

Actuarial Mathematics

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March 24, 2023

Abstract

Actuarial Mathematics is a theory in applied mathematics, which is mainly used for determining the prices of insurance products and evaluating the liability of a company associating with insurance contracts. It is related to calculus, probability theory and financial theory, etc.

In this entry, I formalize the very basic part of Actuarial Mathematics in Isabelle/HOL. The first formalization is about the theory of interest which deals with interest rates, present value factors, an annuity certain, etc.

I have already formalized the basic part of Actuarial Mathematics in Coq (<https://github.com/Yosuke-Ito-345/Actuary>). This entry is currently the partial translation and a little generalization of the Coq formalization. The further translation in Isabelle/HOL is now proceeding.

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theory *Preliminaries*

imports *HOL-Analysis.Analysis*

begin

notation *powr* (infixr \cdot^{\wedge} 80)

1 Preliminary Definitions and Lemmas

lemma *seq-part-multiple*: **fixes** $m\ n :: \text{nat}$ **assumes** $m \neq 0$ **defines** $A \equiv \lambda i :: \text{nat}.$
 $\{i * m ..< (i+1) * m\}$
shows $\forall i\ j. i \neq j \longrightarrow A\ i \cap A\ j = \{\}$ **and** $(\bigcup_{i < n}. A\ i) = \{..< n * m\}$
(*proof*)

lemma (in field) divide-mult-cancel[simp]: fixes a b assumes b ≠ 0
 shows a / b * b = a
 ⟨proof⟩

lemma inverse-powr: (1/a).[^]b = a.[^]-b if a > 0 for a b :: real
 ⟨proof⟩

lemma powr-eq-one-iff-gen[simp]: a.[^]x = 1 ↔ x = 0 if a > 0 a ≠ 1 for a x :: real
 ⟨proof⟩

lemma powr-less-cancel2: 0 < a ⇒ 0 < x ⇒ 0 < y ⇒ x.[^]a < y.[^]a ⇒ x < y
 for a x y :: real
 ⟨proof⟩

lemma geometric-increasing-sum-aux: (1-r)[^]2 * (∑ k<n. (k+1)*r[^]k) = 1 - (n+1)*r[^]n + n*r[^](n+1)
 for n::nat and r::real
 ⟨proof⟩

lemma geometric-increasing-sum: (∑ k<n. (k+1)*r[^]k) = (1 - (n+1)*r[^]n + n*r[^](n+1)) / (1-r)[^]2
 if r ≠ 1 for n::nat and r::real
 ⟨proof⟩

lemma Reals-UNIV[simp]: ℝ = {x::real. True}
 ⟨proof⟩

lemma DERIV-fun-powr2:
 fixes a::real
 assumes a-pos: a > 0
 and f: DERIV f x :=> r
 shows DERIV (λx. a.[^](f x)) x :=> a.[^](f x) * r * ln a
 ⟨proof⟩

lemma has-real-derivative-powr2:
 assumes a-pos: a > 0
 shows ((λx. a.[^]x) has-real-derivative a.[^]x * ln a) (at x)
 ⟨proof⟩

lemma has-integral-powr2-from-0:
 fixes a c :: real
 assumes a-pos: a > 0 and a-neg-1: a ≠ 1 and c-nneg: c ≥ 0
 shows ((λx. a.[^]x) has-integral ((a.[^]c - 1) / (ln a)) {0..c})
 ⟨proof⟩

lemma integrable-on-powr2-from-0:
 fixes a c :: real

assumes $a\text{-pos}: a > 0$ **and** $a\text{-neg-1}: a \neq 1$ **and** $c\text{-nneg}: c \geq 0$
shows $(\lambda x. a. \hat{x})$ *integrable-on* $\{0..c\}$
 \langle *proof* \rangle

lemma *integrable-on-powr2-from-0-general*:
fixes $a\ c :: \text{real}$
assumes $a\text{-pos}: a > 0$ **and** $c\text{-nneg}: c \geq 0$
shows $(\lambda x. a. \hat{x})$ *integrable-on* $\{0..c\}$
 \langle *proof* \rangle

lemma *has-integral-null-interval*: **fixes** $a\ b :: \text{real}$ **and** $f :: \text{real} \Rightarrow \text{real}$ **assumes** $a \geq b$
shows $(f \text{ has-integral } 0)$ $\{a..b\}$
 \langle *proof* \rangle

lemma *has-integral-interval-reverse*: **fixes** $f :: \text{real} \Rightarrow \text{real}$ **and** $a\ b :: \text{real}$
assumes $a \leq b$
and *continuous-on* $\{a..b\}$ f
shows $((\lambda x. f (a+b-x)) \text{ has-integral } (\text{integral } \{a..b\} f))$ $\{a..b\}$
 \langle *proof* \rangle

