

# The Akra–Bazzi theorem and the Master theorem

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## Abstract

This article contains a formalisation of the Akra–Bazzi method [1] based on a proof by Leighton [2]. It is a generalisation of the well-known Master Theorem for analysing the complexity of Divide & Conquer algorithms. We also include a generalised version of the Master theorem based on the Akra–Bazzi theorem, which is easier to apply than the Akra–Bazzi theorem itself.

Some proof methods that facilitate applying the Master theorem are also included. For a more detailed explanation of the formalisation and the proof methods, see the accompanying paper (publication forthcoming).

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## 1 Auxiliary lemmas

**theory** *Akra-Bazzi-Library*

**imports**

*Complex-Main*

*Landau-Symbols.Landau-More*

*Pure-ex.Guess*

**begin**

**lemma** *ln-mono*:  $0 < x \implies 0 < y \implies x \leq y \implies \ln (x::real) \leq \ln y$   
 <proof>

**lemma** *ln-mono-strict*:  $0 < x \implies 0 < y \implies x < y \implies \ln (x::real) < \ln y$   
 <proof>

**declare** *DERIV-pow*[*THEN DERIV-chain2, derivative-intros*]

**lemma** *sum-pos'*:

**assumes** *finite I*

**assumes**  $\exists x \in I. f x > (0 :: - :: \text{linordered-ab-group-add})$

**assumes**  $\bigwedge x. x \in I \implies f x \geq 0$

**shows**  $\text{sum } f I > 0$

<proof>

**lemma** *min-mult-left*:

**assumes**  $(x::real) > 0$

**shows**  $x * \text{min } y z = \text{min } (x*y) (x*z)$

<proof>

**lemma** *max-mult-left*:

**assumes**  $(x::real) > 0$

**shows**  $x * \text{max } y z = \text{max } (x*y) (x*z)$

<proof>

**lemma** *DERIV-nonneg-imp-mono*:

**assumes**  $\bigwedge t. t \in \{x..y\} \implies (f \text{ has-field-derivative } f' t) (at t)$

**assumes**  $\bigwedge t. t \in \{x..y\} \implies f' t \geq 0$

**assumes**  $(x::real) \leq y$

**shows**  $(f x :: real) \leq f y$

*<proof>*

**lemma** *eventually-conjE*: *eventually*  $(\lambda x. P x \wedge Q x) F \implies (\text{eventually } P F \implies \text{eventually } Q F \implies R) \implies R$   
*<proof>*

**lemma** *real-natfloor-nat*:  $x \in \mathbf{N} \implies \text{real } (\text{nat } \lfloor x \rfloor) = x$  *<proof>*

**lemma** *eventually-natfloor*:  
**assumes** *eventually*  $P$  (*at-top* :: *nat filter*)  
**shows** *eventually*  $(\lambda x. P (\text{nat } \lfloor x \rfloor))$  (*at-top* :: *real filter*)  
*<proof>*

**lemma** *tendsto-0-smallo-1*:  $f \in o(\lambda x. 1 :: \text{real}) \implies (f \longrightarrow 0)$  *at-top*  
*<proof>*

**lemma** *smallo-1-tendsto-0*:  $(f \longrightarrow 0)$  *at-top*  $\implies f \in o(\lambda x. 1 :: \text{real})$   
*<proof>*

**lemma** *filterlim-at-top-smallomega-1*:  
**assumes**  $f \in \omega[F](\lambda x. 1 :: \text{real})$  *eventually*  $(\lambda x. f x > 0)$   $F$   
**shows** *filterlim*  $f$  *at-top*  $F$   
*<proof>*

**lemma** *smallo-imp-abs-less-real*:  
**assumes**  $f \in o[F](g)$  *eventually*  $(\lambda x. g x > (0 :: \text{real}))$   $F$   
**shows** *eventually*  $(\lambda x. |f x| < g x)$   $F$   
*<proof>*

**lemma** *smallo-imp-less-real*:  
**assumes**  $f \in o[F](g)$  *eventually*  $(\lambda x. g x > (0 :: \text{real}))$   $F$   
**shows** *eventually*  $(\lambda x. f x < g x)$   $F$   
*<proof>*

**lemma** *smallo-imp-le-real*:  
**assumes**  $f \in o[F](g)$  *eventually*  $(\lambda x. g x \geq (0 :: \text{real}))$   $F$   
**shows** *eventually*  $(\lambda x. f x \leq g x)$   $F$   
*<proof>*

**lemma** *filterlim-at-right*:  
*filterlim*  $f$  (*at-right*  $a$ )  $F \iff \text{eventually } (\lambda x. f x > a) F \wedge \text{filterlim } f$  (*nhds*  $a$ )  $F$   
*<proof>*

**lemma** *one-plus-x-powr-approx-ex*:  
**assumes**  $x: \text{abs } (x :: \text{real}) \leq 1/2$   
**obtains**  $t$  **where**  $\text{abs } t < 1/2$   $(1 + x)$  *powr*  $p =$   
 $1 + p * x + p * (p - 1) * (1 + t)$  *powr*  $(p - 2) / 2 * x ^ 2$

*<proof>*

**lemma** *powr-lower-bound*:  $\llbracket (l::\text{real}) > 0; l \leq x; x \leq u \rrbracket \implies \min (l \text{ powr } z) (u \text{ powr } z) \leq x \text{ powr } z$   
*<proof>*

**lemma** *powr-upper-bound*:  $\llbracket (l::\text{real}) > 0; l \leq x; x \leq u \rrbracket \implies \max (l \text{ powr } z) (u \text{ powr } z) \geq x \text{ powr } z$   
*<proof>*

**lemma** *one-plus-x-powr-Taylor2*:

**obtains** *k* **where**  $\bigwedge x. \text{abs } (x::\text{real}) \leq 1/2 \implies \text{abs } ((1 + x) \text{ powr } p - 1 - p*x) \leq k*x^2$   
*<proof>*

**lemma** *one-plus-x-powr-Taylor2-bigo*:

**assumes** *lim*:  $(f \longrightarrow 0) F$   
**shows**  $(\lambda x. (1 + f x) \text{ powr } (p::\text{real}) - 1 - p * f x) \in O[F](\lambda x. f x^2)$   
*<proof>*

**lemma** *one-plus-x-powr-Taylor1-bigo*:

**assumes** *lim*:  $(f \longrightarrow 0) F$   
**shows**  $(\lambda x. (1 + f x) \text{ powr } (p::\text{real}) - 1) \in O[F](\lambda x. f x)$   
*<proof>*

**lemma** *x-times-x-minus-1-nonneg*:  $x \leq 0 \vee x \geq 1 \implies (x:::\text{linordered-idom}) * (x - 1) \geq 0$   
*<proof>*

**lemma** *x-times-x-minus-1-nonpos*:  $x \geq 0 \implies x \leq 1 \implies (x:::\text{linordered-idom}) * (x - 1) \leq 0$   
*<proof>*

**lemma** *real-powr-at-bot*:

**assumes**  $(a::\text{real}) > 1$   
**shows**  $((\lambda x. a \text{ powr } x) \longrightarrow 0) \text{ at-bot}$   
*<proof>*

**lemma** *real-powr-at-bot-neg*:

**assumes**  $(a::\text{real}) > 0 \ a < 1$   
**shows** *filterlim*  $(\lambda x. a \text{ powr } x) \text{ at-top at-bot}$   
*<proof>*

**lemma** *real-powr-at-top-neg*:

**assumes**  $(a::\text{real}) > 0 \ a < 1$   
**shows**  $((\lambda x. a \text{ powr } x) \longrightarrow 0) \text{ at-top}$   
*<proof>*

**lemma** *eventually-nat-real*:

```

assumes eventually  $P$  (at-top :: real filter)
shows eventually  $(\lambda x. P (\text{real } x))$  (at-top :: nat filter)
  <proof>

```

**end**

## 2 Asymptotic bounds

**theory** *Akra-Bazzi-Asymptotics*

**imports**

*Complex-Main*

*Akra-Bazzi-Library*

*HOL-Library.Landau-Symbols*

**begin**

**locale** *akra-bazzi-asymptotics-bep* =

**fixes**  $b\ e\ p\ hb$  :: *real*

**assumes** *bep*:  $b > 0\ b < 1\ e > 0\ hb > 0$

**begin**

**context**

**begin**

Functions that are negligible w.r.t.  $\ln (b * x)\ \text{powr } (e / 2 + 1)$ .

**private abbreviation** (*input*) *negl* :: (*real*  $\Rightarrow$  *real*)  $\Rightarrow$  *bool* **where**

$\text{negl } f \equiv f \in o(\lambda x. \ln (b*x)\ \text{powr } (-(e/2 + 1)))$

**private lemma** *neglD*:  $\text{negl } f \Longrightarrow c > 0 \Longrightarrow \text{eventually } (\lambda x. |f\ x| \leq c / \ln (b*x)\ \text{powr } (e/2+1))\ \text{at-top}$

<proof> **lemma** *negl-mult*:  $\text{negl } f \Longrightarrow \text{negl } g \Longrightarrow \text{negl } (\lambda x. f\ x * g\ x)$

<proof> **lemma** *ev4*:

**assumes** *g*: *negl g*

**shows** eventually  $(\lambda x. \ln (b*x)\ \text{powr } (-e/2) - \ln x\ \text{powr } (-e/2) \geq g\ x)\ \text{at-top}$

<proof> **lemma** *ev1*:

$\text{negl } (\lambda x. (1 + c * \text{inverse } b * \ln x\ \text{powr } (-(1+e)))\ \text{powr } p - 1)$

<proof> **lemma** *ev2-aux*:

**defines**  $f \equiv \lambda x. (1 + 1/\ln (b*x) * \ln (1 + hb / b * \ln x\ \text{powr } (-1-e)))\ \text{powr } (-e/2)$

**obtains** *h* **where** eventually  $(\lambda x. f\ x \geq 1 + h\ x)\ \text{at-top}\ h \in o(\lambda x. 1 / \ln x)$

<proof> **lemma** *ev2*:

**defines**  $f \equiv \lambda x. \ln (b * x + hb * x / \ln x\ \text{powr } (1 + e))\ \text{powr } (-e/2)$

**obtains** *h* **where**

*negl h*

eventually  $(\lambda x. f\ x \geq \ln (b * x)\ \text{powr } (-e/2) + h\ x)\ \text{at-top}$

eventually  $(\lambda x. |\ln (b * x)\ \text{powr } (-e/2) + h\ x| < 1)\ \text{at-top}$

<proof> **lemma** *ev21*:

**obtains** *g* **where**

*negl g*

eventually  $(\lambda x. 1 + \ln (b * x + hb * x / \ln x\ \text{powr } (1 + e))\ \text{powr } (-e/2) \geq$

$1 + \ln (b * x) \text{ powr } (-e/2) + g x) \text{ at-top}$   
 $\text{eventually } (\lambda x. 1 + \ln (b * x) \text{ powr } (-e/2) + g x > 0) \text{ at-top}$   
 <proof> **lemma ev22:**  
**obtains g where**  
 $\text{negl } g$   
 $\text{eventually } (\lambda x. 1 - \ln (b * x + hb * x / \ln x \text{ powr } (1 + e)) \text{ powr } (-e/2) \leq$   
 $1 - \ln (b * x) \text{ powr } (-e/2) - g x) \text{ at-top}$   
 $\text{eventually } (\lambda x. 1 - \ln (b * x) \text{ powr } (-e/2) - g x > 0) \text{ at-top}$   
 <proof>

**lemma asymptotics1:**  
**shows eventually**  $(\lambda x.$   
 $(1 + c * \text{inverse } b * \ln x \text{ powr } -(1+e)) \text{ powr } p *$   
 $(1 + \ln (b * x + hb * x / \ln x \text{ powr } (1 + e)) \text{ powr } (- e / 2)) \geq$   
 $1 + (\ln x \text{ powr } (-e/2))) \text{ at-top}$   
 <proof>

**lemma asymptotics2:**  
**shows eventually**  $(\lambda x.$   
 $(1 + c * \text{inverse } b * \ln x \text{ powr } -(1+e)) \text{ powr } p *$   
 $(1 - \ln (b * x + hb * x / \ln x \text{ powr } (1 + e)) \text{ powr } (- e / 2)) \leq$   
 $1 - (\ln x \text{ powr } (-e/2))) \text{ at-top}$   
 <proof>

**lemma asymptotics3:**  $\text{eventually } (\lambda x. (1 + (\ln x \text{ powr } (-e/2))) / 2 \leq 1) \text{ at-top}$   
 (is eventually  $(\lambda x. ?f x \leq 1) -$ )  
 <proof>

**lemma asymptotics4:**  $\text{eventually } (\lambda x. (1 - (\ln x \text{ powr } (-e/2))) * 2 \geq 1) \text{ at-top}$   
 (is eventually  $(\lambda x. ?f x \geq 1) -$ )  
 <proof>

**lemma asymptotics5:**  $\text{eventually } (\lambda x. \ln (b*x - hb*x*\ln x \text{ powr } -(1+e)) \text{ powr } (-e/2) < 1) \text{ at-top}$   
 <proof>

**lemma asymptotics6:**  $\text{eventually } (\lambda x. hb / \ln x \text{ powr } (1 + e) < b/2) \text{ at-top}$   
**and asymptotics7:**  $\text{eventually } (\lambda x. hb / \ln x \text{ powr } (1 + e) < (1 - b) / 2) \text{ at-top}$   
**and asymptotics8:**  $\text{eventually } (\lambda x. x*(1 - b - hb / \ln x \text{ powr } (1 + e)) > 1)$   
 $\text{at-top}$   
 <proof>

**end**  
**end**

**definition akra-bazzi-asymptotic1**  $b \ hb \ e \ p \ x \longleftrightarrow$   
 $(1 - hb * \text{inverse } b * \ln x \text{ powr } -(1+e)) \text{ powr } p * (1 + \ln (b*x + hb*x/\ln x$

$\text{powr } (1+e) \text{ powr } (-e/2)$   
 $\geq 1 + (\ln x \text{ powr } (-e/2) :: \text{real})$   
**definition** *akra-bazzi-asymptotic1'*  $b \text{ hb } e \text{ p } x \longleftrightarrow$   
 $(1 + \text{hb} * \text{inverse } b * \ln x \text{ powr } -(1+e)) \text{ powr } p * (1 + \ln (b*x + \text{hb}*x/\ln x$   
 $\text{powr } (1+e) \text{ powr } (-e/2))$   
 $\geq 1 + (\ln x \text{ powr } (-e/2) :: \text{real})$   
**definition** *akra-bazzi-asymptotic2*  $b \text{ hb } e \text{ p } x \longleftrightarrow$   
 $(1 + \text{hb} * \text{inverse } b * \ln x \text{ powr } -(1+e)) \text{ powr } p * (1 - \ln (b*x + \text{hb}*x/\ln x$   
 $\text{powr } (1+e) \text{ powr } (-e/2))$   
 $\leq 1 - \ln x \text{ powr } (-e/2 :: \text{real})$   
**definition** *akra-bazzi-asymptotic2'*  $b \text{ hb } e \text{ p } x \longleftrightarrow$   
 $(1 - \text{hb} * \text{inverse } b * \ln x \text{ powr } -(1+e)) \text{ powr } p * (1 - \ln (b*x + \text{hb}*x/\ln x$   
 $\text{powr } (1+e) \text{ powr } (-e/2))$   
 $\leq 1 - \ln x \text{ powr } (-e/2 :: \text{real})$   
**definition** *akra-bazzi-asymptotic3*  $e \text{ x} \longleftrightarrow (1 + (\ln x \text{ powr } (-e/2))) / 2 \leq$   
 $(1 :: \text{real})$   
**definition** *akra-bazzi-asymptotic4*  $e \text{ x} \longleftrightarrow (1 - (\ln x \text{ powr } (-e/2))) * 2 \geq$   
 $(1 :: \text{real})$   
**definition** *akra-bazzi-asymptotic5*  $b \text{ hb } e \text{ x} \longleftrightarrow$   
 $\ln (b*x - \text{hb}*x*\ln x \text{ powr } -(1+e)) \text{ powr } (-e/2 :: \text{real}) < 1$   
  
