (*BEGIN i*) **using** *BEGIN-def* **by** *auto*
ultimately show *?thesis* **using** *intv1 before-def NEST-BEGIN assms* **by** *blast*
next
assume *iu*:$(i,j) \in ov$
obtain *l* **where** *li*:$(l,i) \in m$ **using** *M3 m meets-wd assms* **by** *blast*
with *iu* **have** $(l,j) \in b$ **using** *cmov* **by** *auto*
moreover from *li* **have** $l \in$ (*BEGIN i*) **using** *BEGIN-def* **by** *auto*
ultimately show *?thesis* **using** *intv1 before-def NEST-BEGIN assms* **by** *blast*
next
assume *iu*:$(j,i) \in f$
obtain *l* **where** *li*:$(l,i) \in m$ **using** *M3 m meets-wd assms* **by** *blast*
with *iu* **have** $(l,j) \in b$ **using** *cmfi* **by** *auto*
moreover from *li* **have** $l \in$ (*BEGIN i*) **using** *BEGIN-def* **by** *auto*
ultimately show *?thesis* **using** *intv1 before-def NEST-BEGIN assms* **by** *blast*
next
assume *iu*:$(j,i) \in d$
obtain *l* **where** *li*:$(l,i) \in m$ **using** *M3 m meets-wd assms* **by** *blast*
with *iu* **have** $(l,j) \in b$ **using** *cmdi* **by** *auto*
moreover from *li* **have** $l \in$ (*BEGIN i*) **using** *BEGIN-def* **by** *auto*
ultimately show *?thesis* **using** *intv1 before-def NEST-BEGIN assms* **by** *blast*

qed
qed

lemma *BEGIN-END-before*:
fixes *i j*
assumes $\mathcal{I}$ *i* **and** $\mathcal{I}$ *j*
shows *BEGIN i* $\ll$ *END j* = $((i,j) \in e \cup b \cup m \cup ov \cup ov\hat{}{-1} \cup s \cup s\hat{}{-1} \cup f \cup f^{-1} \cup d \cup d^{-1})$
proof
assume *a3*:*BEGIN i* $\ll$ *END j*

then obtain p q **where** pa:$p \in$ *BEGIN* i **and** qc:$q \in$ *END* j **and** pq:$(p,q) \in b$ **unfolding** *before-def* **by** *auto*
then obtain r **where** $p \| r$ **and** $r \| q$ **using** b **by** *auto*
then have pr:$(p,r) \in m$ **and** rq:$(r,q) \in m$ **using** m **by** *auto*
from pa **have** pi:$(p,i) \in ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}$ **unfolding** *BEGIN-def* **by** *auto*
moreover with pr **have** $(r,p) \in m\hat{}{-1}$ **by** *simp*
ultimately have $(r,i) \in d \cup f \cup ov\hat{}{-1} \cup e \cup f\hat{}{-1} \cup m\hat{}{-1} \cup b\hat{}{-1} \cup s \cup s\hat{}{-1}$ **using** *cmiov cmis cmim cmifi cmidi e cmisi*
by (*simp-all*,*elim disjE*, *auto simp*:*e*)

then have ir:$(i,r) \in d\hat{}{-1} \cup f\hat{}{-1} \cup ov \cup e \cup f \cup m \cup b \cup s\hat{}{-1} \cup s$
by (*metis* (*mono-tags*, *lifting*) *converseD converse-Un converse-converse eei*)

from qc **have** $(q,j) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$ **unfolding** *END-def* **by** *auto*
with rq **have** rj:$(r,j) \in m \cup ov \cup s \cup d \cup b \cup f\hat{}{-1} \cup f \cup e$ **using** *cm-alpha1ialpha4mi* **by** *blast*