end
theory *Interest*
imports *Preliminaries*
begin

2 List of Actuarial Notations (Global Scope)

definition $i\text{-nom} :: \text{real} \Rightarrow \text{nat} \Rightarrow \text{real}$ $(\$i[-] \hat{\{}} [0,0] 200)$
where $\$i[i] \hat{\{m\}} \equiv m * ((1+i). \hat{\{1/m\}} - 1)$ — nominal interest rate
definition $i\text{-force} :: \text{real} \Rightarrow \text{real}$ $(\$d[-] [0] 200)$
where $\$d[i] \equiv \ln (1+i)$ — force of interest
definition $d\text{-nom} :: \text{real} \Rightarrow \text{nat} \Rightarrow \text{real}$ $(\$d[-] \hat{\{}} [0,0] 200)$
where $\$d[i] \hat{\{m\}} \equiv \$i[i] \hat{\{m\}} / (1 + \$i[i] \hat{\{m\}}/m)$ — discount rate
abbreviation $d\text{-nom-yr} :: \text{real} \Rightarrow \text{real}$ $(\$d[-] [0] 200)$
where $\$d[i] \equiv \$d[i] \hat{\{1\}}$ — Post-fix "yr" stands for "year".
definition $v\text{-pres} :: \text{real} \Rightarrow \text{real}$ $(\$v[-] [0] 200)$
where $\$v[i] \equiv 1 / (1+i)$ — present value factor
definition $ann :: \text{real} \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow \text{real}$ $(\$a[-] \hat{\{}}'-- [0,0,101] 200)$
where $\$a[i] \hat{\{m\}}-n \equiv \sum_{k < n * m}. \$v[i]. \hat{\{(k+1::nat)/m\}} / m$
— present value of an immediate annuity
abbreviation $ann\text{-yr} :: \text{real} \Rightarrow \text{nat} \Rightarrow \text{real}$ $(\$a[-]'-- [0,101] 200)$
where $\$a[i]-n \equiv \$a[i] \hat{\{1\}}-n$
definition $acc :: \text{real} \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow \text{real}$ $(\$s[-] \hat{\{}}'-- [0,0,101] 200)$
where $\$s[i] \hat{\{m\}}-n \equiv \sum_{k < n * m}. (1+i). \hat{\{(k::nat)/m\}} / m$
— future value of an immediate annuity
— The name "acc" stands for "accumulation".
abbreviation $acc\text{-yr} :: \text{real} \Rightarrow \text{nat} \Rightarrow \text{real}$ $(\$s[-]'-- [0] 200)$
where $\$s[i]-n \equiv \$s[i] \hat{\{1\}}-n$

definition *ann-due* :: $real \Rightarrow nat \Rightarrow nat \Rightarrow real$ ($\$a''''[-]\hat{\sim}\{-}\'-- [0,0,101] 200$)
where $\$a''[i]\hat{\sim}\{m\}-n \equiv \sum k < n * m. \$v[i]. \hat{\sim}(k::nat)/m) / m$
— present value of an annuity-due

abbreviation *ann-due-yr* :: $real \Rightarrow nat \Rightarrow real$ ($\$a''''[-]\'-- [0,101] 200$)
where $\$a''[i]-n \equiv \$a''[i]\hat{\sim}\{1\}-n$

definition *acc-due* :: $real \Rightarrow nat \Rightarrow nat \Rightarrow real$ ($\$s''''[-]\hat{\sim}\{-}\'-- [0,0,101] 200$)
where $\$s''[i]\hat{\sim}\{m\}-n \equiv \sum k < n * m. (1+i). \hat{\sim}(k+1::nat)/m) / m$
— future value of an annuity-due

abbreviation *acc-due-yr* :: $real \Rightarrow nat \Rightarrow real$ ($\$s''''[-]\'-- [0,101] 200$)
where $\$s''[i]-n \equiv \$s''[i]\hat{\sim}\{1\}-n$

definition *ann-cont* :: $real \Rightarrow real \Rightarrow real$ ($\$a''[-]\'-- [0,101] 200$)
where $\$a''[i]-n \equiv integral \{0..n\} (\lambda t::real. \$v[i]. \hat{\sim}t)$
— present value of a continuous annuity

definition *acc-cont* :: $real \Rightarrow real \Rightarrow real$ ($\$s''[-]\'-- [0,101] 200$)
where $\$s''[i]-n \equiv integral \{0..n\} (\lambda t::real. (1+i). \hat{\sim}t)$
— future value of a continuous annuity

definition *perp* :: $real \Rightarrow nat \Rightarrow real$ ($\$a[-]\hat{\sim}\{-}\'--\infty [0,0] 200$)
where $\$a[i]\hat{\sim}\{m\}-\infty \equiv 1 / \$i[i]\hat{\sim}\{m\}$
— present value of a perpetual annuity

abbreviation *perp-yr* :: $real \Rightarrow real$ ($\$a[-]\'--\infty [0] 200$)
where $\$a[i]-\infty \equiv \$a[i]\hat{\sim}\{1\}-\infty$

definition *perp-due* :: $real \Rightarrow nat \Rightarrow real$ ($\$a''''[-]\hat{\sim}\{-}\'--\infty [0,0] 200$)
where $\$a''[i]\hat{\sim}\{m\}-\infty \equiv 1 / \$d[i]\hat{\sim}\{m\}$
— present value of a perpetual annuity-due

abbreviation *perp-due-yr* :: $real \Rightarrow real$ ($\$a''''[-]\'--\infty [0] 200$)
where $\$a''[i]-\infty \equiv \$a''[i]\hat{\sim}\{1\}-\infty$

definition *ann-incr* :: $nat \Rightarrow real \Rightarrow nat \Rightarrow nat \Rightarrow real$
($\$(I\hat{\sim}\{-}a)\hat{\sim}\{-}\'-- [0,0,0,101] 200$)
where $\$(I\hat{\sim}\{l\}a)[i]\hat{\sim}\{m\}-n \equiv \sum k < n * m. \$v[i]. \hat{\sim}(k+1::nat)/m) * [l*(k+1::nat)/m]$
/ ($l*m$)
— present value of an increasing annuity
— This is my original definition.
— Here, "l" represents the number of increments per unit time.