**definition** *akra-bazzi-asymptotic6*  $b \text{ hb } e \text{ x} \longleftrightarrow \text{hb} / \ln x \text{ powr } (1 + e :: \text{real}) <$   
 $b/2$   
**definition** *akra-bazzi-asymptotic7*  $b \text{ hb } e \text{ x} \longleftrightarrow \text{hb} / \ln x \text{ powr } (1 + e :: \text{real}) <$   
 $(1 - b) / 2$   
**definition** *akra-bazzi-asymptotic8*  $b \text{ hb } e \text{ x} \longleftrightarrow x*(1 - b - \text{hb} / \ln x \text{ powr } (1 +$   
 $e :: \text{real})) > 1$   
  
**definition** *akra-bazzi-asymptotics*  $b \text{ hb } e \text{ p } x \longleftrightarrow$   
 $\text{akra-bazzi-asymptotic1 } b \text{ hb } e \text{ p } x \wedge \text{akra-bazzi-asymptotic1}' b \text{ hb } e \text{ p } x \wedge$   
 $\text{akra-bazzi-asymptotic2 } b \text{ hb } e \text{ p } x \wedge \text{akra-bazzi-asymptotic2}' b \text{ hb } e \text{ p } x \wedge$   
 $\text{akra-bazzi-asymptotic3 } e \text{ x} \wedge \text{akra-bazzi-asymptotic4 } e \text{ x} \wedge \text{akra-bazzi-asymptotic5}$   
 $b \text{ hb } e \text{ x} \wedge$   
 $\text{akra-bazzi-asymptotic6 } b \text{ hb } e \text{ x} \wedge \text{akra-bazzi-asymptotic7 } b \text{ hb } e \text{ x} \wedge$   
 $\text{akra-bazzi-asymptotic8 } b \text{ hb } e \text{ x}$   
  
**lemmas** *akra-bazzi-asymptotic-defs* =  
 $\text{akra-bazzi-asymptotic1-def } \text{akra-bazzi-asymptotic1}'\text{-def}$   
 $\text{akra-bazzi-asymptotic2-def } \text{akra-bazzi-asymptotic2}'\text{-def } \text{akra-bazzi-asymptotic3-def}$   
 $\text{akra-bazzi-asymptotic4-def } \text{akra-bazzi-asymptotic5-def } \text{akra-bazzi-asymptotic6-def}$   
 $\text{akra-bazzi-asymptotic7-def } \text{akra-bazzi-asymptotic8-def } \text{akra-bazzi-asymptotics-def}$   
  
**lemma** *akra-bazzi-asymptotics*:  
**assumes**  $\bigwedge b. b \in \text{set } bs \implies b \in \{0 < .. < 1\}$   
**assumes**  $\text{hb} > 0 \text{ e} > 0$   
**shows** *eventually*  $(\lambda x. \forall b \in \text{set } bs. \text{akra-bazzi-asymptotics } b \text{ hb } e \text{ p } x)$  *at-top*  
*<proof>*  
  
**end**

### 3 The continuous Akra-Bazzi theorem

```

theory Akra-Bazzi-Real
imports
  Complex-Main
  Akra-Bazzi-Asymptotics
begin

```

We want to be generic over the integral definition used; we fix some arbitrary notions of integrability and integral and assume just the properties we need. The user can then instantiate the theorems with any desired integral definition.

```

locale akra-bazzi-integral =
  fixes integrable :: (real  $\Rightarrow$  real)  $\Rightarrow$  real  $\Rightarrow$  real  $\Rightarrow$  bool
  and integral :: (real  $\Rightarrow$  real)  $\Rightarrow$  real  $\Rightarrow$  real  $\Rightarrow$  real
  assumes integrable-const:  $c \geq 0 \implies \text{integrable } (\lambda-. c) a b$ 
  and integral-const:  $c \geq 0 \implies a \leq b \implies \text{integral } (\lambda-. c) a b = (b - a) * c$ 
  and integrable-subinterval:
     $\text{integrable } f a b \implies a \leq a' \implies b' \leq b \implies \text{integrable } f a' b'$ 
  and integral-le:
     $\text{integrable } f a b \implies \text{integrable } g a b \implies (\bigwedge x. x \in \{a..b\} \implies f x \leq g x)$ 
 $\implies$ 
     $\text{integral } f a b \leq \text{integral } g a b$ 
  and integral-combine:
     $a \leq c \implies c \leq b \implies \text{integrable } f a b \implies$ 
     $\text{integral } f a c + \text{integral } f c b = \text{integral } f a b$ 
begin
lemma integral-nonneg:
   $a \leq b \implies \text{integrable } f a b \implies (\bigwedge x. x \in \{a..b\} \implies f x \geq 0) \implies \text{integral } f a b \geq 0$ 
   $\langle \text{proof} \rangle$ 
end

```

```

declare sum.cong[fundef-cong]

```

```

lemma strict-mono-imp-ex1-real:
  fixes f :: real  $\Rightarrow$  real
  assumes lim-neg-inf: LIM  $x$  at-bot.  $f x \text{ :> } \text{at-top}$ 
  assumes lim-inf:  $(f \longrightarrow z) \text{ at-top}$ 
  assumes mono:  $\bigwedge a b. a < b \implies f b < f a$ 
  assumes cont:  $\bigwedge x. \text{isCont } f x$ 
  assumes y-greater-z:  $z < y$ 
  shows  $\exists !x. f x = y$ 
   $\langle \text{proof} \rangle$ 

```

The parameter  $p$  in the Akra-Bazzi theorem always exists and is unique.

```

definition akra-bazzi-exponent :: real list  $\Rightarrow$  real list  $\Rightarrow$  real where
  akra-bazzi-exponent  $as bs \equiv (\text{THE } p. (\sum i < \text{length } as. as!i * bs!i \text{ powr } p) = 1)$ 

```

```

locale akra-bazzi-params =
  fixes  $k :: \text{nat}$  and  $as\ bs :: \text{real list}$ 
  assumes  $\text{length-as: length } as = k$ 
  and  $\text{length-bs: length } bs = k$ 
  and  $k\text{-not-0: } k \neq 0$ 
  and  $a\text{-ge-0: } a \in \text{set } as \implies a \geq 0$ 
  and  $b\text{-bounds: } b \in \text{set } bs \implies b \in \{0 < .. < 1\}$ 
begin

abbreviation  $p :: \text{real}$  where  $p \equiv \text{akra-bazzi-exponent } as\ bs$ 

lemma  $p\text{-def: } p = (\text{THE } p. (\sum_{i < k}. as!i * bs!i \text{ powr } p) = 1)$ 
   $\langle \text{proof} \rangle$ 

lemma  $b\text{-pos: } b \in \text{set } bs \implies b > 0$  and  $b\text{-less-1: } b \in \text{set } bs \implies b < 1$ 
   $\langle \text{proof} \rangle$ 

lemma  $as\text{-nonempty [simp]: } as \neq []$  and  $bs\text{-nonempty [simp]: } bs \neq []$ 
   $\langle \text{proof} \rangle$ 

lemma  $a\text{-in-as [intro, simp]: } i < k \implies as ! i \in \text{set } as$ 
   $\langle \text{proof} \rangle$ 

lemma  $b\text{-in-bs [intro, simp]: } i < k \implies bs ! i \in \text{set } bs$ 
   $\langle \text{proof} \rangle$ 

end

locale akra-bazzi-params-nonzero =
  fixes  $k :: \text{nat}$  and  $as\ bs :: \text{real list}$ 
  assumes  $\text{length-as: length } as = k$ 
  and  $\text{length-bs: length } bs = k$ 
  and  $a\text{-ge-0: } a \in \text{set } as \implies a \geq 0$ 
  and  $ex\text{-a-pos: } \exists a \in \text{set } as. a > 0$ 
  and  $b\text{-bounds: } b \in \text{set } bs \implies b \in \{0 < .. < 1\}$ 
begin

sublocale akra-bazzi-params  $k\ as\ bs$ 
   $\langle \text{proof} \rangle$ 

lemma  $akra\text{-bazzi-p-strict-mono:}$ 
  assumes  $x < y$ 
  shows  $(\sum_{i < k}. as!i * bs!i \text{ powr } y) < (\sum_{i < k}. as!i * bs!i \text{ powr } x)$ 
   $\langle \text{proof} \rangle$ 

lemma  $akra\text{-bazzi-p-mono:}$ 
  assumes  $x \leq y$ 

```

**shows**  $(\sum_{i < k}. as!i * bs!i \text{ powr } y) \leq (\sum_{i < k}. as!i * bs!i \text{ powr } x)$   
*<proof>*

**lemma** *akra-bazzi-p-unique*:  
 $\exists! p. (\sum_{i < k}. as!i * bs!i \text{ powr } p) = 1$   
*<proof>*

**lemma** *p-props*:  $(\sum_{i < k}. as!i * bs!i \text{ powr } p) = 1$   
**and** *p-unique*:  $(\sum_{i < k}. as!i * bs!i \text{ powr } p') = 1 \implies p = p'$   
*<proof>*

**lemma** *p-greaterI*:  $1 < (\sum_{i < k}. as!i * bs!i \text{ powr } p') \implies p' < p$   
*<proof>*

**lemma** *p-lessI*:  $1 > (\sum_{i < k}. as!i * bs!i \text{ powr } p') \implies p' > p$   
*<proof>*

**lemma** *p-geI*:  $1 \leq (\sum_{i < k}. as!i * bs!i \text{ powr } p') \implies p' \leq p$   
*<proof>*

**lemma** *p-leI*:  $1 \geq (\sum_{i < k}. as!i * bs!i \text{ powr } p') \implies p' \geq p$   
*<proof>*

**lemma** *p-boundsI*:  $(\sum_{i < k}. as!i * bs!i \text{ powr } x) \leq 1 \wedge (\sum_{i < k}. as!i * bs!i \text{ powr } y) \geq 1 \implies p \in \{y..x\}$   
*<proof>*

**lemma** *p-boundsI'*:  $(\sum_{i < k}. as!i * bs!i \text{ powr } x) < 1 \wedge (\sum_{i < k}. as!i * bs!i \text{ powr } y) > 1 \implies p \in \{y < .. < x\}$   
*<proof>*

**lemma** *p-nonneg*: *sum-list as*  $\geq 1 \implies p \geq 0$   
*<proof>*

**end**

**locale** *akra-bazzi-real-recursion* =

**fixes** *as bs* :: *real list* **and** *hs* :: (*real*  $\implies$  *real*) *list* **and** *k* :: *nat* **and** *x<sub>0</sub> x<sub>1</sub> hb e p*  
:: *real*

**assumes** *length-as*: *length as* = *k*

**and** *length-bs*: *length bs* = *k*

**and** *length-hs*: *length hs* = *k*

**and** *k-not-0*: *k*  $\neq 0$

**and** *a-ge-0*: *a*  $\in$  *set as*  $\implies a \geq 0$

**and** *b-bounds*: *b*  $\in$  *set bs*  $\implies b \in \{0 < .. < 1\}$

**and**  $x0\text{-ge-1}$ :  $x_0 \geq 1$   
**and**  $x0\text{-le-x1}$ :  $x_0 \leq x_1$   
**and**  $x1\text{-ge}$ :  $b \in \text{set } bs \implies x_1 \geq 2 * x_0 * \text{inverse } b$

**and**  $e\text{-pos}$ :  $e > 0$   
**and**  $h\text{-bounds}$ :  $x \geq x_1 \implies h \in \text{set } hs \implies |h x| \leq hb * x / \ln x \text{ powr } (1 + e)$

**and**  $asymptotics$ :  $x \geq x_0 \implies b \in \text{set } bs \implies \text{akra-bazzi-asymptotics } b \text{ } hb \text{ } e \text{ } p \text{ } x$   
**begin**

**sublocale**  $\text{akra-bazzi-params } k \text{ as } bs$   
 $\langle \text{proof} \rangle$

**lemma**  $h\text{-in-hs}[\text{intro}, \text{simp}]$ :  $i < k \implies hs ! i \in \text{set } hs$   
 $\langle \text{proof} \rangle$

**lemma**  $x1\text{-gt-1}$ :  $x_1 > 1$   
 $\langle \text{proof} \rangle$

**lemma**  $x1\text{-ge-1}$ :  $x_1 \geq 1$   $\langle \text{proof} \rangle$

**lemma**  $x1\text{-pos}$ :  $x_1 > 0$   $\langle \text{proof} \rangle$

**lemma**  $bx\text{-le-x}$ :  $x \geq 0 \implies b \in \text{set } bs \implies b * x \leq x$   
 $\langle \text{proof} \rangle$

**lemma**  $x0\text{-pos}$ :  $x_0 > 0$   $\langle \text{proof} \rangle$

**lemma**  
**assumes**  $x \geq x_0 \text{ } b \in \text{set } bs$   
**shows**  $x0\text{-hb-bound0}$ :  $hb / \ln x \text{ powr } (1 + e) < b/2$   
**and**  $x0\text{-hb-bound1}$ :  $hb / \ln x \text{ powr } (1 + e) < (1 - b) / 2$   
**and**  $x0\text{-hb-bound2}$ :  $x*(1 - b - hb / \ln x \text{ powr } (1 + e)) > 1$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{step-diff}$ :  
**assumes**  $i < k \text{ } x \geq x_1$   
**shows**  $bs ! i * x + (hs ! i) x + 1 < x$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{step-le-x}$ :  $i < k \implies x \geq x_1 \implies bs ! i * x + (hs ! i) x \leq x$   
 $\langle \text{proof} \rangle$

**lemma**  $x0\text{-hb-bound0'}$ :  $\bigwedge x b. x \geq x_0 \implies b \in \text{set } bs \implies hb / \ln x \text{ powr } (1 + e) < b$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{step-pos}$ :

**assumes**  $i < k$   $x \geq x_1$   
**shows**  $bs ! i * x + (hs ! i) x > 0$   
 ⟨proof⟩

**lemma** *step-nonneg*:  $i < k \implies x \geq x_1 \implies bs ! i * x + (hs ! i) x \geq 0$   
 ⟨proof⟩

**lemma** *step-nonneg'*:  $i < k \implies x \geq x_1 \implies bs ! i + (hs ! i) x / x \geq 0$   
 ⟨proof⟩

**lemma** *hb-nonneg*:  $hb \geq 0$   
 ⟨proof⟩

**lemma** *x0-hb-bound3*:  
**assumes**  $x \geq x_1$   $i < k$   
**shows**  $x - (bs ! i * x + (hs ! i) x) \leq x$   
 ⟨proof⟩

**lemma** *x0-hb-bound4*:  
**assumes**  $x \geq x_1$   $i < k$   
**shows**  $(bs ! i + (hs ! i) x / x) > bs ! i / 2$   
 ⟨proof⟩