with ir **show** $c1$:$(i,j) \in e \cup b \cup m \cup ov \cup ov\hat{}{-1} \cup s \cup s\hat{}{-1} \cup f \cup f^{-1} \cup d \cup d^{-1}$
using *difimov* **by** *blast*
next
assume $a4$:$(i, j) \in e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1}$
then show *BEGIN* $i \ll$ *END* j
proof (*simp-all*,*elim disjE*, *simp-all*)
assume $(i,j) \in e$
obtain l k **where** l:$l \| i$ **and** $i \| k$ **using** *M3 meets-wd assms* **by** *blast*
with ‹$(i,j) \in e$› **have** k:$j \| k$ **by** (*simp add*: e)
from l k **have** $(l,i) \in m$ **and** $(k,j) \in m\hat{}{-1}$ **using** m **by** *auto*
then have $l \in$ *BEGIN* i **and** $k \in$ *END* j **using** *BEGIN-def END-def* **by** *auto*
moreover from l ‹$i \| k$› **have** $(l,k) \in b$ **using** b **by** *auto*
ultimately show *?thesis* **using** *before-def assms NEST-BEGIN NEST-END* **by** *blast*
next
assume $(i,j) \in b$
then show *?thesis* **using** *before-def assms NEST-BEGIN NEST-END intv1*[*of i*] *intv2*[*of j*] **by** *auto*
next
assume $(i,j) \in m$
obtain l **where** $l \| i$ **using** *M3 assms* **by** *blast*
then have $l \in$*BEGIN* i **using** m *BEGIN-def* **by** *auto*
moreover from ‹$(i,j) \in m$› ‹$l \| i$› **have** $(l,j) \in b$ **using** b m **by** *blast*
ultimately show *?thesis* **using** *intv2*[*of j*] *assms NEST-BEGIN NEST-END before-def* **by** *blast*
next
assume $(i,j) \in ov$
then obtain l k **where** li:$l \| i$ **and** lk:$l \| k$ **and** ku:$k \| j$ **using** ov **by** *blast*

from *li* **have** $l \in$ *BEGIN i* **using** *m BEGIN-def* **by** *auto*
moreover from *lk ku* **have** $(l,j) \in b$ **using** *b* **by** *auto*
ultimately show *?thesis* **using** *intv2*[*of j*] *assms NEST-BEGIN NEST-END before-def* **by** *blast*
next
assume $(j,i) \in ov$
then obtain *l k v* **where** *uv*:$j \| v$ **and** *lk*:$l \| k$ **and** *kv*:$k \| v$ **and** *li*:$l \| i$ **using** *ov* **by** *blast*
from *li* **have** $l \in$ *BEGIN i* **using** *m BEGIN-def* **by** *auto*
moreover from *uv* **have** $v \in$ *END j* **using** *m END-def* **by** *auto*
moreover from *lk kv* **have** $(l,v) \in b$ **using** *b* **by** *auto*
ultimately show *?thesis* **using** *assms NEST-BEGIN NEST-END before-def* **by** *blast*
next
assume $(i,j) \in s$
then obtain *v v′* **where** *iv*:$i \| v$ **and** *vvp*:$v \| v'$ **and** $j \| v'$ **using** *s* **by** *blast*
then have $v' \in$ *END j* **using** *END-def m* **by** *auto*
moreover from *iv vvp* **have** $(i,v') \in b$ **using** *b* **by** *auto*
ultimately show *?thesis* **using** *intv1*[*of i*] *assms NEST-BEGIN NEST-END before-def* **by** *blast*
next
assume $(j,i) \in s$
then obtain *l v* **where** *li*:$l \| i$ **and** *lu*:$l \| j$ **and** $j \| v$ **using** *s* **by** *blast*
then have $v \in$ *END j* **using** *m END-def* **by** *auto*
moreover from *li* **have** $l \in$ *BEGIN i* **using** *m BEGIN-def* **by** *auto*
moreover from *lu* ‹$j \| v$› **have** $(l,v) \in b$ **using** *b* **by** *auto*
ultimately show *?thesis* **using** *assms NEST-BEGIN NEST-END before-def* **by** *blast*
next
assume $(i,j) : f$
then obtain *l v* **where** *li*:$l \| i$ **and** *iv*:$i \| v$ **and** $j \| v$ **using** *f* **by** *blast*
then have $v \in$ *END j* **using** *m END-def* **by** *auto*
moreover from *li* **have** $l \in$ *BEGIN i* **using** *m BEGIN-def* **by** *auto*
moreover from *iv li* **have** $(l,v) \in b$ **using** *b* **by** *auto*
ultimately show *?thesis* **using** *assms NEST-BEGIN NEST-END before-def* **by** *blast*
next
assume $(j,i) \in f$
then obtain *l v* **where** *li*:$l \| i$ **and** *iv*:$i \| v$ **and** $j \| v$ **using** *f* **by** *blast*
then have $v \in$ *END j* **using** *m END-def* **by** *auto*
moreover from *li* **have** $l \in$ *BEGIN i* **using** *m BEGIN-def* **by** *auto*
moreover from *iv li* **have** $(l,v) \in b$ **using** *b* **by** *auto*
ultimately show *?thesis* **using** *assms NEST-BEGIN NEST-END before-def* **by** *blast*
next
assume $(i,j) : d$
then obtain *k v* **where** *ik*:$i \| k$ **and** *kv*:$k \| v$ **and** $j \| v$ **using** *d* **by** *blast*
then have $v \in$ *END j* **using** *END-def m* **by** *auto*
moreover from *ik kv* **have** $(i,v) \in b$ **using** *b* **by** *auto*