abbreviation *ann-incr-lvl* :: $real \Rightarrow nat \Rightarrow nat \Rightarrow real$
($\$(Ia)\hat{\sim}\{-}\'-- [0,0,101] 200$)
where $\$(Ia)[i]\hat{\sim}\{m\}-n \equiv \$(I\hat{\sim}\{1\}a)[i]\hat{\sim}\{m\}-n$
— The post-fix "lvl" stands for "level".

abbreviation *ann-incr-yr* :: $real \Rightarrow nat \Rightarrow real$ ($\$(Ia)\hat{\sim}\{-}\'-- [0,101] 200$)
where $\$(Ia)[i]-n \equiv \$(Ia)[i]\hat{\sim}\{1\}-n$

definition *acc-incr* :: $nat \Rightarrow real \Rightarrow nat \Rightarrow nat \Rightarrow real$
($\$(I\hat{\sim}\{-}s)\hat{\sim}\{-}\'-- [0,0,0,101] 200$)
where $\$(I\hat{\sim}\{l\}s)[i]\hat{\sim}\{m\}-n \equiv \sum k < n * m. (1+i). \hat{\sim}(n-(k+1::nat)/m) * [l*(k+1::nat)/m]$
/ ($l*m$)
— future value of an increasing annuity

abbreviation *acc-incr-lvl* :: $real \Rightarrow nat \Rightarrow nat \Rightarrow real$
($\$(Is)\hat{\sim}\{-}\'-- [0,0,101] 200$)
where $\$(Is)[i]\hat{\sim}\{m\}-n \equiv \$(I\hat{\sim}\{1\}s)[i]\hat{\sim}\{m\}-n$

abbreviation *acc-incr-yr* :: $real \Rightarrow nat \Rightarrow real$ ($\$(Is)\hat{\sim}\{-}\'-- [0,101] 200$)
where $\$(Is)[i]-n \equiv \$(Is)[i]\hat{\sim}\{1\}-n$

definition *ann-due-incr* :: $\text{nat} \Rightarrow \text{real} \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow \text{real}$
 $(\$'(I\{\cdot\}a''''')[-]\{\cdot\}'-- [0,0,0,101] 200)$
where $\$(I\{l\}a'')[i]\{m\}-n \equiv \sum_{k < n * m}. \$v[i]. \wedge(k::\text{nat})/m * \lceil l * (k+1::\text{nat})/m \rceil / (l * m)$

abbreviation *ann-due-incr-lvl* :: $\text{real} \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow \text{real}$
 $(\$'(Ia''''')[-]\{\cdot\}'-- [0,0,101] 200)$
where $\$(Ia'')[i]\{m\}-n \equiv \$(I\{1\}a'')[i]\{m\}-n$

abbreviation *ann-due-incr-yr* :: $\text{real} \Rightarrow \text{nat} \Rightarrow \text{real}$ $(\$'(Ia''''')[-]'-- [0,101] 200)$
where $\$(Ia'')[i]-n \equiv \$(Ia'')[i]\{1\}-n$

definition *acc-due-incr* :: $\text{nat} \Rightarrow \text{real} \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow \text{real}$
 $(\$'(I\{\cdot\}s''''')[-]\{\cdot\}'-- [0,0,0,101] 200)$
where $\$(I\{l\}s'')[i]\{m\}-n \equiv \sum_{k < n * m}. (1+i). \wedge(n-(k::\text{nat})/m) * \lceil l * (k+1::\text{nat})/m \rceil / (l * m)$

abbreviation *acc-due-incr-lvl* :: $\text{real} \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow \text{real}$
 $(\$'(Is''''')[-]\{\cdot\}'-- [0,0,101] 200)$
where $\$(Is'')[i]\{m\}-n \equiv \$(I\{1\}s'')[i]\{m\}-n$

abbreviation *acc-due-incr-yr* :: $\text{real} \Rightarrow \text{nat} \Rightarrow \text{real}$ $(\$'(Is''''')[-]'-- [0,101] 200)$
where $\$(Is'')[i]-n \equiv \$(Is'')[i]\{1\}-n$

definition *perp-incr* :: $\text{nat} \Rightarrow \text{real} \Rightarrow \text{nat} \Rightarrow \text{real}$ $(\$'(I\{\cdot\}a')[-]\{\cdot\}'-\infty [0,0,0] 200)$
where $\$(I\{l\}a)[i]\{m\}-\infty \equiv \lim (\lambda n. \$(I\{l\}a)[i]\{m\}-n)$