**lemma** *x0-hb-bound4'*:  $x \geq x_1 \implies i < k \implies (bs ! i + (hs ! i) x / x) \geq bs ! i / 2$   
 ⟨proof⟩

**lemma** *x0-hb-bound5*:  
**assumes**  $x \geq x_1$   $i < k$   
**shows**  $(bs ! i + (hs ! i) x / x) \leq bs ! i * 3/2$   
 ⟨proof⟩

**lemma** *x0-hb-bound6*:  
**assumes**  $x \geq x_1$   $i < k$   
**shows**  $x * ((1 - bs ! i) / 2) \leq x - (bs ! i * x + (hs ! i) x)$   
 ⟨proof⟩

**lemma** *x0-hb-bound7*:  
**assumes**  $x \geq x_1$   $i < k$   
**shows**  $bs ! i * x + (hs ! i) x > x_0$   
 ⟨proof⟩

**lemma** *x0-hb-bound7'*:  $x \geq x_1 \implies i < k \implies bs ! i * x + (hs ! i) x > 1$   
 ⟨proof⟩

**lemma** *x0-hb-bound8*:  
**assumes**  $x \geq x_1$   $i < k$   
**shows**  $bs ! i * x - hb * x / \ln x \text{ powr } (1+e) > x_0$   
 ⟨proof⟩

**lemma** *x0-hb-bound8'*:

**assumes**  $x \geq x_1$   $i < k$

**shows**  $bs!i*x + hb * x / \ln x \text{ powr } (1+e) > x_0$

*<proof>*

**lemma**

**assumes**  $x \geq x_0$

**shows** *asymptotics1*:  $i < k \implies 1 + \ln x \text{ powr } (-e/2) \leq$   
 $(1 - hb * \text{inverse } (bs!i) * \ln x \text{ powr } -(1+e)) \text{ powr } p *$   
 $(1 + \ln (bs!i*x + hb*x/\ln x \text{ powr } (1+e)) \text{ powr } (-e/2))$

**and** *asymptotics2*:  $i < k \implies 1 - \ln x \text{ powr } (-e/2) \geq$   
 $(1 + hb * \text{inverse } (bs!i) * \ln x \text{ powr } -(1+e)) \text{ powr } p *$   
 $(1 - \ln (bs!i*x + hb*x/\ln x \text{ powr } (1+e)) \text{ powr } (-e/2))$

**and** *asymptotics1'*:  $i < k \implies 1 + \ln x \text{ powr } (-e/2) \leq$   
 $(1 + hb * \text{inverse } (bs!i) * \ln x \text{ powr } -(1+e)) \text{ powr } p *$   
 $(1 + \ln (bs!i*x + hb*x/\ln x \text{ powr } (1+e)) \text{ powr } (-e/2))$

**and** *asymptotics2'*:  $i < k \implies 1 - \ln x \text{ powr } (-e/2) \geq$   
 $(1 - hb * \text{inverse } (bs!i) * \ln x \text{ powr } -(1+e)) \text{ powr } p *$   
 $(1 - \ln (bs!i*x + hb*x/\ln x \text{ powr } (1+e)) \text{ powr } (-e/2))$

**and** *asymptotics3*:  $(1 + \ln x \text{ powr } (-e/2)) / 2 \leq 1$

**and** *asymptotics4*:  $(1 - \ln x \text{ powr } (-e/2)) * 2 \geq 1$

**and** *asymptotics5*:  $i < k \implies \ln (bs!i*x - hb*x*\ln x \text{ powr } -(1+e)) \text{ powr } (-e/2) < 1$   
*<proof>*

**lemma** *x0-hb-bound9*:

**assumes**  $x \geq x_1$   $i < k$

**shows**  $\ln (bs!i*x + (hs!i) x) \text{ powr } -(e/2) < 1$

*<proof>*

**definition** *akra-bazzi-measure* :: *real*  $\implies$  *nat* **where**

*akra-bazzi-measure*  $x = \text{nat } \lceil x \rceil$

**lemma** *akra-bazzi-measure-decreases*:

**assumes**  $x \geq x_1$   $i < k$

**shows** *akra-bazzi-measure*  $(bs!i*x + (hs!i) x) < \text{akra-bazzi-measure } x$

*<proof>*

**lemma** *akra-bazzi-induct*[*consumes 1, case-names base rec*]:

**assumes**  $x \geq x_0$

**assumes** *base*:  $\bigwedge x. x \geq x_0 \implies x \leq x_1 \implies P x$

**assumes** *rec*:  $\bigwedge x. x > x_1 \implies (\bigwedge i. i < k \implies P (bs!i*x + (hs!i) x)) \implies P x$

**shows**  $P x$

*<proof>*

**end**

**locale** *akra-bazzi-real* = *akra-bazzi-real-recursion* +  
**fixes** *integrable integral*  
**assumes** *integral: akra-bazzi-integral integrable integral*  
**fixes**  $f :: \text{real} \Rightarrow \text{real}$   
**and**  $g :: \text{real} \Rightarrow \text{real}$   
**and**  $C :: \text{real}$   
**assumes** *p-props:*  $(\sum_{i < k. \text{as!}i * \text{bs!}i \text{ powr } p) = 1$   
**and** *f-base:*  $x \geq x_0 \Longrightarrow x \leq x_1 \Longrightarrow f x \geq 0$   
**and** *f-rec:*  $x > x_1 \Longrightarrow f x = g x + (\sum_{i < k. \text{as!}i * f (\text{bs!}i * x + (\text{hs!}i$   
*x))*  
**and** *g-nonneg:*  $x \geq x_0 \Longrightarrow g x \geq 0$   
**and** *C-bound:*  $b \in \text{set } \text{bs} \Longrightarrow x \geq x_1 \Longrightarrow C * x \leq b * x - \text{hb} * x / \ln x \text{ powr } (1 + e)$   
**and** *g-integrable:*  $x \geq x_0 \Longrightarrow \text{integrable } (\lambda u. g u / u \text{ powr } (p + 1)) x_0 x$   
**begin**

**interpretation** *akra-bazzi-integral integrable integral*  $\langle \text{proof} \rangle$

**lemma** *akra-bazzi-integrable:*

$a \geq x_0 \Longrightarrow a \leq b \Longrightarrow \text{integrable } (\lambda x. g x / x \text{ powr } (p + 1)) a b$   
 $\langle \text{proof} \rangle$

**definition** *g-approx* ::  $\text{nat} \Rightarrow \text{real} \Rightarrow \text{real}$  **where**

$g\text{-approx } i x = x \text{ powr } p * \text{integral } (\lambda u. g u / u \text{ powr } (p + 1)) (\text{bs!}i * x + (\text{hs!}i$   
*x) x*

**lemma** *f-nonneg:*  $x \geq x_0 \Longrightarrow f x \geq 0$

$\langle \text{proof} \rangle$

**definition** *f-approx* ::  $\text{real} \Rightarrow \text{real}$  **where**

$f\text{-approx } x = x \text{ powr } p * (1 + \text{integral } (\lambda u. g u / u \text{ powr } (p + 1)) x_0 x)$

**lemma** *f-approx-aux:*

**assumes**  $x \geq x_0$

**shows**  $1 + \text{integral } (\lambda u. g u / u \text{ powr } (p + 1)) x_0 x \geq 1$

$\langle \text{proof} \rangle$

**lemma** *f-approx-pos:*  $x \geq x_0 \Longrightarrow f\text{-approx } x > 0$

$\langle \text{proof} \rangle$

**lemma** *f-approx-nonneg:*  $x \geq x_0 \Longrightarrow f\text{-approx } x \geq 0$

$\langle \text{proof} \rangle$

**lemma** *f-approx-bounded-below:*

**obtains**  $c$  **where**  $\bigwedge x. x \geq x_0 \Longrightarrow x \leq x_1 \Longrightarrow f\text{-approx } x \geq c \ c > 0$

*<proof>*

**lemma** *asymptotics-aux*:

**assumes**  $x \geq x_1$   $i < k$

**assumes**  $s \equiv (\text{if } p \geq 0 \text{ then } 1 \text{ else } -1)$

**shows**  $(bs!i*x - s*hb*x*ln\ x\ \text{powr}\ -(1+e))\ \text{powr}\ p \leq (bs!i*x + (hs!i)\ x)\ \text{powr}\ p$   
**(is ?thesis1)**

**and**  $(bs!i*x + (hs!i)\ x)\ \text{powr}\ p \leq (bs!i*x + s*hb*x*ln\ x\ \text{powr}\ -(1+e))\ \text{powr}\ p$   
**(is ?thesis2)**

*<proof>*

**lemma** *asymptotics1'*:

**assumes**  $x \geq x_1$   $i < k$

**shows**  $(bs!i*x)\ \text{powr}\ p * (1 + ln\ x\ \text{powr}\ (-e/2)) \leq$

$(bs!i*x + (hs!i)\ x)\ \text{powr}\ p * (1 + ln\ (bs!i*x + (hs!i)\ x)\ \text{powr}\ (-e/2))$

*<proof>*

**lemma** *asymptotics2'*:

**assumes**  $x \geq x_1$   $i < k$

**shows**  $(bs!i*x + (hs!i)\ x)\ \text{powr}\ p * (1 - ln\ (bs!i*x + (hs!i)\ x)\ \text{powr}\ (-e/2))$   
 $\leq$

$(bs!i*x)\ \text{powr}\ p * (1 - ln\ x\ \text{powr}\ (-e/2))$

*<proof>*

**lemma** *Cx-le-step*:

**assumes**  $i < k$   $x \geq x_1$

**shows**  $C*x \leq bs!i*x + (hs!i)\ x$

*<proof>*

**end**

**locale** *akra-bazzi-nat-to-real = akra-bazzi-real-recursion +*

**fixes**  $f :: \text{nat} \Rightarrow \text{real}$

**and**  $g :: \text{real} \Rightarrow \text{real}$

**assumes**  $f\text{-base}: \text{real } x \geq x_0 \Longrightarrow \text{real } x \leq x_1 \Longrightarrow f\ x \geq 0$

**and**  $f\text{-rec}: \text{real } x > x_1 \Longrightarrow$

$f\ x = g\ (\text{real } x) + (\sum\ i < k. as!i * f\ (\text{nat } \lfloor bs!i * x + (hs!i)$

$(\text{real } x) \rfloor))$

**and**  $x0\text{-int}: \text{real } (\text{nat } \lfloor x_0 \rfloor) = x_0$

**begin**

**function**  $f' :: \text{real} \Rightarrow \text{real}$  **where**

$x \leq x_1 \Longrightarrow f'\ x = f\ (\text{nat } \lfloor x \rfloor)$

$| x > x_1 \Longrightarrow f'\ x = g\ x + (\sum\ i < k. as!i * f'\ (bs!i * x + (hs!i)\ x))$

*<proof>*

**termination** *<proof>*

**lemma** *f'-base*:  $x \geq x_0 \implies x \leq x_1 \implies f' x \geq 0$   
 ⟨*proof*⟩

**lemmas** *f'-rec* = *f'.simps*(2)

**end**

**locale** *akra-bazzi-real-lower* = *akra-bazzi-real* +  
**fixes** *fb2 gb2 c2* :: *real*  
**assumes** *f-base2*:  $x \geq x_0 \implies x \leq x_1 \implies f x \geq fb2$   
**and** *fb2-pos*:  $fb2 > 0$   
**and** *g-growth2*:  $\forall x \geq x_1. \forall u \in \{C * x..x\}. c2 * g x \geq g u$   
**and** *c2-pos*:  $c2 > 0$   
**and** *g-bounded*:  $x \geq x_0 \implies x \leq x_1 \implies g x \leq gb2$   
**begin**

**interpretation** *akra-bazzi-integral integrable integral* ⟨*proof*⟩

**lemma** *gb2-nonneg*:  $gb2 \geq 0$  ⟨*proof*⟩

**lemma** *g-growth2'*:  
**assumes**  $x \geq x_1 \ i < k \ u \in \{bs!i*x+(hs!i) x..x\}$   
**shows**  $c2 * g x \geq g u$   
 ⟨*proof*⟩

**lemma** *g-bounds2*:  
**obtains** *c4* **where**  $\bigwedge x \ i. x \geq x_1 \implies i < k \implies g\text{-approx } i \ x \leq c4 * g x \ c4 > 0$   
 ⟨*proof*⟩

**lemma** *f-approx-bounded-above*:  
**obtains** *c* **where**  $\bigwedge x. x \geq x_0 \implies x \leq x_1 \implies f\text{-approx } x \leq c \ c > 0$   
 ⟨*proof*⟩

**lemma** *f-bounded-below*:  
**assumes** *c'*:  $c' > 0$   
**obtains** *c* **where**  $\bigwedge x. x \geq x_0 \implies x \leq x_1 \implies 2 * (c * f\text{-approx } x) \leq f x \ c \leq c'$   
 $c > 0$   
 ⟨*proof*⟩

**lemma** *akra-bazzi-lower*:  
**obtains** *c5* **where**  $\bigwedge x. x \geq x_0 \implies f x \geq c5 * f\text{-approx } x \ c5 > 0$   
 ⟨*proof*⟩

**lemma** *akra-bazzi-bigomega*:  
 $f \in \Omega(\lambda x. x \text{ powr } p * (1 + \text{integral } (\lambda u. g u / u \text{ powr } (p + 1)) x_0 x))$   
 ⟨*proof*⟩

**end**

```

locale akra-bazzi-real-upper = akra-bazzi-real +
  fixes fb1 c1 :: real
  assumes f-base1:  $x \geq x_0 \implies x \leq x_1 \implies f x \leq fb1$ 
  and g-growth1:  $\forall x \geq x_1. \forall u \in \{C * x..x\}. c1 * g x \leq g u$ 
  and c1-pos:  $c1 > 0$ 
begin

interpretation akra-bazzi-integral integrable integral <proof>

lemma g-growth1':
  assumes  $x \geq x_1 \ i < k \ u \in \{bs!i*x+(hs!i) x..x\}$ 
  shows  $c1 * g x \leq g u$ 
  <proof>

lemma g-bounds1:
  obtains c3 where
     $\bigwedge x \ i. x \geq x_1 \implies i < k \implies c3 * g x \leq g\text{-approx } i \ x \ c3 > 0$ 
  <proof>

lemma f-bounded-above:
  assumes c':  $c' > 0$ 
  obtains c where  $\bigwedge x. x \geq x_0 \implies x \leq x_1 \implies f x \leq (1/2) * (c * f\text{-approx } x) \ c$ 
   $\geq c' \ c > 0$ 
  <proof>

lemma akra-bazzi-upper:
  obtains c6 where  $\bigwedge x. x \geq x_0 \implies f x \leq c6 * f\text{-approx } x \ c6 > 0$ 
  <proof>

lemma akra-bazzi-bigo:
   $f \in O(\lambda x. x \text{ powr } p * (1 + \text{integral } (\lambda u. g u / u \text{ powr } (p + 1)) x_0 x))$ 
  <proof>

end

end

```

## 4 The discrete Akra-Bazzi theorem

```

theory Akra-Bazzi
imports
  Complex-Main
  HOL-Library.Landau-Symbols
  Akra-Bazzi-Real
begin

```

**lemma** *ex-mono*:  $(\exists x. P x) \implies (\bigwedge x. P x \implies Q x) \implies (\exists x. Q x)$  *<proof>*

**lemma** *x-over-ln-mono*:

**assumes**  $(e::real) > 0$

**assumes**  $x > \exp e$

**assumes**  $x \leq y$

**shows**  $x / \ln x \text{ powr } e \leq y / \ln y \text{ powr } e$

*<proof>*

**definition** *akra-bazzi-term* ::  $nat \Rightarrow nat \Rightarrow real \Rightarrow (nat \Rightarrow nat) \Rightarrow bool$  **where**