ultimately show *?thesis* **using** *intv1[of i]* *assms NEST-BEGIN NEST-END before-def* **by** *blast*
next
assume $(j,i) \in d$
then obtain l k **where** $l \| i$ **and** $lk{:}l \| k$ **and** $ku{:}k \| j$ **using** d **by** *blast*
then have $l \in$ *BEGIN i* **using** *BEGIN-def m* **by** *auto*
moreover from *lk ku* **have** $(l,j) \in b$ **using** b **by** *auto*
ultimately show *?thesis* **using** *intv2[of j]* *assms NEST-BEGIN NEST-END before-def* **by** *blast*
qed
qed

lemma *END-BEGIN-before*:
fixes i j
assumes $\mathcal{I}$ i **and** $\mathcal{I}$ j
shows *END* $i \ll$ *BEGIN* $j = ((i,j) \in b)$
proof
assume *a3*:*END* $i \ll$ *BEGIN* j
from *a3* **obtain** p q **where** $pa{:}p \in$ *END* i **and** $qc{:}q \in$ *BEGIN* j **and** $pq{:}(p,q) \in b$ **unfolding** *before-def* **by** *auto*
then obtain r **where** $p \| r$ **and** $r \| q$ **using** b **by** *auto*
then have $pr{:}(p,r) \in m$ **and** $rq{:}(r,q) \in m$ **using** m **by** *auto*
from *pa* **have** $pi{:}(p,i) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$ **unfolding** *END-def* **by** *auto*
moreover with *pr* **have** $(r,p) \in m\hat{\ }{-1}$ **by** *simp*
ultimately have $(r,i) \in m\hat{\ }{-1} \cup b\hat{\ }{-1}$ **using** *e cmiovi cmisi cmidi cmif cmifi cmimi*
by (*simp-all*,*elim disjE*, *auto simp*:*e*)

then have $ir{:}(i,r) \in m \cup b$ **by** *simp*

from *qc* **have** $(q,j) \in ov \cup s \cup m \cup f^{-1} \cup d^{-1} \cup e \cup s^{-1}$ **unfolding** *BEGIN-def* **by** *auto*
with *rq* **have** $rj{:}(r,j) \in b \cup m$ **using** *cmov cms cmm cmfi cmdi e cmsi*
by (*simp-all*, *elim disjE*, *auto simp*:*e*)

with *ir* **show** $(i,j) \in b$ **using** *cmb cmm cbm cbb* **by** *auto*

next
assume $(i,j) \in b$ **thus** *END* $i \ll$ *BEGIN* j **using** *intv1[of j]* *intv2[of i]* *assms before-def NEST-END NEST-BEGIN* **by** *auto*
qed

lemma *END-END-before*:
fixes i j
assumes $\mathcal{I}$ i **and** $\mathcal{I}$ j
shows *END* $i \ll$ *END* $j = ((i,j) \in b \cup m \cup ov \cup s \cup d)$
proof
assume *a3*:*END* $i \ll$ *END* j