abbreviation *perp-incr-lvl* :: $\text{real} \Rightarrow \text{nat} \Rightarrow \text{real}$ $(\$'(Ia')[-]\{\cdot\}'-\infty [0,0] 200)$
where $\$(Ia)[i]\{m\}-\infty \equiv \$(I\{1\}a)[i]\{m\}-\infty$

abbreviation *perp-incr-yr* :: $\text{real} \Rightarrow \text{real}$ $(\$'(Ia')[-]'-\infty [0] 200)$
where $\$(Ia)[i]-\infty \equiv \$(Ia)[i]\{1\}-\infty$

definition *perp-due-incr* :: $\text{nat} \Rightarrow \text{real} \Rightarrow \text{nat} \Rightarrow \text{real}$ $(\$'(I\{\cdot\}a''''')[-]\{\cdot\}'-\infty [0,0,0] 200)$
where $\$(I\{l\}a'')[i]\{m\}-\infty \equiv \lim (\lambda n. \$(I\{l\}a'')[i]\{m\}-n)$

abbreviation *perp-due-incr-lvl* :: $\text{real} \Rightarrow \text{nat} \Rightarrow \text{real}$ $(\$'(Ia''''')[-]\{\cdot\}'-\infty [0,0] 200)$
where $\$(Ia'')[i]\{m\}-\infty \equiv \$(I\{1\}a'')[i]\{m\}-\infty$

abbreviation *perp-due-incr-yr* :: $\text{real} \Rightarrow \text{real}$ $(\$'(Ia''''')[-]'-\infty [0] 200)$
where $\$(Ia'')[i]-\infty \equiv \$(Ia'')[i]\{1\}-\infty$

3 Theory of Interest

locale *interest* =

fixes $i :: \text{real}$ — i stands for an interest rate.

assumes *v-futr-pos*: $1 + i > 0$ — Assume that the future value is positive.

context *interest*

begin

abbreviation *i-nom'* :: $\text{nat} \Rightarrow \text{real}$ $(\$i\{\cdot\} [0] 200)$

where $\$i\{m\} \equiv \$i[i]\{m\}$

abbreviation *i-force'* :: real $(\$ \delta)$

where $\$ \delta \equiv \$ \delta[i]$

abbreviation *d-nom'* :: $\text{nat} \Rightarrow \text{real}$ $(\$d\{\cdot\} [0] 200)$

where $\$d\{m\} \equiv \$d[i]\{m\}$

abbreviation $d\text{-nom-yr}' :: \text{real } (\$d)$
where $\$d \equiv \$d[i]$
abbreviation $v\text{-pres}' :: \text{real } (\$v)$
where $\$v \equiv \$v[i]$
abbreviation $\text{ann}' :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{real } (\$a\{\}\text{'-- } [0,101] \text{ } 200)$
where $\$a\{m\}\text{-n} \equiv \$a[i]\{m\}\text{-n}$
abbreviation $\text{ann-yr}' :: \text{nat} \Rightarrow \text{real } (\$a'\text{'-- } [101] \text{ } 200)$
where $\$a\text{-n} \equiv \$a[i]\text{-n}$
abbreviation $\text{acc}' :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{real } (\$s\{\}\text{'-- } [0,101] \text{ } 200)$
where $\$s\{m\}\text{-n} \equiv \$s[i]\{m\}\text{-n}$
abbreviation $\text{acc-yr}' :: \text{nat} \Rightarrow \text{real } (\$s'\text{'-- } [101] \text{ } 200)$
where $\$s\text{-n} \equiv \$s[i]\text{-n}$
abbreviation $\text{ann-due}' :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{real } (\$a''''\{\}\text{'-- } [0,101] \text{ } 200)$
where $\$a''\{m\}\text{-n} \equiv \$a''[i]\{m\}\text{-n}$
abbreviation $\text{ann-due-yr}' :: \text{nat} \Rightarrow \text{real } (\$a'''''\text{'-- } [101] \text{ } 200)$
where $\$a''\text{-n} \equiv \$a''[i]\text{-n}$
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where $\$s''\text{-n} \equiv \$s''[i]\text{-n}$
abbreviation $\text{ann-cont}' :: \text{real} \Rightarrow \text{real } (\$a'''\text{'-- } [101] \text{ } 200)$
where $\$a'\text{-n} \equiv \$a'[i]\text{-n}$
abbreviation $\text{acc-cont}' :: \text{real} \Rightarrow \text{real } (\$s'''\text{'-- } [101] \text{ } 200)$
where $\$s'\text{-n} \equiv \$s'[i]\text{-n}$
abbreviation $\text{perp}' :: \text{nat} \Rightarrow \text{real } (\$a\{\}\text{'-- } [0] \text{ } 200)$
where $\$a\{m\}\text{-}\infty \equiv \$a[i]\{m\}\text{-}\infty$
abbreviation $\text{perp-yr}' :: \text{real } (\$a'\text{'-- } \infty)$
where $\$a\text{-}\infty \equiv \$a[i]\text{-}\infty$
abbreviation $\text{perp-due}' :: \text{nat} \Rightarrow \text{real } (\$a''''\{\}\text{'-- } [0] \text{ } 200)$
where $\$a''\{m\}\text{-}\infty \equiv \$a''[i]\{m\}\text{-}\infty$
abbreviation $\text{perp-due-yr}' :: \text{real } (\$a'''''\text{'-- } \infty)$
where $\$a''\text{-}\infty \equiv \$a''[i]\text{-}\infty$
abbreviation $\text{ann-incr}' :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow \text{real } (\$(I\{\}a')\{\}\text{'-- } [0,0,101] \text{ } 200)$
where $\$(I\{l\}a)\{m\}\text{-n} \equiv \$(I\{l\}a)[i]\{m\}\text{-n}$
abbreviation $\text{ann-incr-lvl}' :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{real } (\$(Ia')\{\}\text{'-- } [0,101] \text{ } 200)$
where $\$(Ia)\{m\}\text{-n} \equiv \$(Ia)[i]\{m\}\text{-n}$
abbreviation $\text{ann-incr-yr}' :: \text{nat} \Rightarrow \text{real } (\$(Ia')'\text{'-- } [101] \text{ } 200)$
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where $\$(Is)\{m\}\text{-n} \equiv \$(Is)[i]\{m\}\text{-n}$
abbreviation $\text{acc-incr-yr}' :: \text{nat} \Rightarrow \text{real } (\$(Is')'\text{'-- } [101] \text{ } 200)$
where $\$(Is)\text{-n} \equiv \$(Is)[i]\text{-n}$
abbreviation $\text{ann-due-incr}' :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow \text{real } (\$(I\{\}a''''')\{\}\text{'-- } [0,0,101] \text{ } 200)$
where $\$(I\{l\}a'')\{m\}\text{-n} \equiv \$(I\{l\}a'')[i]\{m\}\text{-n}$