*akra-bazzi-term*  $x_0 x_1 b t =$

$(\exists e h. e > 0 \wedge (\lambda x. h x) \in O(\lambda x. real\ x / \ln (real\ x) \text{ powr } (1+e))) \wedge$   
 $(\forall x \geq x_1. t\ x \geq x_0 \wedge t\ x < x \wedge b*x + h\ x = real\ (t\ x))$

**lemma** *akra-bazzi-termI* [*intro?*]:

**assumes**  $e > 0 (\lambda x. h x) \in O(\lambda x. real\ x / \ln (real\ x) \text{ powr } (1+e))$

$\bigwedge x. x \geq x_1 \implies t\ x \geq x_0 \bigwedge x. x \geq x_1 \implies t\ x < x$

$\bigwedge x. x \geq x_1 \implies b*x + h\ x = real\ (t\ x)$

**shows** *akra-bazzi-term*  $x_0 x_1 b t$

*<proof>*

**lemma** *akra-bazzi-term-imp-less*:

**assumes** *akra-bazzi-term*  $x_0 x_1 b t x \geq x_1$

**shows**  $t\ x < x$

*<proof>*

**lemma** *akra-bazzi-term-imp-less'*:

**assumes** *akra-bazzi-term*  $x_0 (Suc\ x_1) b t x > x_1$

**shows**  $t\ x < x$

*<proof>*

**locale** *akra-bazzi-recursion* =

**fixes**  $x_0 x_1 k :: nat$  **and** *as* *bs* :: *real list* **and** *ts* ::  $(nat \Rightarrow nat)$  *list* **and** *f* ::  $nat \Rightarrow real$

**assumes** *k-not-0*:  $k \neq 0$

**and** *length-as*:  $length\ as = k$

**and** *length-bs*:  $length\ bs = k$

**and** *length-ts*:  $length\ ts = k$

**and** *a-ge-0*:  $a \in set\ as \implies a \geq 0$

**and** *b-bounds*:  $b \in set\ bs \implies b \in \{0 < .. < 1\}$

**and** *ts*:  $i < length\ bs \implies \text{akra-bazzi-term } x_0 x_1 (bs!i) (ts!i)$

**begin**

**sublocale** *akra-bazzi-params*  $k\ as\ bs$

*<proof>*

**lemma** *ts-nonempty*:  $ts \neq []$   $\langle$ proof $\rangle$

**definition** *e-hs* ::  $real \times (nat \Rightarrow real)$  list **where**

$e-hs = (SOME (e,hs)).$   
 $e > 0 \wedge length\ hs = k \wedge (\forall h \in set\ hs. (\lambda x. h\ x) \in O(\lambda x. real\ x / ln\ (real\ x)$   
 $powr\ (1+e))) \wedge$   
 $(\forall t \in set\ ts. \forall x \geq x_1. t\ x \geq x_0 \wedge t\ x < x) \wedge$   
 $(\forall i < k. \forall x \geq x_1. (bs!i)*x + (hs!i)\ x = real\ ((ts!i)\ x))$   
 $)$

**definition**  $e = (case\ e-hs\ of\ (e,-) \Rightarrow e)$

**definition**  $hs = (case\ e-hs\ of\ (-,hs) \Rightarrow hs)$

**lemma** *filterlim-powr-zero-cong*:

$filterlim\ (\lambda x. P\ (x::real)\ (x\ powr\ (0::real)))\ F\ at-top = filterlim\ (\lambda x. P\ x\ 1)\ F$   
 $at-top$   
 $\langle$ proof $\rangle$

**lemma** *e-hs-aux*:

$0 < e \wedge length\ hs = k \wedge$   
 $(\forall h \in set\ hs. (\lambda x. h\ x) \in O(\lambda x. real\ x / ln\ (real\ x)\ powr\ (1 + e))) \wedge$   
 $(\forall t \in set\ ts. \forall x \geq x_1. x_0 \leq t\ x \wedge t\ x < x) \wedge$   
 $(\forall i < k. \forall x \geq x_1. (bs!i)*x + (hs!i)\ x = real\ ((ts!i)\ x))$   
 $\langle$ proof $\rangle$

**lemma**

*e-pos*:  $e > 0$  **and** *length-hs*:  $length\ hs = k$  **and**  
*hs-growth*:  $\bigwedge h. h \in set\ hs \Longrightarrow (\lambda x. h\ x) \in O(\lambda x. real\ x / ln\ (real\ x)\ powr\ (1 + e))$  **and**  
*step-ge-x0*:  $\bigwedge t\ x. t \in set\ ts \Longrightarrow x \geq x_1 \Longrightarrow x_0 \leq t\ x$  **and**  
*step-less*:  $\bigwedge t\ x. t \in set\ ts \Longrightarrow x \geq x_1 \Longrightarrow t\ x < x$  **and**  
*decomp*:  $\bigwedge i\ x. i < k \Longrightarrow x \geq x_1 \Longrightarrow (bs!i)*x + (hs!i)\ x = real\ ((ts!i)\ x)$   
 $\langle$ proof $\rangle$

**lemma** *h-in-hs* [*intro*, *simp*]:  $i < k \Longrightarrow hs\ !\ i \in set\ hs$

$\langle$ proof $\rangle$

**lemma** *t-in-ts* [*intro*, *simp*]:  $i < k \Longrightarrow ts\ !\ i \in set\ ts$

$\langle$ proof $\rangle$

**lemma** *x0-less-x1*:  $x_0 < x_1$  **and** *x0-le-x1*:  $x_0 \leq x_1$

$\langle$ proof $\rangle$

**lemma** *akra-bazzi-induct* [*consumes 1*, *case-names base rec*]:

**assumes**  $x \geq x_0$

**assumes** *base*:  $\bigwedge x. x \geq x_0 \Longrightarrow x < x_1 \Longrightarrow P\ x$

**assumes** *rec*:  $\bigwedge x. x \geq x_1 \Longrightarrow (\bigwedge t. t \in set\ ts \Longrightarrow P\ (t\ x)) \Longrightarrow P\ x$

**shows**  $P\ x$

*<proof>*

**end**

**locale** *akra-bazzi-function* = *akra-bazzi-recursion* +  
  **fixes** *integrable integral*  
  **assumes** *integral: akra-bazzi-integral integrable integral*  
  **fixes** *g :: nat ⇒ real*  
  **assumes** *f-nonneg-base: x ≥ x<sub>0</sub> ⇒ x < x<sub>1</sub> ⇒ f x ≥ 0*  
  **and** *f-rec: x ≥ x<sub>1</sub> ⇒ f x = g x + (∑ i<k. as!i \* f ((ts!i) x))*  
  **and** *g-nonneg: x ≥ x<sub>1</sub> ⇒ g x ≥ 0*  
  **and** *ex-pos-a: ∃ a∈set as. a > 0*  
**begin**

**lemma** *ex-pos-a'*:  $\exists i < k. as!i > 0$   
*<proof>*

**sublocale** *akra-bazzi-params-nonzero*  
*<proof>*

**definition** *g-real :: real ⇒ real where g-real x = g (nat [x])*

**lemma** *g-real-real[simp]*:  $g\text{-real } (real\ x) = g\ x$  *<proof>*

**lemma** *f-nonneg*:  $x \geq x_0 \implies f\ x \geq 0$   
*<proof>*

**definition** *hs' = map (λh x. h (nat [x::real])) hs*

**lemma** *length-hs'*:  $length\ hs' = k$  *<proof>*

**lemma** *hs'-real*:  $i < k \implies (hs'!i)\ (real\ x) = (hs!i)\ x$   
*<proof>*

**lemma** *h-bound*:  
  **obtains** *hb where hb > 0 and*  
    *eventually (λx. ∀ h∈set hs'. |h x| ≤ hb \* x / ln x powr (1 + e)) at-top*  
*<proof>*

**lemma** *C-bound*:  
  **assumes**  $\bigwedge b. b \in set\ bs \implies C < b\ hb > 0$   
  **shows** *eventually (λx::real. ∀ b∈set bs. C\*x ≤ b\*x - hb\*x/ln x powr (1+e))*  
*at-top*  
*<proof>*

**end**

**locale** *akra-bazzi-lower* = *akra-bazzi-function* +  
**fixes**  $g' :: \text{real} \Rightarrow \text{real}$   
**assumes** *f-pos*: *eventually*  $(\lambda x. f\ x > 0)$  *at-top*  
**and** *g-growth2*:  $\exists C\ c2. c2 > 0 \wedge C < \text{Min}(\text{set } bs) \wedge$   
*eventually*  $(\lambda x. \forall u \in \{C * x..x\}. g'\ u \leq c2 * g'\ x)$  *at-top*  
**and** *g'-integrable*:  $\exists a. \forall b \geq a. \text{integrable}(\lambda u. g'\ u / u^{\text{powr}(p+1)})\ a\ b$   
**and** *g'-bounded*: *eventually*  $(\lambda a :: \text{real}. (\forall b > a. \exists c. \forall x \in \{a..b\}. g'\ x \leq c))$  *at-top*  
**and** *g-bigomega*:  $g \in \Omega(\lambda x. g'\ (\text{real } x))$   
**and** *g'-nonneg*: *eventually*  $(\lambda x. g'\ x \geq 0)$  *at-top*  
**begin**

**definition** *gc2*  $\equiv \text{SOME } gc2. gc2 > 0 \wedge \text{eventually}(\lambda x. g\ x \geq gc2 * g'\ (\text{real } x))$   
*at-top*

**lemma** *gc2*:  $gc2 > 0$  *eventually*  $(\lambda x. g\ x \geq gc2 * g'\ (\text{real } x))$  *at-top*  
 $\langle \text{proof} \rangle$

**definition** *gx0*  $\equiv \max\ x_1\ (\text{SOME } gx0. \forall x \geq gx0. g\ x \geq gc2 * g'\ (\text{real } x) \wedge f\ x > 0 \wedge g'\ (\text{real } x) \geq 0)$

**definition** *gx1*  $\equiv \max\ gx0\ (\text{SOME } gx1. \forall x \geq gx1. \forall i < k. (ts!i)\ x \geq gx0)$

**lemma** *gx0*:  
**assumes**  $x \geq gx0$   
**shows**  $g\ x \geq gc2 * g'\ (\text{real } x) \wedge f\ x > 0 \wedge g'\ (\text{real } x) \geq 0$   
 $\langle \text{proof} \rangle$

**lemma** *gx1*:  
**assumes**  $x \geq gx1 \wedge i < k$   
**shows**  $(ts!i)\ x \geq gx0$   
 $\langle \text{proof} \rangle$

**lemma** *gx0-ge-x1*:  $gx0 \geq x_1$   $\langle \text{proof} \rangle$

**lemma** *gx0-le-gx1*:  $gx0 \leq gx1$   $\langle \text{proof} \rangle$

**function** *f2'*  $:: \text{nat} \Rightarrow \text{real}$  **where**  
 $x < gx1 \implies f2'\ x = \max\ 0\ (f\ x / gc2)$   
 $| x \geq gx1 \implies f2'\ x = g'\ (\text{real } x) + (\sum\ i < k. as!i * f2'\ ((ts!i)\ x))$   
 $\langle \text{proof} \rangle$   
**termination**  $\langle \text{proof} \rangle$

**lemma** *f2'-nonneg*:  $x \geq gx0 \implies f2'\ x \geq 0$   
 $\langle \text{proof} \rangle$

**lemma** *f2'-le-f*:  $x \geq x_0 \implies gc2 * f2'\ x \leq f\ x$   
 $\langle \text{proof} \rangle$

**lemma** *f2'-pos*: *eventually*  $(\lambda x. f2'\ x > 0)$  *at-top*  
 $\langle \text{proof} \rangle$

**lemma** *bigomega-f-aux*:

**obtains a where**  $a \geq A \forall a' \geq a. a' \in \mathbf{N} \longrightarrow$

$f \in \Omega(\lambda x. x \text{ powr } p * (1 + \text{integral } (\lambda u. g' u / u \text{ powr } (p + 1)) a' x))$

*<proof>*

**lemma** *bigomega-f*:

**obtains a where**  $a \geq A f \in \Omega(\lambda x. x \text{ powr } p * (1 + \text{integral } (\lambda u. g' u / u \text{ powr } (p+1)) a x))$

*<proof>*

**end**

**locale** *akra-bazzi-upper* = *akra-bazzi-function* +

**fixes**  $g' :: \text{real} \Rightarrow \text{real}$

**assumes** *g'-integrable*:  $\exists a. \forall b \geq a. \text{integrable } (\lambda u. g' u / u \text{ powr } (p + 1)) a b$

**and** *g-growth1*:  $\exists C c1. c1 > 0 \wedge C < \text{Min } (\text{set } bs) \wedge$

$\text{eventually } (\lambda x. \forall u \in \{C * x..x\}. g' u \geq c1 * g' x) \text{ at-top}$

**and** *g-bigo*:  $g \in O(g')$

**and** *g'-nonneg*:  $\text{eventually } (\lambda x. g' x \geq 0) \text{ at-top}$

**begin**

**definition** *gc1*  $\equiv \text{SOME } gc1. gc1 > 0 \wedge \text{eventually } (\lambda x. g x \leq gc1 * g' (\text{real } x)) \text{ at-top}$

**lemma** *gc1*:  $gc1 > 0 \text{ eventually } (\lambda x. g x \leq gc1 * g' (\text{real } x)) \text{ at-top}$

*<proof>*

**definition** *gx3*  $\equiv \max x_1 (\text{SOME } gx0. \forall x \geq gx0. g x \leq gc1 * g' (\text{real } x))$

**lemma** *gx3*:

**assumes**  $x \geq gx3$

**shows**  $g x \leq gc1 * g' (\text{real } x)$

*<proof>*

**lemma** *gx3-ge-x1*:  $gx3 \geq x_1$  *<proof>*

**function** *f'*  $:: \text{nat} \Rightarrow \text{real}$  **where**

$x < gx3 \Longrightarrow f' x = \max 0 (f x / gc1)$

$| x \geq gx3 \Longrightarrow f' x = g' (\text{real } x) + (\sum_{i < k. as!i * f' ((ts!i) x))$

*<proof>*

**termination** *<proof>*

**lemma** *f'-ge-f*:  $x \geq x_0 \Longrightarrow gc1 * f' x \geq f x$

*<proof>*

**lemma** *bigo-f-aux*:

**obtains a where**  $a \geq A \forall a' \geq a. a' \in \mathbf{N} \longrightarrow$

$f \in O(\lambda x. x \text{ powr } p *(1 + \text{integral } (\lambda u. g' u / u \text{ powr } (p + 1)) a' x))$

*<proof>*

**lemma** *bigo-f*:

**obtains a where**  $a > A f \in O(\lambda x. x \text{ powr } p *(1 + \text{integral } (\lambda u. g' u / u \text{ powr } (p + 1)) a x))$

*<proof>*

**end**

**locale** *akra-bazzi* = *akra-bazzi-function* +

**fixes**  $g' :: \text{real} \Rightarrow \text{real}$

**assumes** *f-pos*: *eventually*  $(\lambda x. f x > 0)$  *at-top*

**and** *g'-nonneg*: *eventually*  $(\lambda x. g' x \geq 0)$  *at-top*

**assumes** *g'-integrable*:  $\exists a. \forall b \geq a. \text{integrable } (\lambda u. g' u / u \text{ powr } (p + 1)) a b$