from *a3* **obtain** *p q* **where** *pa*:$p \in$ *END i* **and** *qc*:$q \in$ *END j* **and** *pq*:$(p,q) \in b$ **unfolding** *before-def* **by** *auto*
then obtain *r* **where** $p\|r$ **and** $r\|q$ **using** *b* **by** *auto*
then have *pr*:$(p,r) \in m$ **and** *rq*:$(r,q) \in m$ **using** *m* **by** *auto*
from *pa* **have** *pi*:$(p,i) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$ **unfolding** *END-def* **by** *auto*
moreover with *pr* **have** $(r,p) \in m\hat{}{-1}$ **by** *simp*
ultimately have $(r,i) \in m\hat{}{-1} \cup b\hat{}{-1}$ **using** *e cmiovi cmisi cmidi cmif cmifi cmimi*
by (*simp-all*,*elim disjE*, *auto simp*:*e*)

then have *ir*:$(i,r) \in m \cup b$ **by** *simp*

from *qc* **have** $(q,j) \in e \cup ov^{-1} \cup s^{-1} \cup d^{-1} \cup f \cup f^{-1} \cup m^{-1}$ **unfolding** *END-def* **by** *auto*
with *rq* **have** *rj*:$(r,j) \in m \cup ov \cup s \cup d \cup b \cup f\hat{}{-1} \cup e \cup f$ **using** *e cmovi cmsi cmdi cmf cmfi cmmi*
by (*simp-all*, *elim disjE*, *auto simp*:*e*)

with *ir* **show** $(i,j) \in b \cup m \cup ov \cup s \cup d$ **using** *cmm cmov cms cmd cmb cmfi e cmf cbm cbov cbs cbd cbb cbfi cbf*
by (*simp-all*, *elim disjE*, *auto simp*:*e*)
next
assume $(i, j) \in b \cup m \cup ov \cup s \cup d$
then show *END i* $\ll$ *END j*
proof (*simp-all*,*elim disjE*, *simp-all*)
assume $(i,j) \in b$ **thus** *?thesis* **using** *intv2*[*of i*] *intv2*[*of j*] *assms NEST-END before-def* **by** *blast*
next
assume $(i,j) \in m$
obtain *v* **where** $j\|v$ **using** *M3 assms* **by** *blast*
with ‹$(i,j) \in m$› **have** $(i,v) \in b$ **using** *b m* **by** *blast*
moreover from ‹$j\|v$› **have** $v \in$ *END j* **using** *m END-def* **by** *auto*
ultimately show *?thesis* **using** *intv2*[*of i*] *assms NEST-END before-def* **by** *blast*
next
assume (i,j) : *ov*
then obtain *v v'* **where** *iv*:$i\|v$ **and** *vvp*:$v\|v'$ **and** $j\|v'$ **using** *ov* **by** *blast*
then have $v' \in$ *END j* **using** *m END-def* **by** *auto*
moreover from *iv vvp* **have** $(i,v') \in b$ **using** *b* **by** *auto*
ultimately show *?thesis* **using** *intv2*[*of i*] *assms NEST-END before-def* **by** *blast*
next
assume $(i,j) \in s$
then obtain *v v'* **where** *iv*:$i\|v$ **and** *vvp*:$v\|v'$ **and** $j\|v'$ **using** *s* **by** *blast*
then have $v' \in$ *END j* **using** *m END-def* **by** *auto*
moreover from *iv vvp* **have** $(i,v') \in b$ **using** *b* **by** *auto*
ultimately show *?thesis* **using** *intv2*[*of i*] *assms NEST-END before-def* **by** *blast*

next
assume $(i,j) \in d$
then obtain $v\ v'$ **where** $iv{:}i\|v$ **and** $vvp{:}v\|v'$ **and** $j\|v'$ **using** d **by** *blast*
then have $v' \in END\ j$ **using** $m\ END\text{-}def$ **by** *auto*
moreover from $iv\ vvp$ **have** $(i,v') \in b$ **using** b **by** *auto*
ultimately show *?thesis* **using** $intv2[of\ i]$ *assms NEST-END before-def* **by** *blast*
qed
qed

lemma *overlaps*:
assumes $\mathcal{I}\ i$ **and** $\mathcal{I}\ j$
shows $(i,j) \in ov = ((BEGIN\ i \ll BEGIN\ j) \wedge (BEGIN\ j \ll END\ i) \wedge (END\ i \ll END\ j))$
proof