abbreviation $\text{ann-due-incr-lvl}' :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{real } (\$(Ia''''') \wedge \{-}\text{'-- } [0,101] \text{ } 200)$
where $\$(Ia'') \wedge \{m\}\text{-n} \equiv \$(Ia'')[i] \wedge \{m\}\text{-n}$
abbreviation $\text{ann-due-incr-yr}' :: \text{nat} \Rightarrow \text{real } (\$(Ia''''')'\text{'-- } [101] \text{ } 200)$
where $\$(Ia'')\text{-n} \equiv \$(Ia'')[i]\text{-n}$
abbreviation $\text{acc-due-incr}' :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow \text{real } (\$(I \wedge \{-}\text{'s''''') \wedge \{-}\text{'-- } [0,0,101] \text{ } 200)$
where $\$(I \wedge \{l\} s'') \wedge \{m\}\text{-n} \equiv \$(I \wedge \{l\} s'')[i] \wedge \{m\}\text{-n}$
abbreviation $\text{acc-due-incr-lvl}' :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{real } (\$(Is''''') \wedge \{-}\text{'-- } [0,101] \text{ } 200)$
where $\$(Is'') \wedge \{m\}\text{-n} \equiv \$(Is'')[i] \wedge \{m\}\text{-n}$
abbreviation $\text{acc-due-incr-yr}' :: \text{nat} \Rightarrow \text{real } (\$(Is''''')'\text{'-- } [101] \text{ } 200)$
where $\$(Is'')\text{-n} \equiv \$(Is'')[i]\text{-n}$
abbreviation $\text{perp-incr}' :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{real } (\$(I \wedge \{-}\text{'a}') \wedge \{-}\text{'-\infty } [0,0] \text{ } 200)$
where $\$(I \wedge \{l\} a) \wedge \{m\}\text{-\infty} \equiv \$(I \wedge \{l\} a)[i] \wedge \{m\}\text{-\infty}$
abbreviation $\text{perp-incr-lvl}' :: \text{nat} \Rightarrow \text{real } (\$(Ia') \wedge \{-}\text{'-\infty } [0] \text{ } 200)$
where $\$(Ia) \wedge \{m\}\text{-\infty} \equiv \$(Ia)[i] \wedge \{m\}\text{-\infty}$
abbreviation $\text{perp-incr-yr}' :: \text{real } (\$(Ia')'\text{'-\infty})$
where $\$(Ia)\text{-\infty} \equiv \$(Ia)[i]\text{-\infty}$
abbreviation $\text{perp-due-incr}' :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{real } (\$(I \wedge \{-}\text{'a''''') \wedge \{-}\text{'-\infty } [0,0] \text{ } 200)$
where $\$(I \wedge \{l\} a'') \wedge \{m\}\text{-\infty} \equiv \$(I \wedge \{l\} a'')[i] \wedge \{m\}\text{-\infty}$
abbreviation $\text{perp-due-incr-lvl}' :: \text{nat} \Rightarrow \text{real } (\$(Ia''''') \wedge \{-}\text{'-\infty } [0] \text{ } 200)$
where $\$(Ia'') \wedge \{m\}\text{-\infty} \equiv \$(Ia'')[i] \wedge \{m\}\text{-\infty}$
abbreviation $\text{perp-due-incr-yr}' :: \text{real } (\$(Ia''''')'\text{'-\infty})$
where $\$(Ia'')\text{-\infty} \equiv \$(Ia'')[i]\text{-\infty}$