**and** *g-growth1*:  $\exists C c1. c1 > 0 \wedge C < \text{Min } (\text{set } bs) \wedge$   
*eventually*  $(\lambda x. \forall u \in \{C * x .. x\}. g' u \geq c1 * g' x)$  *at-top*

**and** *g-growth2*:  $\exists C c2. c2 > 0 \wedge C < \text{Min } (\text{set } bs) \wedge$   
*eventually*  $(\lambda x. \forall u \in \{C * x .. x\}. g' u \leq c2 * g' x)$  *at-top*

**and** *g-bounded*: *eventually*  $(\lambda a :: \text{real}. (\forall b > a. \exists c. \forall x \in \{a..b\}. g' x \leq c))$  *at-top*

**and** *g-bigtheta*:  $g \in \Theta(g')$

**begin**

**sublocale** *akra-bazzi-lower* *<proof>*

**sublocale** *akra-bazzi-upper* *<proof>*

**lemma** *bigtheta-f*:

**obtains a where**  $a > A f \in \Theta(\lambda x. x \text{ powr } p *(1 + \text{integral } (\lambda u. g' u / u \text{ powr } (p + 1)) a x))$

*<proof>*

**end**

**named-theorems** *akra-bazzi-term-intros* *introduction rules for Akra–Bazzi terms*

**lemma** *akra-bazzi-term-floor-add* [*akra-bazzi-term-intros*]:

**assumes**  $(b :: \text{real}) > 0 b < 1 \text{ real } x_0 \leq b * \text{real } x_1 + c c < (1 - b) * \text{real } x_1 x_1 > 0$

**shows** *akra-bazzi-term*  $x_0 x_1 b (\lambda x. \text{nat } \lfloor b * \text{real } x + c \rfloor)$

*<proof>*

**lemma** *akra-bazzi-term-floor-add'* [*akra-bazzi-term-intros*]:

**assumes**  $(b :: \text{real}) > 0 b < 1 \text{ real } x_0 \leq b * \text{real } x_1 + \text{real } c \text{ real } c < (1 - b) * \text{real } x_1 x_1 > 0$

**shows** *akra-bazzi-term*  $x_0 x_1 b (\lambda x. \text{nat } \lfloor b * \text{real } x \rfloor + c)$

*<proof>*

**lemma** *akra-bazzi-term-floor-subtract* [*akra-bazzi-term-intros*]:

**assumes**  $(b::real) > 0$   $b < 1$   $real\ x_0 \leq b * real\ x_1 - c$   $0 < c + (1 - b) * real\ x_1$   $x_1 > 0$

**shows** *akra-bazzi-term*  $x_0\ x_1\ b\ (\lambda x. nat\ [b * real\ x - c])$

*<proof>*

**lemma** *akra-bazzi-term-floor-subtract'* [*akra-bazzi-term-intros*]:

**assumes**  $(b::real) > 0$   $b < 1$   $real\ x_0 \leq b * real\ x_1 - real\ c$   $0 < real\ c + (1 - b) * real\ x_1$   $x_1 > 0$

**shows** *akra-bazzi-term*  $x_0\ x_1\ b\ (\lambda x. nat\ [b * real\ x] - c)$

*<proof>*

**lemma** *akra-bazzi-term-floor* [*akra-bazzi-term-intros*]:

**assumes**  $(b::real) > 0$   $b < 1$   $real\ x_0 \leq b * real\ x_1$   $0 < (1 - b) * real\ x_1$   $x_1 > 0$

**shows** *akra-bazzi-term*  $x_0\ x_1\ b\ (\lambda x. nat\ [b * real\ x])$

*<proof>*

**lemma** *akra-bazzi-term-ceiling-add* [*akra-bazzi-term-intros*]:

**assumes**  $(b::real) > 0$   $b < 1$   $real\ x_0 \leq b * real\ x_1 + c$   $c + 1 \leq (1 - b) * x_1$

**shows** *akra-bazzi-term*  $x_0\ x_1\ b\ (\lambda x. nat\ [b * real\ x + c])$

*<proof>*

**lemma** *akra-bazzi-term-ceiling-add'* [*akra-bazzi-term-intros*]:

**assumes**  $(b::real) > 0$   $b < 1$   $real\ x_0 \leq b * real\ x_1 + real\ c$   $real\ c + 1 \leq (1 - b) * x_1$

**shows** *akra-bazzi-term*  $x_0\ x_1\ b\ (\lambda x. nat\ [b * real\ x] + c)$

*<proof>*

**lemma** *akra-bazzi-term-ceiling-subtract* [*akra-bazzi-term-intros*]:

**assumes**  $(b::real) > 0$   $b < 1$   $real\ x_0 \leq b * real\ x_1 - c$   $1 \leq c + (1 - b) * x_1$

**shows** *akra-bazzi-term*  $x_0\ x_1\ b\ (\lambda x. nat\ [b * real\ x - c])$

*<proof>*

**lemma** *akra-bazzi-term-ceiling-subtract'* [*akra-bazzi-term-intros*]:

**assumes**  $(b::real) > 0$   $b < 1$   $real\ x_0 \leq b * real\ x_1 - real\ c$   $1 \leq real\ c + (1 - b) * x_1$

**shows** *akra-bazzi-term*  $x_0\ x_1\ b\ (\lambda x. nat\ [b * real\ x] - c)$

*<proof>*

**lemma** *akra-bazzi-term-ceiling* [*akra-bazzi-term-intros*]:

**assumes**  $(b::real) > 0$   $b < 1$   $real\ x_0 \leq b * real\ x_1$   $1 \leq (1 - b) * x_1$

**shows** *akra-bazzi-term*  $x_0\ x_1\ b\ (\lambda x. nat\ [b * real\ x])$

*<proof>*

end

## 5 The Master theorem

**theory** *Master-Theorem*

**imports**

*HOL-Analysis.Equivalence-Lebesgue-Henstock-Integration*

*Akra-Bazzi-Library*

*Akra-Bazzi*

**begin**

**lemma** *fundamental-theorem-of-calculus-real*:

$a \leq b \implies \forall x \in \{a..b\}. (f \text{ has-real-derivative } f' x) \text{ (at } x \text{ within } \{a..b\}) \implies$   
 $(f' \text{ has-integral } (f b - f a)) \{a..b\}$   
*<proof>*

**lemma** *integral-powr*:

$y \neq -1 \implies a \leq b \implies a > 0 \implies \text{integral } \{a..b\} (\lambda x. x \text{ powr } y :: \text{real}) =$   
 $\text{inverse } (y + 1) * (b \text{ powr } (y + 1) - a \text{ powr } (y + 1))$   
*<proof>*

**lemma** *integral-ln-powr-over-x*:

$y \neq -1 \implies a \leq b \implies a > 1 \implies \text{integral } \{a..b\} (\lambda x. \ln x \text{ powr } y / x :: \text{real}) =$   
 $\text{inverse } (y + 1) * (\ln b \text{ powr } (y + 1) - \ln a \text{ powr } (y + 1))$   
*<proof>*

**lemma** *integral-one-over-x-ln-x*:

$a \leq b \implies a > 1 \implies \text{integral } \{a..b\} (\lambda x. \text{inverse } (x * \ln x) :: \text{real}) = \ln (\ln b)$   
 $- \ln (\ln a)$   
*<proof>*

**lemma** *akra-bazzi-integral-kurzweil-henstock*:

$\text{akra-bazzi-integral } (\lambda f a b. f \text{ integrable-on } \{a..b\}) (\lambda f a b. \text{integral } \{a..b\} f)$   
*<proof>*

**locale** *master-theorem-function = akra-bazzi-recursion +*

**fixes**  $g :: \text{nat} \Rightarrow \text{real}$

**assumes**  $f\text{-nonneg-base}: x \geq x_0 \implies x < x_1 \implies f x \geq 0$

**and**  $f\text{-rec}: x \geq x_1 \implies f x = g x + (\sum_{i < k}. \text{as! } i * f ((\text{ts! } i) x))$

**and**  $g\text{-nonneg}: x \geq x_1 \implies g x \geq 0$

**and**  $ex\text{-pos-}a: \exists a \in \text{set } \text{as}. a > 0$

**begin**

**interpretation** *akra-bazzi-integral*  $\lambda f a b. f \text{ integrable-on } \{a..b\} \lambda f a b. \text{integral } \{a..b\} f$

*<proof>*

**sublocale** *akra-bazzi-function*  $x_0 x_1 k$  as  $bs$   $ts$   $f$   $\lambda f a b. f$  *integrable-on*  $\{a..b\}$   
 $\lambda f a b. \text{integral } \{a..b\} f g$   
 ⟨proof⟩

**context**  
**begin**

**private lemma** *g-nonneg'*: *eventually*  $(\lambda x. g x \geq 0)$  *at-top*

⟨proof⟩ **lemma** *g-pos*:

**assumes**  $g \in \Omega(h)$

**assumes** *eventually*  $(\lambda x. h x > 0)$  *at-top*

**shows** *eventually*  $(\lambda x. g x > 0)$  *at-top*

⟨proof⟩ **lemma** *f-pos*:

**assumes**  $g \in \Omega(h)$

**assumes** *eventually*  $(\lambda x. h x > 0)$  *at-top*

**shows** *eventually*  $(\lambda x. f x > 0)$  *at-top*

⟨proof⟩

**lemma** *bs-lower-bound*:  $\exists C > 0. \forall b \in \text{set } bs. C < b$

⟨proof⟩ **lemma** *powr-growth2*:

$\exists C c2. 0 < c2 \wedge C < \text{Min } (\text{set } bs) \wedge$

*eventually*  $(\lambda x. \forall u \in \{C * x..x\}. c2 * x \text{ powr } p' \geq u \text{ powr } p')$  *at-top*

⟨proof⟩ **lemma** *powr-growth1*:

$\exists C c1. 0 < c1 \wedge C < \text{Min } (\text{set } bs) \wedge$

*eventually*  $(\lambda x. \forall u \in \{C * x..x\}. c1 * x \text{ powr } p' \leq u \text{ powr } p')$  *at-top*

⟨proof⟩ **lemma** *powr-ln-powr-lower-bound*:

$a > 1 \implies a \leq x \implies x \leq b \implies$

$\min (a \text{ powr } p) (b \text{ powr } p) * \min (\ln a \text{ powr } p') (\ln b \text{ powr } p') \leq x \text{ powr } p * \ln x \text{ powr } p'$

⟨proof⟩ **lemma** *powr-ln-powr-upper-bound*:

$a > 1 \implies a \leq x \implies x \leq b \implies$

$\max (a \text{ powr } p) (b \text{ powr } p) * \max (\ln a \text{ powr } p') (\ln b \text{ powr } p') \geq x \text{ powr } p * \ln x \text{ powr } p'$

⟨proof⟩ **lemma** *powr-ln-powr-upper-bound'*:

*eventually*  $(\lambda a. \forall b > a. \exists c. \forall x \in \{a..b\}. x \text{ powr } p * \ln x \text{ powr } p' \leq c)$  *at-top*

⟨proof⟩ **lemma** *powr-upper-bound'*:

*eventually*  $(\lambda a::\text{real}. \forall b > a. \exists c. \forall x \in \{a..b\}. x \text{ powr } p' \leq c)$  *at-top*

⟨proof⟩

**lemmas** *bounds* =

*powr-ln-powr-lower-bound powr-ln-powr-upper-bound powr-ln-powr-upper-bound'*  
*powr-upper-bound'*

**private lemma** *eventually-ln-const*:

**assumes**  $(C::\text{real}) > 0$

**shows** *eventually*  $(\lambda x. \ln (C*x) / \ln x > 1/2)$  *at-top*

⟨proof⟩ **lemma** *powr-ln-powr-growth1*:  $\exists C c1. 0 < c1 \wedge C < \text{Min } (\text{set } bs) \wedge$

*eventually*  $(\lambda x. \forall u \in \{C * x..x\}. c1 * (x \text{ powr } r * \ln x \text{ powr } r') \leq u \text{ powr } r * \ln$

$u \text{ powr } r')$  at-top  
 ⟨proof⟩ **lemma** *powr-ln-powr-growth2*:  $\exists C c1. 0 < c1 \wedge C < \text{Min } (\text{set } bs) \wedge$   
     *eventually*  $(\lambda x. \forall u \in \{C * x..x\}. c1 * (x \text{ powr } r * \ln x \text{ powr } r') \geq u \text{ powr } r * \ln$   
 $u \text{ powr } r')$  at-top  
 ⟨proof⟩

**lemmas** *growths* = *powr-growth1 powr-growth2 powr-ln-powr-growth1 powr-ln-powr-growth2*

**private lemma** *master-integrable*:

$\exists a::\text{real}. \forall b \geq a. (\lambda u. u \text{ powr } r * \ln u \text{ powr } s / u \text{ powr } t) \text{ integrable-on } \{a..b\}$

$\exists a::\text{real}. \forall b \geq a. (\lambda u. u \text{ powr } r / u \text{ powr } s) \text{ integrable-on } \{a..b\}$

⟨proof⟩ **lemma** *master-integral*:

**fixes**  $a p p' :: \text{real}$

**assumes**  $p: p \neq p'$  **and**  $a: a > 0$

**obtains**  $c d$  **where**  $c \neq 0 p > p' \longrightarrow d \neq 0$

$(\lambda x::\text{nat}. x \text{ powr } p * (1 + \text{integral } \{a..x\} (\lambda u. u \text{ powr } p' / u \text{ powr } (p+1)))) \in$   
 $\Theta(\lambda x::\text{nat}. d * x \text{ powr } p + c * x \text{ powr } p')$

⟨proof⟩ **lemma** *master-integral'*:

**fixes**  $a p p' :: \text{real}$

**assumes**  $p': p' \neq 0$  **and**  $a: a > 1$

**obtains**  $c d :: \text{real}$  **where**  $p' < 0 \longrightarrow c \neq 0 d \neq 0$

$(\lambda x::\text{nat}. x \text{ powr } p * (1 + \text{integral } \{a..x\} (\lambda u. u \text{ powr } p * \ln u \text{ powr } (p'-1) / u$   
 $\text{powr } (p+1)))) \in$   
 $\Theta(\lambda x::\text{nat}. c * x \text{ powr } p + d * x \text{ powr } p * \ln x \text{ powr } p')$

⟨proof⟩ **lemma** *master-integral''*:

**fixes**  $a p p' :: \text{real}$

**assumes**  $a: a > 1$

**shows**  $(\lambda x::\text{nat}. x \text{ powr } p * (1 + \text{integral } \{a..x\} (\lambda u. u \text{ powr } p * \ln u \text{ powr } -$   
 $1 / u \text{ powr } (p+1)))) \in$

$\Theta(\lambda x::\text{nat}. x \text{ powr } p * \ln (\ln x))$

⟨proof⟩

**lemma** *master1-bigo*:

**assumes** *g-bigo*:  $g \in O(\lambda x. \text{real } x \text{ powr } p')$

**assumes** *less-p'*:  $(\sum i < k. \text{as}!i * \text{bs}!i \text{ powr } p') > 1$

**shows**  $f \in O(\lambda x. \text{real } x \text{ powr } p)$

⟨proof⟩

**lemma** *master1*:

**assumes** *g-bigo*:  $g \in O(\lambda x. \text{real } x \text{ powr } p')$

**assumes** *less-p'*:  $(\sum i < k. \text{as}!i * \text{bs}!i \text{ powr } p') > 1$

**assumes** *f-pos*: *eventually*  $(\lambda x. f x > 0)$  at-top

**shows**  $f \in \Theta(\lambda x. \text{real } x \text{ powr } p)$

⟨proof⟩

**lemma** *master2-3*:  
**assumes** *g-bigtheta*:  $g \in \Theta(\lambda x. \text{real } x \text{ powr } p * \ln (\text{real } x) \text{ powr } (p' - 1))$   
**assumes** *p'*:  $p' > 0$   
**shows**  $f \in \Theta(\lambda x. \text{real } x \text{ powr } p * \ln (\text{real } x) \text{ powr } p')$   
 $\langle \text{proof} \rangle$