assume $a{:}(i,j) \in ov$
then obtain $n\ t\ q\ \ u\ v$ **where** $nt{:}n\|t$ **and** $tj{:}t\|j$ **and** $tq{:}t\|q$ **and** $qu{:}q\|u$ **and** $iu{:}i\|u$ **and** $uv{:}u\|v$ **and** $jv{:}j\|v$ **and** $n\|i$ **using** ov **by** *blast*
then have $ni{:}(n,i) \in m$ **using** m **by** *blast*
then have $n{:}n \in BEGIN\ i$ **unfolding** $BEGIN\text{-}def$ **by** *auto*
from $nt\ tj$ **have** $nj{:}(n,j) \in b$ **using** b **by** *auto*
have $j \in BEGIN\ j$ **using** $assms(2)$ **by** $(simp\ add{:}\ intv1)$
with $assms\ n\ nj$ **have** $c1{:}BEGIN\ i \ll BEGIN\ j$ **unfolding** *before-def* **using** *NEST-BEGIN* **by** *blast*

from tj **have** $a1{:}(t,j) \in m$ **and** $a2{:}t \in BEGIN\ j$ **using** $m\ BEGIN\text{-}def$ **by** *auto*
from iu **have** $(u,i) \in m\hat{}{-1}$ **and** $u \in END\ i$ **using** $m\ END\text{-}def$ **by** *auto*
with $assms\ \ tq\ qu\ a2$ **have** $c2{:}BEGIN\ j \ll END\ i$ **unfolding** *before-def* **using** b *NEST-BEGIN NEST-END* **by** *blast*

have $i \in END\ i$ **by** $(simp\ add{:}\ assms\ \ intv2)$
moreover with jv **have** $v \in END\ j$ **using** $m\ END\text{-}def$ **by** *auto*
moreover with $iu\ uv$ **have** $(i,v) \in b$ **using** b **by** *auto*
ultimately have $c3{:}END\ i \ll END\ j$ **using** *assms NEST-END before-def* **by** *blast*

show $((BEGIN\ i \ll BEGIN\ j) \wedge (BEGIN\ j \ll END\ i) \wedge (END\ i \ll END\ j))$ **using** $c1\ c2\ c3$ **by** *simp*
next
assume $a0{:}((BEGIN\ i \ll BEGIN\ j) \wedge (BEGIN\ j \ll END\ i) \wedge (END\ i \ll END\ j))$
then have $(i,j) \in b \cup m \cup ov \cup f^{-1} \cup d^{-1} \wedge (i,j) \in e \cup b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1} \cup ov \cup s\hat{}{-1} \cup s \cup f\hat{}{-1} \cup f \cup d\hat{}{-1} \cup d$
$\wedge$
$(i,j) \in b \cup m \cup ov \cup s \cup d$
using *BEGIN-before BEGIN-END-before END-END-before assms*
by (*metis* (*no-types*, *lifting*) *converseD converse-Un converse-converse eei*)
then have $(i,j) \in\ (b \cup m \cup ov \cup f^{-1} \cup d^{-1}) \cap (e \cup b\hat{}{-1} \cup m\hat{}{-1} \cup ov\hat{}{-1}$

$\cup$ *ov* $\cup$ *s*^*−1* $\cup$ *s* $\cup$ *f*^*−1* $\cup$ *f* $\cup$ *d*^*−1* $\cup$ *d*) $\cap$ (*b* $\cup$ *m* $\cup$ *ov* $\cup$ *s* $\cup$ *d*)
  by (*auto*)
  then show (*i*,*j*) $\in$ *ov*
  using *inter-ov* **by** *blast*

qed

7.3 Ordering of nests

class *strict-order* =
fixes *ls*::′*a nest* $\Rightarrow$ ′*a nest* $\Rightarrow$ *bool*
assumes
 irrefl:$\neg$ *ls a a* **and**
 trans:*ls a c* $\Longrightarrow$ *ls c g* $\Longrightarrow$ *ls a g* **and**
 asym:*ls a c* $\Longrightarrow$ $\neg$ *ls c a*

class *total-strict-order* = *strict-order* +
assumes *trichotomy*: *a* = *c* $\Longrightarrow$ ($\neg$ (*ls a c*) $\wedge$ $\neg$ (*ls c a*))

interpretation *nest*:*total-strict-order* ($\ll$)
proof
{ fix *a*::′*a nest*
show $\neg$ *a* $\ll$ *a*
by (*simp add*: *before-irrefl*) **} note** *irrefl-nest* = *this*