lemma $v\text{-futr-}m\text{-pos}$: $1 + \$(i \wedge \{m\})/m > 0$ **if** $m \neq 0$ **for** $m::\text{nat}$
<proof>

lemma $i\text{-nom-}1$ [*simp*]: $\$(i \wedge \{1\}) = i$
<proof>

lemma $i\text{-nom-eff}$: $(1 + \$(i \wedge \{m\})/m) \wedge^m = 1 + i$ **if** $m \neq 0$ **for** $m::\text{nat}$
<proof>

lemma $i\text{-nom-}i$: $1 + \$(i \wedge \{m\})/m = (1+i) \wedge (1/m)$ **if** $m \neq 0$ **for** $m::\text{nat}$
<proof>

lemma $i\text{-nom-}0\text{-iff-}i\text{-}0$: $\$(i \wedge \{m\}) = 0 \iff i = 0$ **if** $m \neq 0$ **for** $m::\text{nat}$
<proof>

lemma $i\text{-nom-pos-iff-}i\text{-pos}$: $\$(i \wedge \{m\}) > 0 \iff i > 0$ **if** $m \neq 0$ **for** $m::\text{nat}$
<proof>

lemma $e\text{-delta}$: $\exp \$(\delta) = 1 + i$
<proof>

lemma $\text{delta-}0\text{-iff-}i\text{-}0$: $\$(\delta) = 0 \iff i = 0$
<proof>

lemma *lim-i-nom*: $(\lambda m. \$i\{m\}) \longrightarrow \δ
 ⟨proof⟩

lemma *d-nom-0-iff-i-0*: $\$d\{m\} = 0 \longleftrightarrow i = 0$ **if** $m \neq 0$ **for** $m::nat$
 ⟨proof⟩

lemma *d-nom-pos-iff-i-pos*: $\$d\{m\} > 0 \longleftrightarrow i > 0$ **if** $m \neq 0$ **for** $m::nat$
 ⟨proof⟩

lemma *d-nom-i-nom*: $1 - \$d\{m\}/m = 1 / (1 + \$i\{m\}/m)$ **if** $m \neq 0$ **for** $m::nat$
 ⟨proof⟩

lemma *lim-d-nom*: $(\lambda m. \$d\{m\}) \longrightarrow \δ
 ⟨proof⟩

lemma *v-pos*: $\$v > 0$
 ⟨proof⟩

lemma *v-1-iff-i-0*: $\$v = 1 \longleftrightarrow i = 0$
 ⟨proof⟩

lemma *v-lt-1-iff-i-pos*: $\$v < 1 \longleftrightarrow i > 0$
 ⟨proof⟩

lemma *v-i-nom*: $\$v = (1 + \$i\{m\}/m).\hat{-}m$ **if** $m \neq 0$ **for** $m::nat$
 ⟨proof⟩

lemma *i-v*: $1 + i = \$v.\hat{-}1$
 ⟨proof⟩

lemma *i-v-powr*: $(1 + i).\hat{a} = \$v.\hat{-}a$ **for** $a::real$
 ⟨proof⟩

lemma *v-delta*: $\ln \$v = - \δ
 ⟨proof⟩

lemma *is-derive-vpow*: *DERIV* $(\lambda t. \$v.\hat{t}) t :> - \$\delta * \$v.\hat{t}$
 ⟨proof⟩

lemma *d-nom-v*: $\$d\{m\} = m * (1 - \$v.\hat{(1/m)})$ **if** $m \neq 0$ **for** $m::nat$
 ⟨proof⟩

lemma *d-nom-i-nom-v*: $\$d\{m\} = \$i\{m\} * \$v.\hat{(1/m)}$ **if** $m \neq 0$ **for** $m::nat$
 ⟨proof⟩

lemma *a-calc*: $\$a\{m\}-n = (1 - \$v.\hat{n}) / \$i\{m\}$ **if** $m \neq 0$ $i \neq 0$ **for** $n m :: nat$
 ⟨proof⟩

lemma *a-calc-i-0*: $\$a\{m\}-n = n$ **if** $m \neq 0$ $i = 0$ **for** $n m :: nat$

<proof>

lemma *s-calc-i-0*: $\$s^{\wedge}\{m\}\text{-}n = n$ **if** $m \neq 0$ **if** $i = 0$ **for** $n m :: \text{nat}$
<proof>

lemma *s-a*: $\$s^{\wedge}\{m\}\text{-}n = (1+i)^{\wedge}n * \$a^{\wedge}\{m\}\text{-}n$ **if** $m \neq 0$ **for** $n m :: \text{nat}$
<proof>

lemma *s-calc*: $\$s^{\wedge}\{m\}\text{-}n = ((1+i)^{\wedge}n - 1) / \$i^{\wedge}\{m\}$ **if** $m \neq 0$ **if** $i \neq 0$ **for** $n m :: \text{nat}$
<proof>

lemma *a''-a*: $\$a''^{\wedge}\{m\}\text{-}n = (1+i)^{\wedge}(1/m) * \$a^{\wedge}\{m\}\text{-}n$ **if** $m \neq 0$ **for** $m :: \text{nat}$
<proof>