**lemma** *master2-1*:  
**assumes** *g-bigtheta*:  $g \in \Theta(\lambda x. \text{real } x \text{ powr } p * \ln (\text{real } x) \text{ powr } p')$   
**assumes** *p'*:  $p' < -1$   
**shows**  $f \in \Theta(\lambda x. \text{real } x \text{ powr } p)$   
 $\langle \text{proof} \rangle$

**lemma** *master2-2*:  
**assumes** *g-bigtheta*:  $g \in \Theta(\lambda x. \text{real } x \text{ powr } p / \ln (\text{real } x))$   
**shows**  $f \in \Theta(\lambda x. \text{real } x \text{ powr } p * \ln (\ln (\text{real } x)))$   
 $\langle \text{proof} \rangle$

**lemma** *master3*:  
**assumes** *g-bigtheta*:  $g \in \Theta(\lambda x. \text{real } x \text{ powr } p')$   
**assumes** *p'-greater'*:  $(\sum_{i < k}. \text{as!}i * \text{bs!}i \text{ powr } p') < 1$   
**shows**  $f \in \Theta(\lambda x. \text{real } x \text{ powr } p')$   
 $\langle \text{proof} \rangle$

**end**  
**end**

**end**

## 6 Evaluating expressions with rational numerals

**theory** *Eval-Numeral*  
**imports**  
*Complex-Main*  
**begin**

**lemma** *real-numeral-to-Ratreal*:  
 $(0::\text{real}) = \text{Ratreal } (\text{Frct } (0, 1))$   
 $(1::\text{real}) = \text{Ratreal } (\text{Frct } (1, 1))$   
 $(\text{numeral } x :: \text{real}) = \text{Ratreal } (\text{Frct } (\text{numeral } x, 1))$   
 $(1::\text{int}) = \text{numeral } \text{Num.One}$   
 $\langle \text{proof} \rangle$

**lemma** *real-equals-code*:  $\text{Ratreal } x = \text{Ratreal } y \longleftrightarrow x = y$   
 $\langle \text{proof} \rangle$

**lemma** *Rat-normalize-idempotent*:  $\text{Rat.normalize } (\text{Rat.normalize } x) = \text{Rat.normalize } x$   
 $\langle \text{proof} \rangle$

**lemma** *uminus-pow-Numerals1*:  $(-(x:::\text{monoid-mult})) \wedge \text{Numerals1} = -x$  *<proof>*

**lemmas** *power-numeral-simps* = *power-0 uminus-pow-Numerals1 power-minus-Bit0 power-minus-Bit1*

**lemma** *Fract-normalize*:  $\text{Fract} (\text{fst} (\text{Rat.normalize} (x,y))) (\text{snd} (\text{Rat.normalize} (x,y))) = \text{Fract} x y$   
*<proof>*

**lemma** *Fract-add*:  $\text{Frct} (a, \text{numeral } b) + \text{Frct} (c, \text{numeral } d) = \text{Frct} (\text{Rat.normalize} (a * \text{numeral } d + c * \text{numeral } b, \text{numeral } (b*d)))$   
*<proof>*

**lemma** *Frct-uminus*:  $-(\text{Frct} (a,b)) = \text{Frct} (-a,b)$  *<proof>*

**lemma** *Frct-diff*:  $\text{Frct} (a, \text{numeral } b) - \text{Frct} (c, \text{numeral } d) = \text{Frct} (\text{Rat.normalize} (a * \text{numeral } d - c * \text{numeral } b, \text{numeral } (b*d)))$   
*<proof>*

**lemma** *Frct-mult*:  $\text{Frct} (a, \text{numeral } b) * \text{Frct} (c, \text{numeral } d) = \text{Frct} (a*c, \text{numeral } (b*d))$   
*<proof>*

**lemma** *Frct-inverse*:  $\text{inverse} (\text{Frct} (a, b)) = \text{Frct} (b, a)$  *<proof>*

**lemma** *Frct-divide*:  $\text{Frct} (a, \text{numeral } b) / \text{Frct} (c, \text{numeral } d) = \text{Frct} (a*\text{numeral } d, \text{numeral } b * c)$   
*<proof>*

**lemma** *Frct-pow*:  $\text{Frct} (a, \text{numeral } b) \wedge c = \text{Frct} (a \wedge c, \text{numeral } b \wedge c)$   
*<proof>*

**lemma** *Frct-less*:  $\text{Frct} (a, \text{numeral } b) < \text{Frct} (c, \text{numeral } d) \iff a * \text{numeral } d < c * \text{numeral } b$   
*<proof>*

**lemma** *Frct-le*:  $\text{Frct} (a, \text{numeral } b) \leq \text{Frct} (c, \text{numeral } d) \iff a * \text{numeral } d \leq c * \text{numeral } b$   
*<proof>*

**lemma** *Frct-equals*:  $\text{Frct} (a, \text{numeral } b) = \text{Frct} (c, \text{numeral } d) \iff a * \text{numeral } d = c * \text{numeral } b$   
*<proof>*

**lemma** *real-power-code*:  $(\text{Ratreal } x) \wedge y = \text{Ratreal} (x \wedge y)$  *<proof>*

**lemmas** *real-arith-code* =  
*real-plus-code real-minus-code real-times-code real-uminus-code real-inverse-code*  
*real-divide-code real-power-code real-less-code real-less-eq-code real-equals-code*

**lemmas** *rat-arith-code* =  
*Frct-add Frct-uminus Frct-diff Frct-mult Frct-inverse Frct-divide Frct-pow*  
*Frct-less Frct-le Frct-equals*

**lemma** *gcd-numeral-red*:  $\text{gcd} (\text{numeral } x::\text{int}) (\text{numeral } y) = \text{gcd} (\text{numeral } y)$   
 $(\text{numeral } x \text{ mod numeral } y)$   
 ⟨proof⟩

**lemma** *divmod-one*:  
 $\text{divmod} (\text{Num.One}) (\text{Num.One}) = (\text{Numeral1}, 0)$   
 $\text{divmod} (\text{Num.One}) (\text{Num.Bit0 } x) = (0, \text{Numeral1})$   
 $\text{divmod} (\text{Num.One}) (\text{Num.Bit1 } x) = (0, \text{Numeral1})$   
 $\text{divmod } x (\text{Num.One}) = (\text{numeral } x, 0)$   
 ⟨proof⟩

**lemmas** *divmod-numeral-simps* =  
*div-0 div-by-0 mod-0 mod-by-0*  
*fst-divmod [symmetric]*  
*snd-divmod [symmetric]*  
*divmod-cancel*  
*divmod-steps [simplified rel-simps if-True] divmod-trivial*  
*rel-simps*

**lemma** *Suc-0-to-numeral*:  $\text{Suc } 0 = \text{Numeral1}$  ⟨proof⟩  
**lemmas** *Suc-to-numeral* = *Suc-0-to-numeral Num.Suc-1 Num.Suc-numeral*

**lemma** *rat-powr*:  
 $0 \text{ powr } y = 0$   
 $x > 0 \implies x \text{ powr } \text{Ratreal} (\text{Frct } (0, \text{Numeral1})) = \text{Ratreal} (\text{Frct} (\text{Numeral1}, \text{Numeral1}))$   
 $x > 0 \implies x \text{ powr } \text{Ratreal} (\text{Frct} (\text{numeral } a, \text{Numeral1})) = x \wedge \text{numeral } a$   
 $x > 0 \implies x \text{ powr } \text{Ratreal} (\text{Frct} (-\text{numeral } a, \text{Numeral1})) = \text{inverse} (x \wedge \text{numeral } a)$   
 ⟨proof⟩

**lemmas** *eval-numeral-simps* =  
*real-numeral-to-Ratreal real-arith-code rat-arith-code Num.arith-simps*  
*Rat.normalize-def fst-conv snd-conv gcd-0-int gcd-0-left-int gcd.bottom-right-bottom*  
*gcd.bottom-left-bottom*  
*gcd-neg1-int gcd-neg2-int gcd-numeral-red zmod-numeral-Bit0 zmod-numeral-Bit1*  
*power-numeral-simps*  
*divmod-numeral-simps numeral-One [symmetric] Groups.Let-0 Num.Let-numeral*  
*Suc-to-numeral power-numeral*  
*greaterThanLessThan-iff atLeastAtMost-iff atLeastLessThan-iff greaterThanAt-*  
*Most-iff rat-powr*

*Num.pow.simps Num.sqr.simps Product-Type.split of-int-numeral of-int-neg-numeral of-nat-numeral*

*<ML>*

**lemma**  $21254387548659589512 * 314213523632464357453884361 * 2342523623324234 * 56432743858724173474$   
 $12561712738645824362329316482973164398214286 \text{ powr } 2 /$   
 $(1130246312978423123 + 231212374631082764842731842 * 122474378389424362347451251263)$   
 $>$   
 $(12313244512931247243543279768645745929475829310651205623844 :: \text{real})$   
*<proof>*

**end**

## 7 The proof methods

### 7.1 Master theorem and termination

**theory** *Akra-Bazzi-Method*

**imports**

*Complex-Main*

*Akra-Bazzi*

*Master-Theorem*

*Eval-Numeral*

**begin**

**lemma** *landau-symbol-ge-3-cong*:

**assumes** *landau-symbol*  $L L' Lr$

**assumes**  $\bigwedge x :: 'a :: \text{linordered-semidom. } x \geq 3 \implies f x = g x$

**shows**  $L \text{ at-top } (f) = L \text{ at-top } (g)$

*<proof>*

**lemma** *exp-1-lt-3*:  $\text{exp } (1 :: \text{real}) < 3$

*<proof>*

**lemma** *ln-ln-pos*:

**assumes**  $(x :: \text{real}) \geq 3$

**shows**  $\ln (\ln x) > 0$

*<proof>*

**definition** *akra-bazzi-terms where*

*akra-bazzi-terms*  $x_0 x_1 bs ts = (\forall i < \text{length } bs. \text{akra-bazzi-term } x_0 x_1 (bs!i) (ts!i))$

**lemma** *akra-bazzi-termsI*:

$(\bigwedge i. i < \text{length } bs \implies \text{akra-bazzi-term } x_0 x_1 (bs!i) (ts!i)) \implies \text{akra-bazzi-terms } x_0 x_1 bs ts$

*<proof>*

**lemma** *master-theorem-functionI*:

**assumes**  $\forall x \in \{x_0..<x_1\}. f\ x \geq 0$   
**assumes**  $\forall x \geq x_1. f\ x = g\ x + (\sum i < k. as\ !\ i * f\ ((ts\ !\ i)\ x))$   
**assumes**  $\forall x \geq x_1. g\ x \geq 0$   
**assumes**  $\forall a \in set\ as. a \geq 0$   
**assumes** *list-ex*  $(\lambda a. a > 0)\ as$   
**assumes**  $\forall b \in set\ bs. b \in \{0 < .. < 1\}$   
**assumes**  $k \neq 0$   
**assumes** *length*  $as = k$   
**assumes** *length*  $bs = k$   
**assumes** *length*  $ts = k$   
**assumes** *akra-bazzi-terms*  $x_0\ x_1\ bs\ ts$   
**shows** *master-theorem-function*  $x_0\ x_1\ k\ as\ bs\ ts\ f\ g$   
*<proof>*

**lemma** *akra-bazzi-term-measure*:

$x \geq x_1 \implies akra-bazzi-term\ 0\ x_1\ b\ t \implies (t\ x, x) \in Wellfounded.measure\ (\lambda n :: nat. n)$   
 $x > x_1 \implies akra-bazzi-term\ 0\ (Suc\ x_1)\ b\ t \implies (t\ x, x) \in Wellfounded.measure\ (\lambda n :: nat. n)$   
*<proof>*

**lemma** *measure-prod-conv*:

$((a, b), (c, d)) \in Wellfounded.measure\ (\lambda x. t\ (fst\ x)) \longleftrightarrow (a, c) \in Wellfounded.measure\ t$   
 $((e, f), (g, h)) \in Wellfounded.measure\ (\lambda x. t\ (snd\ x)) \longleftrightarrow (f, h) \in Wellfounded.measure\ t$   
*<proof>*

**lemmas** *measure-prod-conv'* = *measure-prod-conv* [where  $t = \lambda x. x$ ]

**lemma** *akra-bazzi-termination-simps*:

**fixes**  $x :: nat$   
**shows**  $a * real\ x / b = a/b * real\ x$   $real\ x / b = 1/b * real\ x$   
*<proof>*

**lemma** *akra-bazzi-params-nonzeroI*:

$length\ as = length\ bs \implies$   
 $(\forall a \in set\ as. a \geq 0) \implies (\forall b \in set\ bs. b \in \{0 < .. < 1\}) \implies (\exists a \in set\ as. a > 0) \implies$   
*akra-bazzi-params-nonzero*  $(length\ as)\ as\ bs$  *<proof>*

**lemmas** *akra-bazzi-p-rel-intros* =

*akra-bazzi-params-nonzero.p-lessI* [rotated, OF - *akra-bazzi-params-nonzeroI*]  
*akra-bazzi-params-nonzero.p-greaterI* [rotated, OF - *akra-bazzi-params-nonzeroI*]  
*akra-bazzi-params-nonzero.p-leI* [rotated, OF - *akra-bazzi-params-nonzeroI*]  
*akra-bazzi-params-nonzero.p-geI* [rotated, OF - *akra-bazzi-params-nonzeroI*]  
*akra-bazzi-params-nonzero.p-boundsI* [rotated, OF - *akra-bazzi-params-nonzeroI*]  
*akra-bazzi-params-nonzero.p-boundsI'* [rotated, OF - *akra-bazzi-params-nonzeroI*]

**lemma** *eval-length*:  $length\ [] = 0$   $length\ (x \# xs) = Suc\ (length\ xs)$  *<proof>*

**lemma** *eval-akra-bazzi-sum*:

$$\begin{aligned} & (\sum i < 0. as!i * bs!i \text{ powr } x) = 0 \\ & (\sum i < \text{Suc } 0. (a\#as)!i * (b\#bs)!i \text{ powr } x) = a * b \text{ powr } x \\ & (\sum i < \text{Suc } k. (a\#as)!i * (b\#bs)!i \text{ powr } x) = a * b \text{ powr } x + (\sum i < k. as!i * bs!i \\ & \text{ powr } x) \\ & \langle \text{proof} \rangle \end{aligned}$$

**lemma** *eval-akra-bazzi-sum'*:

$$\begin{aligned} & (\sum i < 0. as!i * f ((ts!i) x)) = 0 \\ & (\sum i < \text{Suc } 0. (a\#as)!i * f (((t\#ts)!i) x)) = a * f (t x) \\ & (\sum i < \text{Suc } k. (a\#as)!i * f (((t\#ts)!i) x)) = a * f (t x) + (\sum i < k. as!i * f ((ts!i) \\ & x)) \\ & \langle \text{proof} \rangle \end{aligned}$$