{fix *a c*::′*a nest*
assume *a* = *c*
show $\neg$ *a* $\ll$ *c* $\wedge$ $\neg$ *c* $\ll$ *a*
by (*simp add*: ‹*a* = *c*› *irrefl-nest*)**}** **note** *trichotomy-nest* = *this*

{fix *a c g*::′*a nest*
assume *a*:*a* $\ll$ *c* **and** *c*: *c* $\ll$ *g*
show *a* $\ll$ *g*
proof −
  from *a c* **have** *na*:*NEST a* **and** *nc*:*NEST c* **and** *ng*:*NEST g* **unfolding** *before-def* **by** *auto*
  from *na* **obtain** *i* **where** *i*:*a* = *BEGIN i* $\vee$ *a* = *END i* **and** *wdi*:$\mathcal{I}$ *i* **unfolding** *NEST-def* **by** *auto*
  from *nc* **obtain** *j* **where** *j*:*c* = *BEGIN j* $\vee$ *c* = *END j* **and** *wdj*:$\mathcal{I}$ *j* **unfolding** *NEST-def* **by** *auto*
  from *ng* **obtain** *u* **where** *u*:*g* = *BEGIN u* $\vee$ *g* = *END u* **and** *wdu*:$\mathcal{I}$ *u* **unfolding** *NEST-def* **by** *auto*
  from *i j u* **show** *?thesis*
  proof (*elim disjE*, *auto*)
   assume *abi*:*a* = *BEGIN i* **and** *cbj*:*c* = *BEGIN j* **and** *gbu*:*g* = *BEGIN u*
    from *abi cbj a wdi wdj* **have** (*i*,*j*) $\in$ *b* $\cup$ *m* $\cup$ *ov* $\cup$ f^{-1} $\cup$ d^{-1} **using** *BEGIN-before* **by** *auto*
   moreover from *cbj gbu c wdj wdu* **have** (*j*,*u*) $\in$ *b* $\cup$ *m* $\cup$ *ov* $\cup$ f^{-1} $\cup$ d^{-1} **using** *BEGIN-before* **by** *auto*

ultimately have *c1*:(i,u) $\in b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
using *cbeta2-beta2* **by** *blast*

then have $a \ll g$ **by** (*simp add*: *BEGIN-before abi gbu wdi wdu*)

thus *BEGIN i* $\ll$ *BEGIN u* **using** *abi gbu* **by** *auto*
next
assume *abi*:a = *BEGIN i* **and** *cbj*:c = *BEGIN j* **and** *geu*:g = *END u*
from *abi cbj a wdi wdj* **have** (i,j) $\in b \cup m \cup ov \cup f^{-1} \cup d^{-1}$ **using** *BEGIN-before* **by** *auto*
moreover from *cbj geu c wdj wdu* **have** (j,u) : $e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1}$ **using** *BEGIN-END-before* **by** *auto*
ultimately have (i,u) $\in e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1}$
using *cbeta2-gammabm* **by** *blast*

then have $a \ll g$
by (*simp add*: *BEGIN-END-before abi geu wdi wdj wdu*)

thus *BEGIN i* $\ll$ *END u* **using** *abi geu* **by** *auto*
next
assume *abi*:a = *BEGIN i* **and** *cej*:c = *END j* **and** *gbu*:g = *BEGIN u*
from *abi cej a wdi wdj* **have** *ij*:(i,j) : $e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1}$ **using** *BEGIN-END-before* **by** *auto*
from *cej gbu c wdj wdu* **have** (j,u) $\in b$ **using** *END-BEGIN-before* **by** *auto*
with *ij* **have** (i,u) $\in b \cup m \cup ov \cup f\hat{}{-1} \cup d\hat{}{-1}$
using *ebmovovissifsiddib* **by** (*auto*)