lemma *a-a''*: $\$a^{\wedge}\{m\}\text{-}n = \$v^{\wedge}(1/m) * \$a''^{\wedge}\{m\}\text{-}n$ **if** $m \neq 0$ **for** $m :: \text{nat}$
<proof>

lemma *a''-calc-i-0*: $\$a''^{\wedge}\{m\}\text{-}n = n$ **if** $m \neq 0$ **if** $i = 0$ **for** $n m :: \text{nat}$
<proof>

lemma *s''-calc-i-0*: $\$s''^{\wedge}\{m\}\text{-}n = n$ **if** $m \neq 0$ **if** $i = 0$ **for** $n m :: \text{nat}$
<proof>

lemma *a''-calc*: $\$a''^{\wedge}\{m\}\text{-}n = (1 - \$v^{\wedge}n) / \$d^{\wedge}\{m\}$ **if** $m \neq 0$ **if** $i \neq 0$ **for** $n m :: \text{nat}$
<proof>

lemma *s''-s*: $\$s''^{\wedge}\{m\}\text{-}n = (1+i)^{\wedge}(1/m) * \$s^{\wedge}\{m\}\text{-}n$ **if** $m \neq 0$ **for** $m :: \text{nat}$
<proof>

lemma *s-s''*: $\$s^{\wedge}\{m\}\text{-}n = \$v^{\wedge}(1/m) * \$s''^{\wedge}\{m\}\text{-}n$ **if** $m \neq 0$ **for** $m :: \text{nat}$
<proof>

lemma *s''-calc*: $\$s''^{\wedge}\{m\}\text{-}n = ((1+i)^{\wedge}n - 1) / \$d^{\wedge}\{m\}$ **if** $m \neq 0$ **if** $i \neq 0$ **for** $n m :: \text{nat}$
<proof>

lemma *s''-a''*: $\$s''^{\wedge}\{m\}\text{-}n = (1+i)^{\wedge}n * \$a''^{\wedge}\{m\}\text{-}n$ **if** $m \neq 0$ **for** $m :: \text{nat}$
<proof>

lemma *a'-calc*: $\$a'\text{-}n = (1 - \$v^{\wedge}n) / \$\delta$ **if** $i \neq 0$ **if** $n \geq 0$ **for** $n :: \text{real}$
<proof>

lemma *a'-calc-i-0*: $\$a'\text{-}n = n$ **if** $i = 0$ **if** $n \geq 0$ **for** $n :: \text{real}$
<proof>

lemma *s'-calc*: $\$s'\text{-}n = ((1+i)^{\wedge}n - 1) / \δ **if** $i \neq 0$ **if** $n \geq 0$ **for** $n :: \text{real}$
<proof>

lemma *s'-calc-i-0*: $\$s'-n = n$ if $i = 0$ $n \geq 0$ for $n::real$

<proof>

lemma *s'-a'*: $\$s'-n = (1+i).\widehat{n} * \$a'-n$ if $n \geq 0$ for $n::real$

<proof>

lemma *lim-m-a*: $(\lambda m. \$a\widehat{\{m\}}-n) \longrightarrow \$a'-n$ for $n::nat$

<proof>

lemma *lim-m-a''*: $(\lambda m. \$a''\widehat{\{m\}}-n) \longrightarrow \$a'-n$ for $n::nat$

<proof>

lemma *lim-m-s*: $(\lambda m. \$s\widehat{\{m\}}-n) \longrightarrow \$s'-n$ for $n::nat$

<proof>

lemma *lim-m-s''*: $(\lambda m. \$s''\widehat{\{m\}}-n) \longrightarrow \$s'-n$ for $n::nat$

<proof>

lemma *lim-n-a*: $(\lambda n. \$a\widehat{\{m\}}-n) \longrightarrow \$a\widehat{\{m\}}-\infty$ if $m \neq 0$ $i > 0$ for $m::nat$

<proof>

lemma *lim-n-a''*: $(\lambda n. \$a''\widehat{\{m\}}-n) \longrightarrow \$a''\widehat{\{m\}}-\infty$ if $m \neq 0$ $i > 0$ for $m::nat$

<proof>

lemma *lsm-lam*: $\$(I\widehat{\{l\}}s)\widehat{\{m\}}-n = (1+i).\widehat{n} * \$(I\widehat{\{l\}}a)\widehat{\{m\}}-n$

if $l \neq 0$ $m \neq 0$ for l n $m :: nat$

<proof>

lemma *Iam-calc*: $\$(Ia)\widehat{\{m\}}-n = (\sum j < n. (j+1)/m * (\sum k=j*m..<(j+1)*m. \$v.\widehat{((k+1)/m)}))$

if $m \neq 0$ for n $m :: nat$

<proof>

lemma *Ism-calc*: $\$(Is)\widehat{\{m\}}-n = (\sum j < n. (j+1)/m * (\sum k=j*m..<(j+1)*m. (1+i).\widehat{(n-(k+1)/m)}))$

if $m \neq 0$ for n $m :: nat$

<proof>

lemma *Imam-calc-aux*: $\$(I\widehat{\{m\}}a)\widehat{\{m\}}-n = (\sum k < n*m. \$v.\widehat{((k+1)/m)} * (k+1) / m^2)$

if $m \neq 0$ for $m::nat$

<proof>

lemma *Imam-calc*:

$\$(I\widehat{\{m\}}a)\widehat{\{m\}}-n = (\$v.\widehat{(1/m)} * (1 - (n*m+1)*\$v.\widehat{n} + n*m*\$v.\widehat{(n+1/m)})) / (m*(1-\$v.\widehat{(1/m)}))^2$

if $i \neq 0$ $m \neq 0$ for n $m :: nat$

<proof>

lemma *Imam-calc-i-0*: $\$(I\widehat{\{m\}}a)\widehat{\{m\}}-n = (n*m+1)*n / (2*m)$ if $i = 0$ $m \neq 0$

for $n\ m :: \text{nat}$
 $\langle \text{proof} \rangle$

lemma *Imsm-calc*:

$\$(I\{m\}s)\{m\}\text{-}n = ((1+i).\wedge(n+1/m) - (n*m+1)*(1+i).\wedge(1/m) + n*m) /$
 $(m*((1+i).\wedge(1/m)-1))\wedge 2$
if $i \neq 0\ m \neq 0$ **for** $n\ m :: \text{nat}$
 $\langle \text{proof} \rangle$

lemma *Imsm-calc-i-0*: $\$(I\{m\}s)\{m\}\text{-}n = (n*m+1)*n / (2*m)$ **if** $i = 0\ m \neq 0$
for $n\ m :: \text{nat}$
 $\langle \text{proof} \rangle$

lemma *Ila''m-Ilam*: $\$(I\{l\}a'')\{m\}\text{-}n = (1+i).\wedge(1/m) * \$(I\{l\}a)\{m\}\text{-}n$
if $l \neq 0\ m \neq 0$ **for** $l\ m\ n :: \text{nat}$
 $\langle \text{proof} \rangle$

lemma *Ia''m-calc*: $\$(Ia'')\{m\}\text{-}n = (\sum j < n. (j+1)/m * (\sum k=j*m..<(j+1)*m. \$v.\wedge(k/m)))$
 $\$v.\wedge(k/m))$
if $m \neq 0$ **for** $n\ m :: \text{nat}$
 $\langle \text{proof} \rangle$

lemma *Ima''m-calc-aux*: $\$(I\{m\}a'')\{m\}\text{-}n = (\sum k < n*m. \$v.\wedge(k/m) * (k+1) /$
 $m\wedge 2)$
if $m \neq 0$ **for** $m :: \text{nat}$
 $\langle \text{proof} \rangle$

lemma *Ima''m-calc*: $\$(I\{m\}a'')\{m\}\text{-}n = (1 - (n*m+1)*\$v.\wedge n + n*m*\$v.\wedge(n+1/m))$
 $/ (m*(1-\$v.\wedge(1/m)))\wedge 2$
if $i \neq 0\ m \neq 0$ **for** $n\ m :: \text{nat}$
 $\langle \text{proof} \rangle$

lemma *Ils''m-Ilsm*: $\$(I\{l\}s'')\{m\}\text{-}n = (1+i).\wedge(1/m) * \$(I\{l\}s)\{m\}\text{-}n$
if $l \neq 0\ m \neq 0$ **for** $l\ m\ n :: \text{nat}$
 $\langle \text{proof} \rangle$

lemma *Ims''m-calc*:

$\$(I\{m\}s'')\{m\}\text{-}n =$
 $(1+i).\wedge(1/m) * ((1+i).\wedge(n+1/m) - (n*m+1)*(1+i).\wedge(1/m) + n*m) /$
 $(m*((1+i).\wedge(1/m)-1))\wedge 2$
if $i \neq 0\ m \neq 0$ **for** $n\ m :: \text{nat}$
 $\langle \text{proof} \rangle$

lemma *lim-Imam*: $(\lambda n. \$(I\{m\}a)\{m\}\text{-}n) \longrightarrow 1 / (\$i\{m\}*\$d\{m\})$ **if** $m \neq$
 $0\ i > 0$ **for** $m :: \text{nat}$
 $\langle \text{proof} \rangle$

lemma *perp-incr-calc*: $\$(I\{m\}a)\{m\}\text{-}\infty = 1 / (\$i\{m\}*\$d\{m\})$ **if** $m \neq 0\ i >$
 0 **for** $m :: \text{nat}$

<proof>

lemma *lim-Ima''m*: $(\lambda n. \$ (I^{\wedge}\{m\}a'')^{\wedge}\{m\}-n) \longrightarrow 1 / (\$d^{\wedge}\{m\})^{\wedge}2$ **if** $m \neq 0$
 $i > 0$ **for** $m::nat$
<proof>

lemma *perp-due-incr-calc*: $\$(I^{\wedge}\{m\}a'')^{\wedge}\{m\}-\infty = 1 / (\$d^{\wedge}\{m\})^{\wedge}2$ **if** $m \neq 0$ $i >$
 0 **for** $m::nat$
<proof>

end

end