**lemma** *akra-bazzi-termsI'*:

$$\begin{aligned} & \text{akra-bazzi-terms } x_0 \ x_1 \ [] \ [] \\ & \text{akra-bazzi-term } x_0 \ x_1 \ b \ t \implies \text{akra-bazzi-terms } x_0 \ x_1 \ bs \ ts \implies \text{akra-bazzi-terms} \\ & x_0 \ x_1 \ (b\#bs) \ (t\#ts) \\ & \langle \text{proof} \rangle \end{aligned}$$

**lemma** *ball-set-intros*:  $(\forall x \in \text{set } []. P \ x) \implies (\forall x \in \text{set } xs. P \ x) \implies (\forall x \in \text{set} \\ (x\#xs). P \ x)$   
 $\langle \text{proof} \rangle$

**lemma** *ball-set-simps*:  $(\forall x \in \text{set } []. P \ x) = \text{True} \ (\forall x \in \text{set } (x\#xs). P \ x) = (P \ x \wedge \\ (\forall x \in \text{set } xs. P \ x))$   
 $\langle \text{proof} \rangle$

**lemma** *beX-set-simps*:  $(\exists x \in \text{set } []. P \ x) = \text{False} \ (\exists x \in \text{set } (x\#xs). P \ x) = (P \ x \vee \\ (\exists x \in \text{set } xs. P \ x))$   
 $\langle \text{proof} \rangle$

**lemma** *eval-akra-bazzi-le-list-ex*:

$$\begin{aligned} & \text{list-ex } P \ (x\#y\#xs) \longleftrightarrow P \ x \vee \text{list-ex } P \ (y\#xs) \\ & \text{list-ex } P \ [x] \longleftrightarrow P \ x \\ & \text{list-ex } P \ [] \longleftrightarrow \text{False} \\ & \langle \text{proof} \rangle \end{aligned}$$

**lemma** *eval-akra-bazzi-le-sum-list*:

$$\begin{aligned} & x \leq \text{sum-list } [] \longleftrightarrow x \leq 0 \ x \leq \text{sum-list } (y\#ys) \longleftrightarrow x \leq y + \text{sum-list } ys \\ & x \leq z + \text{sum-list } [] \longleftrightarrow x \leq z \ x \leq z + \text{sum-list } (y\#ys) \longleftrightarrow x \leq z + y + \text{sum-list} \\ & ys \\ & \langle \text{proof} \rangle \end{aligned}$$

**lemma** *atLeastLessThanE*:  $x \in \{a..<b\} \implies (x \geq a \implies x < b \implies P) \implies P$   
 $\langle \text{proof} \rangle$

**lemma** *master-theorem-preprocess*:

$$\begin{aligned}
\Theta(\lambda n::\text{nat}. 1) &= \Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } 0) \\
\Theta(\lambda n::\text{nat}. \text{real } n) &= \Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } 1) \\
O(\lambda n::\text{nat}. 1) &= O(\lambda n::\text{nat}. \text{real } n \text{ powr } 0) \\
O(\lambda n::\text{nat}. \text{real } n) &= O(\lambda n::\text{nat}. \text{real } n \text{ powr } 1)
\end{aligned}$$

$$\begin{aligned}
\Theta(\lambda n::\text{nat}. \ln (\ln (\text{real } n))) &= \Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } 0 * \ln (\ln (\text{real } n))) \\
\Theta(\lambda n::\text{nat}. \text{real } n * \ln (\ln (\text{real } n))) &= \Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } 1 * \ln (\ln (\text{real } n)))
\end{aligned}$$

$$\begin{aligned}
\Theta(\lambda n::\text{nat}. \ln (\text{real } n)) &= \Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } 0 * \ln (\text{real } n) \text{ powr } 1) \\
\Theta(\lambda n::\text{nat}. \text{real } n * \ln (\text{real } n)) &= \Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } 1 * \ln (\text{real } n) \text{ powr } 1) \\
\Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } p * \ln (\text{real } n)) &= \Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } p * \ln (\text{real } n) \\
&\text{powr } 1) \\
\Theta(\lambda n::\text{nat}. \ln (\text{real } n) \text{ powr } p') &= \Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } 0 * \ln (\text{real } n) \text{ powr } p') \\
\Theta(\lambda n::\text{nat}. \text{real } n * \ln (\text{real } n) \text{ powr } p') &= \Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } 1 * \ln (\text{real } n) \\
&\text{powr } p') \\
\langle \text{proof} \rangle
\end{aligned}$$

**lemma** *akra-bazzi-term-imp-size-less*:

$$\begin{aligned}
x_1 \leq x &\implies \text{akra-bazzi-term } 0 \ x_1 \ b \ t \implies \text{size } (t \ x) < \text{size } x \\
x_1 < x &\implies \text{akra-bazzi-term } 0 \ (\text{Suc } x_1) \ b \ t \implies \text{size } (t \ x) < \text{size } x \\
\langle \text{proof} \rangle
\end{aligned}$$

**definition** *CLAMP* ( $f :: \text{nat} \Rightarrow \text{real}$ )  $x = (\text{if } x < 3 \text{ then } 0 \text{ else } f \ x)$

**definition** *CLAMP'* ( $f :: \text{nat} \Rightarrow \text{real}$ )  $x = (\text{if } x < 3 \text{ then } 0 \text{ else } f \ x)$

**definition** *MASTER-BOUND*  $a \ b \ c \ x = \text{real } x \text{ powr } a * \ln (\text{real } x) \text{ powr } b * \ln (\ln (\text{real } x)) \text{ powr } c$

**definition** *MASTER-BOUND'*  $a \ b \ x = \text{real } x \text{ powr } a * \ln (\text{real } x) \text{ powr } b$

**definition** *MASTER-BOUND''*  $a \ x = \text{real } x \text{ powr } a$

**lemma** *ln-1-imp-less-3*:

$$\begin{aligned}
\ln \ x = (1::\text{real}) &\implies x < 3 \\
\langle \text{proof} \rangle
\end{aligned}$$

**lemma** *ln-1-imp-less-3'*:  $\ln (\text{real } (x::\text{nat})) = 1 \implies x < 3 \ \langle \text{proof} \rangle$

**lemma** *ln-ln-nonneg*:  $x \geq (3::\text{real}) \implies \ln (\ln \ x) \geq 0 \ \langle \text{proof} \rangle$

**lemma** *ln-ln-nonneg'*:  $x \geq (3::\text{nat}) \implies \ln (\ln (\text{real } x)) \geq 0 \ \langle \text{proof} \rangle$

**lemma** *MASTER-BOUND-postproc*:

$$\begin{aligned}
&\text{CLAMP } (\text{MASTER-BOUND}' \ a \ 0) = \text{CLAMP } (\text{MASTER-BOUND}'' \ a) \\
&\text{CLAMP } (\text{MASTER-BOUND}' \ a \ 1) = \text{CLAMP } (\lambda x. \text{CLAMP } (\text{MASTER-BOUND}'' \\
&a) \ x * \text{CLAMP } (\lambda x. \ln (\text{real } x)) \ x) \\
&\text{CLAMP } (\text{MASTER-BOUND}' \ a \ (\text{numeral } n)) = \\
&\quad \text{CLAMP } (\lambda x. \text{CLAMP } (\text{MASTER-BOUND}'' \ a) \ x * \text{CLAMP } (\lambda x. \ln (\text{real } x) \\
&\wedge \text{numeral } n) \ x) \\
&\text{CLAMP } (\text{MASTER-BOUND}' \ a \ (-1)) = \\
&\quad \text{CLAMP } (\lambda x. \text{CLAMP } (\text{MASTER-BOUND}'' \ a) \ x / \text{CLAMP } (\lambda x. \ln (\text{real } \\
&x)) \ x) \\
&\text{CLAMP } (\text{MASTER-BOUND}' \ a \ (-\text{numeral } n)) =
\end{aligned}$$

$$\begin{aligned} & \text{CLAMP } (\lambda x. \text{CLAMP } (\text{MASTER-BOUND'' } a) x / \text{CLAMP } (\lambda x. \text{ln } (\text{real } x) \\ & \text{^ numeral } n) x) \\ & \text{CLAMP } (\text{MASTER-BOUND' } a b) = \\ & \text{CLAMP } (\lambda x. \text{CLAMP } (\text{MASTER-BOUND'' } a) x * \text{CLAMP } (\lambda x. \text{ln } (\text{real } x) \\ & \text{powr } b) x) \end{aligned}$$

$$\begin{aligned} & \text{CLAMP } (\text{MASTER-BOUND'' } 0) = \text{CLAMP } (\lambda x. 1) \\ & \text{CLAMP } (\text{MASTER-BOUND'' } 1) = \text{CLAMP } (\lambda x. (\text{real } x)) \\ & \text{CLAMP } (\text{MASTER-BOUND'' } (\text{numeral } n)) = \text{CLAMP } (\lambda x. (\text{real } x) \text{^ numeral } \\ & n) \\ & \text{CLAMP } (\text{MASTER-BOUND'' } (-1)) = \text{CLAMP } (\lambda x. 1 / (\text{real } x)) \\ & \text{CLAMP } (\text{MASTER-BOUND'' } (-\text{numeral } n)) = \text{CLAMP } (\lambda x. 1 / (\text{real } x) \text{^ } \\ & \text{numeral } n) \\ & \text{CLAMP } (\text{MASTER-BOUND'' } a) = \text{CLAMP } (\lambda x. (\text{real } x) \text{ powr } a) \end{aligned}$$

**and MASTER-BOUND-UNCLAMP:**

$$\begin{aligned} & \text{CLAMP } (\lambda x. \text{CLAMP } f x * \text{CLAMP } g x) = \text{CLAMP } (\lambda x. f x * g x) \\ & \text{CLAMP } (\lambda x. \text{CLAMP } f x / \text{CLAMP } g x) = \text{CLAMP } (\lambda x. f x / g x) \\ & \text{CLAMP } (\text{CLAMP } f) = \text{CLAMP } f \\ & \langle \text{proof} \rangle \end{aligned}$$

**context**  
**begin**

**private lemma CLAMP-:**

$$\text{landau-symbol } L L' Lr \implies L \text{ at-top } (f :: \text{nat} \Rightarrow \text{real}) \equiv L \text{ at-top } (\lambda x. \text{CLAMP } f x)$$

$\langle \text{proof} \rangle$  **lemma UNCLAMP'-:**

$$\text{landau-symbol } L L' Lr \implies L \text{ at-top } (\text{CLAMP}' (\text{MASTER-BOUND } a b c)) \equiv L \text{ at-top } (\text{MASTER-BOUND } a b c)$$

$\langle \text{proof} \rangle$  **lemma UNCLAMP-:**

$$\text{landau-symbol } L L' Lr \implies L \text{ at-top } (\text{CLAMP } f) \equiv L \text{ at-top } (f)$$

$\langle \text{proof} \rangle$

**lemmas CLAMP = landau-symbols[THEN CLAMP-]**

**lemmas UNCLAMP' = landau-symbols[THEN UNCLAMP'-]**

**lemmas UNCLAMP = landau-symbols[THEN UNCLAMP-]**

**end**

**lemma propagate-CLAMP:**

$$\text{CLAMP } (\lambda x. f x * g x) = \text{CLAMP}' (\lambda x. \text{CLAMP } f x * \text{CLAMP } g x)$$

$$\text{CLAMP } (\lambda x. f x / g x) = \text{CLAMP}' (\lambda x. \text{CLAMP } f x / \text{CLAMP } g x)$$

$$\text{CLAMP } (\lambda x. \text{inverse } (f x)) = \text{CLAMP}' (\lambda x. \text{inverse } (\text{CLAMP } f x))$$

$$\text{CLAMP } (\lambda x. \text{real } x) = \text{CLAMP}' (\text{MASTER-BOUND } 1 0 0)$$

$$\text{CLAMP } (\lambda x. \text{real } x \text{ powr } a) = \text{CLAMP}' (\text{MASTER-BOUND } a 0 0)$$

$$\text{CLAMP } (\lambda x. \text{real } x \text{^ } a) = \text{CLAMP}' (\text{MASTER-BOUND } (\text{real } a) 0 0)$$

$$\text{CLAMP } (\lambda x. \text{ln } (\text{real } x)) = \text{CLAMP}' (\text{MASTER-BOUND } 0 1 0)$$

$$\text{CLAMP } (\lambda x. \text{ln } (\text{real } x) \text{ powr } b) = \text{CLAMP}' (\text{MASTER-BOUND } 0 b 0)$$

$CLAMP (\lambda x. \ln (\text{real } x) \wedge b') = CLAMP' (MASTER-BOUND 0 (\text{real } b') 0)$   
 $CLAMP (\lambda x. \ln (\ln (\text{real } x))) = CLAMP' (MASTER-BOUND 0 0 1)$   
 $CLAMP (\lambda x. \ln (\ln (\text{real } x)) \text{ powr } c) = CLAMP' (MASTER-BOUND 0 0 c)$   
 $CLAMP (\lambda x. \ln (\ln (\text{real } x)) \wedge c') = CLAMP' (MASTER-BOUND 0 0 (\text{real } c'))$   
 $CLAMP' (CLAMP f) = CLAMP' f$   
 $CLAMP' (\lambda x. CLAMP' (MASTER-BOUND a1 b1 c1) x * CLAMP' (MASTER-BOUND a2 b2 c2) x) =$   
 $CLAMP' (MASTER-BOUND (a1+a2) (b1+b2) (c1+c2))$   
 $CLAMP' (\lambda x. CLAMP' (MASTER-BOUND a1 b1 c1) x / CLAMP' (MASTER-BOUND a2 b2 c2) x) =$   
 $CLAMP' (MASTER-BOUND (a1-a2) (b1-b2) (c1-c2))$   
 $CLAMP' (\lambda x. \text{inverse } (MASTER-BOUND a1 b1 c1) x) = CLAMP' (MASTER-BOUND (-a1) (-b1) (-c1))$   
 <proof>

**lemma** *numeral-assoc-simps*:

$((a::\text{real}) + \text{numeral } b) + \text{numeral } c = a + \text{numeral } (b + c)$   
 $(a + \text{numeral } b) - \text{numeral } c = a + \text{neg-numeral-class.sub } b c$   
 $(a - \text{numeral } b) + \text{numeral } c = a + \text{neg-numeral-class.sub } c b$   
 $(a - \text{numeral } b) - \text{numeral } c = a - \text{numeral } (b + c)$  <proof>

**lemmas** *CLAMP-aux* =

*arith-simps numeral-assoc-simps of-nat-power of-nat-mult of-nat-numeral*  
*one-add-one numeral-One [symmetric]*

**lemmas** *CLAMP-postproc* = *numeral-One*

**context** *master-theorem-function*

**begin**

**lemma** *master1-bigo-automation*:

**assumes**  $g \in O(\lambda x. \text{real } x \text{ powr } p')$   $1 < (\sum i < k. a_s ! i * b_s ! i \text{ powr } p')$   
**shows**  $f \in O(MASTER-BOUND p 0 0)$   
 <proof>

**lemma** *master1-automation*:

**assumes**  $g \in O(MASTER-BOUND'' p')$   $1 < (\sum i < k. a_s ! i * b_s ! i \text{ powr } p')$   
*eventually*  $(\lambda x. f x > 0)$  *at-top*  
**shows**  $f \in \Theta(MASTER-BOUND p 0 0)$   
 <proof>