thus *BEGIN i* $\ll$ *BEGIN u*
by (*simp add*: *BEGIN-before abi gbu wdi wdu*)
next
assume *abi*:a = *BEGIN i* **and** *cej*:c = *END j* **and** *geu*:g = *END u*
with *a* **have** (i,j) $\in e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1}$
using *BEGIN-END-before wdi wdj* **by** *auto*
moreover from *cej geu c wdj wdu* **have** (j,u) $\in b \cup m \cup ov \cup s \cup d$
using *END-END-before* **by** *auto*
ultimately have (i,u) $\in b \cup m \cup ov \cup s \cup d \cup f\hat{}{-1} \cup d\hat{}{-1} \cup ov\hat{}{-1} \cup s^{-1} \cup f \cup e$
using *ebmovovissiffiddibmovsd* **by** *blast*

thus *BEGIN i* $\ll$ *END u* **using** *BEGIN-END-before wdi wdu* **by** *auto*
next
assume *aei*:a = *END i* **and** *cbj*:c = *BEGIN j* **and** *gbu*:g = *BEGIN u*
from *a aei cbj wdi wdj* **have** (i,j) $\in b$
using *END-BEGIN-before* **by** *auto*
moreover from *c cbj gbu wdj wdu* **have** (j,u) $\in b \cup m \cup ov \cup f^{-1} \cup d^{-1}$
using *BEGIN-before* **by** *auto*
ultimately have (i,u) : b **using** *cbb cbm cbov cbfi cbdi*

by (*simp-all*, *elim disjE*, *auto*)
thus *END* $i \ll$ *BEGIN* u **using** *END-BEGIN-before wdi wdu* **by** *auto*
next
assume *aei*:a = *END* i **and** *cbj*:c = *BEGIN* j **and** *geu*:g = *END* u
from a *aei cbj wdi wdj* **have** $(i,j) \in b$
using *END-BEGIN-before* **by** *auto*
moreover from c *cbj geu wdj wdu* **have** $(j,u) \in e \cup b \cup m \cup ov \cup ov^{-1} \cup s \cup s^{-1} \cup f \cup f^{-1} \cup d \cup d^{-1}$
using *BEGIN-END-before* **by** *auto*
ultimately have $(i,u) \in b \cup m \cup ov \cup s \cup d$
using *bebmovovissiffiddi* **by** *blast*
thus *END* $i \ll$ *END* u **using** *END-END-before wdi wdu* **by** *auto*
next
assume *aei*:a = *END* i **and** *cej*:c = *END* j **and** *gbu*:g = *BEGIN* u
from *aei cej wdi wdj* **have** $(i,j) \in b \cup m \cup ov \cup s \cup d$ **using** *END-END-before* a **by** *auto*
moreover from *cej gbu c wdj wdu* **have** $(j,u) \in b$ **using** *END-BEGIN-before* **by** *auto*
ultimately have $(i,u) \in b$
using *cbb cmb covb csb cdb*
by (*simp-all*, *elim disjE*, *auto*)
thus *END* $i \ll$ *BEGIN* u **using** *END-BEGIN-before wdi wdu* **by** *auto*
next
assume *aei*:a = *END* i **and** *cej*:c = *END* j **and** *geu*:g = *END* u
from *aei cej wdi wdj* **have** $(i,j) \in b \cup m \cup ov \cup s \cup d$ **using** *END-END-before* a **by** *auto*
moreover from *cej geu c wdj wdu* **have** $(j,u) \in b \cup m \cup ov \cup s \cup d$ **using** *END-END-before* **by** *auto*
ultimately have $(i,u) \in b \cup m \cup ov \cup s \cup d$
using *calpha1-alpha1* **by** *auto*
thus *END* $i \ll$ *END* u **using** *END-END-before wdi wdu* **by** *auto*
qed
qed} **note** *trans-nest* = *this*

{ **fix** a c::$'a$ *nest*
assume a:$a \ll c$
show $\neg\ c \ll a$
apply (*rule ccontr*, *auto*)
using a *irrefl-nest trans-nest* **by** *blast*}
qed

end