**lemma** *master2-1-automation*:

**assumes**  $g \in \Theta(MASTER-BOUND' p p')$   $p' < -1$   
**shows**  $f \in \Theta(MASTER-BOUND p 0 0)$   
 <proof>

**lemma** *master2-2-automation*:

**assumes**  $g \in \Theta(MASTER-BOUND' p (-1))$   
**shows**  $f \in \Theta(MASTER-BOUND p 0 1)$

*<proof>*

**lemma** *master2-3-automation*:

**assumes**  $g \in \Theta(\text{MASTER-BOUND}' p (p' - 1))$   $p' > 0$

**shows**  $f \in \Theta(\text{MASTER-BOUND } p p' 0)$

*<proof>*

**lemma** *master3-automation*:

**assumes**  $g \in \Theta(\text{MASTER-BOUND}'' p')$   $1 > (\sum_{i < k}. as ! i * bs ! i \text{ powr } p')$

**shows**  $f \in \Theta(\text{MASTER-BOUND } p' 0 0)$

*<proof>*

**lemmas** *master-automation =*

*master1-automation master2-1-automation master2-2-automation*

*master2-2-automation master3-automation*

*<ML>*

**end**

**definition** *arith-consts* ( $x :: \text{real}$ ) ( $y :: \text{nat}$ ) =

*(if  $\neg (-x) + 3 / x * 5 - 1 \leq x \wedge \text{True} \vee \text{True} \longrightarrow \text{True}$  then*

*$x < \text{inverse } 3 \text{ powr } 21$  else  $x = \text{real } (\text{Suc } 0 \wedge 2 +$*

*(if  $42 - x \leq 1 \wedge 1 \text{ div } y = y \text{ mod } 2 \vee y < \text{Numeral1}$  then  $0$  else  $0$ )) + Numeral1)*

*<ML>*

**hide-const** *arith-consts*

*<ML>*

**hide-const** *CLAMP CLAMP' MASTER-BOUND MASTER-BOUND' MASTER-BOUND''*

**end**

**theory** *Akra-Bazzi-Approximation*

**imports**

*Complex-Main*

*Akra-Bazzi-Method*

*HOL-Decision-Procs.Approximation*

**begin**

**context** *akra-bazzi-params-nonzero*

**begin**

**lemma** *sum-alt*:  $(\sum_{i < k}. as ! i * bs ! i \text{ powr } p') = (\sum_{i < k}. as ! i * \text{exp } (p' * \ln (bs ! i)))$

*<proof>*

**lemma** *akra-bazzi-p-rel-intros-aux*:

$1 < (\sum_{i < k}. as!i * \exp (p' * \ln (bs!i))) \implies p' < p$   
 $1 > (\sum_{i < k}. as!i * \exp (p' * \ln (bs!i))) \implies p' > p$   
 $1 \leq (\sum_{i < k}. as!i * \exp (p' * \ln (bs!i))) \implies p' \leq p$   
 $1 \geq (\sum_{i < k}. as!i * \exp (p' * \ln (bs!i))) \implies p' \geq p$   
 $(\sum_{i < k}. as!i * \exp (x * \ln (bs!i))) \leq 1 \wedge (\sum_{i < k}. as!i * \exp (y * \ln (bs!i))) \geq 1 \implies p \in \{y..x\}$   
 $(\sum_{i < k}. as!i * \exp (x * \ln (bs!i))) < 1 \wedge (\sum_{i < k}. as!i * \exp (y * \ln (bs!i))) > 1 \implies p \in \{y < .. < x\}$   
*<proof>*

**end**

**lemmas** *akra-bazzi-p-rel-intros-exp* =

*akra-bazzi-params-nonzero.akra-bazzi-p-rel-intros-aux[rotated, OF - akra-bazzi-params-nonzeroI]*

**lemma** *eval-akra-bazzi-sum*:

$(\sum_{i < 0}. as!i * \exp (x * \ln (bs!i))) = 0$   
 $(\sum_{i < \text{Suc } 0}. (a\#as)!i * \exp (x * \ln ((b\#bs)!i))) = a * \exp (x * \ln b)$   
 $(\sum_{i < \text{Suc } k}. (a\#as)!i * \exp (x * \ln ((b\#bs)!i))) = a * \exp (x * \ln b) +$   
 $(\sum_{i < k}. as!i * \exp (x * \ln (bs!i)))$   
*<proof>*

*<ML>*

**end**

## 8 Examples

**theory** *Master-Theorem-Examples*

**imports**

*Complex-Main*

*Akra-Bazzi-Method*

*Akra-Bazzi-Approximation*

**begin**

### 8.1 Merge sort

**function** *merge-sort-cost* ::  $(\text{nat} \Rightarrow \text{real}) \Rightarrow \text{nat} \Rightarrow \text{real}$  **where**

*merge-sort-cost*  $t$  0 = 0

| *merge-sort-cost*  $t$  1 = 1

|  $n \geq 2 \implies \text{merge-sort-cost } t$   $n =$

*merge-sort-cost*  $t$   $(\text{nat } \lfloor \text{real } n / 2 \rfloor) + \text{merge-sort-cost } t$   $(\text{nat } \lceil \text{real } n / 2 \rceil) + t$

$n$

*<proof>*

**termination**  $\langle proof \rangle$

**lemma** *merge-sort-nonneg[simp]*:  $(\bigwedge n. t\ n \geq 0) \implies \text{merge-sort-cost } t\ x \geq 0$   
 $\langle proof \rangle$

**lemma**  $t \in \Theta(\lambda n. \text{real } n) \implies (\bigwedge n. t\ n \geq 0) \implies \text{merge-sort-cost } t \in \Theta(\lambda n. \text{real } n * \ln(\text{real } n))$   
 $\langle proof \rangle$

## 8.2 Karatsuba multiplication

**function** *karatsuba-cost* ::  $\text{nat} \Rightarrow \text{real}$  **where**

*karatsuba-cost* 0 = 0  
| *karatsuba-cost* 1 = 1  
|  $n \geq 2 \implies \text{karatsuba-cost } n =$   
     $3 * \text{karatsuba-cost } (\text{nat } \lceil \text{real } n / 2 \rceil) + \text{real } n$   
 $\langle proof \rangle$

**termination**  $\langle proof \rangle$

**lemma** *karatsuba-cost-nonneg[simp]*:  $\text{karatsuba-cost } n \geq 0$   
 $\langle proof \rangle$

**lemma**  $\text{karatsuba-cost} \in O(\lambda n. \text{real } n \text{ powr } \log 2 3)$   
 $\langle proof \rangle$

**lemma** *karatsuba-cost-pos*:  $n \geq 1 \implies \text{karatsuba-cost } n > 0$   
 $\langle proof \rangle$

**lemma**  $\text{karatsuba-cost} \in \Theta(\lambda n. \text{real } n \text{ powr } \log 2 3)$   
 $\langle proof \rangle$

## 8.3 Strassen matrix multiplication

**function** *strassen-cost* ::  $\text{nat} \Rightarrow \text{real}$  **where**

*strassen-cost* 0 = 0  
| *strassen-cost* 1 = 1  
|  $n \geq 2 \implies \text{strassen-cost } n = 7 * \text{strassen-cost } (\text{nat } \lceil \text{real } n / 2 \rceil) + \text{real } (n^2)$   
 $\langle proof \rangle$

**termination**  $\langle proof \rangle$

**lemma** *strassen-cost-nonneg[simp]*:  $\text{strassen-cost } n \geq 0$   
 $\langle proof \rangle$

**lemma**  $\text{strassen-cost} \in O(\lambda n. \text{real } n \text{ powr } \log 2 7)$   
 $\langle proof \rangle$

**lemma** *strassen-cost-pos*:  $n \geq 1 \implies \text{strassen-cost } n > 0$   
 $\langle proof \rangle$

**lemma**  $\text{strassen-cost} \in \Theta(\lambda n. \text{real } n \text{ powr } \log 2 7)$

*<proof>*

#### 8.4 Deterministic select

**function** *select-cost* :: *nat*  $\Rightarrow$  *real* **where**

$n \leq 20 \implies \text{select-cost } n = 0$

|  $n > 20 \implies \text{select-cost } n =$

$\text{select-cost } (\text{nat } \lfloor \text{real } n / 5 \rfloor) + \text{select-cost } (\text{nat } \lfloor 7 * \text{real } n / 10 \rfloor + 6) + 12$   
 $* \text{real } n / 5$

*<proof>*

**termination** *<proof>*

**lemma** *select-cost*  $\in \Theta(\lambda n. \text{real } n)$

*<proof>*

#### 8.5 Decreasing function

**function** *dec-cost* :: *nat*  $\Rightarrow$  *real* **where**

$n \leq 2 \implies \text{dec-cost } n = 1$

|  $n > 2 \implies \text{dec-cost } n = 0.5 * \text{dec-cost } (\text{nat } \lfloor \text{real } n / 2 \rfloor) + 1 / \text{real } n$

*<proof>*

**termination** *<proof>*

**lemma** *dec-cost*  $\in \Theta(\lambda x :: \text{nat}. \ln x / x)$

*<proof>*

#### 8.6 Example taken from Drmota and Szpakowski

**function** *drmota1* :: *nat*  $\Rightarrow$  *real* **where**

$n < 20 \implies \text{drmota1 } n = 1$

|  $n \geq 20 \implies \text{drmota1 } n = 2 * \text{drmota1 } (\text{nat } \lfloor \text{real } n / 2 \rfloor) + 8/9 * \text{drmota1 } (\text{nat } \lfloor 3 * \text{real } n / 4 \rfloor) + \text{real } n^2 / \ln (\text{real } n)$

*<proof>*

**termination** *<proof>*

**lemma** *drmota1*  $\in \Theta(\lambda n :: \text{real}. n^2 * \ln (\ln n))$

*<proof>*

**function** *drmota2* :: *nat*  $\Rightarrow$  *real* **where**

$n < 20 \implies \text{drmota2 } n = 1$

|  $n \geq 20 \implies \text{drmota2 } n = 1/3 * \text{drmota2 } (\text{nat } \lfloor \text{real } n / 3 + 1/2 \rfloor) + 2/3 * \text{drmota2 } (\text{nat } \lfloor 2 * \text{real } n / 3 - 1/2 \rfloor) + 1$

*<proof>*

**termination** *<proof>*

**lemma** *drmota2*  $\in \Theta(\lambda x. \ln (\text{real } x))$

*<proof>*

**lemma** *boncelet-phrase-length*:  
**fixes**  $p \delta :: \text{real}$  **assumes**  $p: p > 0 \ p < 1$  **and**  $\delta: \delta > 0 \ \delta < 1 \ 2*p + \delta < 2$   
**fixes**  $d :: \text{nat} \Rightarrow \text{real}$   
**defines**  $q \equiv 1 - p$   
**assumes**  $d\text{-nonneg}: \bigwedge n. d \ n \geq 0$   
**assumes**  $d\text{-rec}: \bigwedge n. n \geq 2 \implies d \ n = 1 + p * d \ (\text{nat} \lfloor p * \text{real } n + \delta \rfloor) + q * d$   
 $(\text{nat} \lfloor q * \text{real } n - \delta \rfloor)$   
**shows**  $d \in \Theta(\lambda x. \ln x)$   
 $\langle \text{proof} \rangle$

## 8.7 Transcendental exponents

**function** *foo-cost*  $:: \text{nat} \Rightarrow \text{real}$  **where**  
 $n < 200 \implies \text{foo-cost } n = 0$   
 $| \ n \geq 200 \implies \text{foo-cost } n =$   
 $\text{foo-cost } (\text{nat} \lfloor \text{real } n / 3 \rfloor) + \text{foo-cost } (\text{nat} \lfloor 3 * \text{real } n / 4 \rfloor + 42) + \text{real } n$   
 $\langle \text{proof} \rangle$   
**termination**  $\langle \text{proof} \rangle$

**lemma** *foo-cost-nonneg*  $[\text{simp}]$ :  $\text{foo-cost } n \geq 0$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{foo-cost} \in \Theta(\lambda n. \text{real } n \text{ powr } \text{akra-bazzi-exponent } [1,1] [1/3,3/4])$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{akra-bazzi-exponent } [1,1] [1/3,3/4] \in \{1.1519623..1.1519624\}$   
 $\langle \text{proof} \rangle$

## 8.8 Functions in locale contexts

**locale** *det-select* =  
**fixes**  $b :: \text{real}$   
**assumes**  $b: b > 0 \ b < 7/10$   
**begin**

**function** *select-cost'*  $:: \text{nat} \Rightarrow \text{real}$  **where**  
 $n \leq 20 \implies \text{select-cost}' \ n = 0$   
 $| \ n > 20 \implies \text{select-cost}' \ n =$   
 $\text{select-cost}' \ (\text{nat} \lfloor \text{real } n / 5 \rfloor) + \text{select-cost}' \ (\text{nat} \lfloor b * \text{real } n \rfloor + 6) + 6 * \text{real}$   
 $n + 5$   
 $\langle \text{proof} \rangle$   
**termination**  $\langle \text{proof} \rangle$

**lemma**  $a \geq 0 \implies \text{select-cost}' \in \Theta(\lambda n. \text{real } n)$   
 $\langle \text{proof} \rangle$

**end**

## 8.9 Non-curried functions

**function** *baz-cost* :: *nat* × *nat* ⇒ *real* **where**

$n \leq 2 \implies \text{baz-cost } (a, n) = 0$

$| n > 2 \implies \text{baz-cost } (a, n) = 3 * \text{baz-cost } (a, \text{nat } \lfloor \text{real } n / 2 \rfloor) + \text{real } a$   
⟨*proof*⟩

**termination** ⟨*proof*⟩

**lemma** *baz-cost-nonneg* [*simp*]:  $a \geq 0 \implies \text{baz-cost } (a, n) \geq 0$

⟨*proof*⟩

**lemma**

**assumes**  $a > 0$

**shows**  $(\lambda x. \text{baz-cost } (a, x)) \in \Theta(\lambda x. x \text{ powr } \log 2 3)$

⟨*proof*⟩

**function** *bar-cost* :: *nat* × *nat* ⇒ *real* **where**

$n \leq 2 \implies \text{bar-cost } (a, n) = 0$

$| n > 2 \implies \text{bar-cost } (a, n) = 3 * \text{bar-cost } (2 * a, \text{nat } \lfloor \text{real } n / 2 \rfloor) + \text{real } a$   
⟨*proof*⟩

**termination** ⟨*proof*⟩

## 8.10 Ham-sandwich trees

**function** *ham-sandwich-cost* :: *nat* ⇒ *real* **where**

$n < 4 \implies \text{ham-sandwich-cost } n = 1$

$| n \geq 4 \implies \text{ham-sandwich-cost } n =$   
 $\text{ham-sandwich-cost } (\text{nat } \lfloor n/4 \rfloor) + \text{ham-sandwich-cost } (\text{nat } \lfloor n/2 \rfloor) + 1$

⟨*proof*⟩

**termination** ⟨*proof*⟩

**lemma** *ham-sandwich-cost-pos* [*simp*]:  $\text{ham-sandwich-cost } n > 0$

⟨*proof*⟩

The golden ratio

**definition**  $\varphi = ((1 + \text{sqrt } 5) / 2 :: \text{real})$

**lemma** *φ-pos* [*simp*]:  $\varphi > 0$  **and** *φ-nonneg* [*simp*]:  $\varphi \geq 0$  **and** *φ-nonzero* [*simp*]:

$\varphi \neq 0$

⟨*proof*⟩

**lemma** *ham-sandwich-cost* ∈  $\Theta(\lambda n. n \text{ powr } (\log 2 \varphi))$

⟨*proof*⟩

**end**

## References

- [1] M. Akra and L. Bazzi. On the solution of linear recurrence equations. *Computational Optimization and Applications*, 10(2):195–210, 1998.
- [2] T. Leighton. Notes on better Master theorems for divide-and-conquer recurrences. 1996.