Ackermann's Function Is Not Primitive Recursive

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Abstract

Ackermann's function is defined in the usual way and a number of its elementary properties are proved. Then, the primitive recursive functions are defined inductively: as a predicate on the functions that map lists of numbers to numbers. It is shown that every primitive recursive function is strictly dominated by Ackermann's function. The formalisation follows an earlier one by Nora Szasz [1].

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Remark. This development was part of the Isabelle distribution from 1997 to 2022. It has been transferred to the AFP, where it may be more useful.

1 Ackermann's Function and the PR Functions

This proof has been adopted from a development by Nora Szasz [1].

theory *Primrec* **imports** *Main* **begin**

1.1 Ackermann's Function

fun *ack* :: [*nat*,*nat*] ⇒ *nat* **where**
  ack 0 n = *Suc n*
| *ack* (*Suc m*) *0* = *ack m 1*
| *ack* (*Suc m*) (*Suc n*) = *ack m* (*ack* (*Suc m*) *n*)

PROPERTY A 4

lemma *less-ack2* [*iff*]: *j* < *ack i j*
  by (*induct i j rule*: *ack.induct*) *simp-all*

PROPERTY A 5-, the single-step lemma

lemma *ack-less-ack-Suc2* [*iff*]: *ack i j* < *ack i* (*Suc j*)
  by (*induct i j rule*: *ack.induct*) *simp-all*

PROPERTY A 5, monotonicity for <

lemma *ack-less-mono2*: *j* < *k* ⟹ *ack i j* < *ack i k*
  by (*simp add*: *lift-Suc-mono-less*)

PROPERTY A 5', monotonicity for ≤

lemma *ack-le-mono2*: *j* ≤ *k* ⟹ *ack i j* ≤ *ack i k*
  by (*simp add*: *ack-less-mono2 less-mono-imp-le-mono*)

PROPERTY A 6

lemma *ack2-le-ack1* [*iff*]: *ack i* (*Suc j*) ≤ *ack* (*Suc i*) *j*
proof (*induct j*)
  case *0* **show** *?case* **by** *simp*
next
  case (*Suc j*) **show** *?case*
    by (*metis Suc ack.simps*(*3*) *ack-le-mono2 le-trans less-ack2 less-eq-Suc-le*)
qed

PROPERTY A 4'? Extra lemma needed for *CONSTANT* case, constant functions

lemma *ack-less-ack-Suc1* [*iff*]: *ack i j* < *ack* (*Suc i*) *j*
  by (*blast intro*: *ack-less-mono2 less-le-trans*)

lemma *less-ack1* [*iff*]: *i* < *ack i j*
  by (*induct i*) (*auto intro*: *less-trans-Suc*)

PROPERTY A 8

lemma *ack-1* [*simp*]: *ack* (*Suc 0*) *j* = *j* + *2*
  by (*induct j*) *simp-all*

PROPERTY A 9. The unary *1* and *2* in *ack* is essential for the rewriting.

lemma *ack-2* [*simp*]: *ack* (*Suc* (*Suc* *0*)) *j* = *2* ∗ *j* + *3*
by (*induct j*) *simp-all*

Added in 2022 just for fun

lemma *ack-3*: *ack* (*Suc* (*Suc* (*Suc* *0*))) *j* = *2* ^ (*j*+*3*) − *3*
proof (*induct j*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc j*)
with *less-le-trans* **show** *?case*
by (*fastforce simp add*: *power-add algebra-simps*)
qed

PROPERTY A 7, monotonicity for < [not clear why *ack-1* is now needed first!]

lemma *ack-less-mono1-aux*: *ack i k* < *ack* (*Suc* (*i*+*j*)) *k*
proof (*induct i k rule*: *ack.induct*)
case (*1 n*) **show** *?case*
using *less-le-trans* **by** *auto*
next
case (*2 m*) **thus** *?case* **by** *simp*
next
case (*3 m n*) **thus** *?case*
using *ack-less-mono2 less-trans* **by** *fastforce*
qed

lemma *ack-less-mono1*: *i* < *j* ⟹ *ack i k* < *ack j k*
using *ack-less-mono1-aux less-iff-Suc-add* **by** *auto*

PROPERTY A 7', monotonicity for ≤

lemma *ack-le-mono1*: *i* ≤ *j* ⟹ *ack i k* ≤ *ack j k*
using *ack-less-mono1 le-eq-less-or-eq* **by** *auto*

PROPERTY A 10

lemma *ack-nest-bound*: *ack i1* (*ack i2 j*) < *ack* (*2* + (*i1* + *i2*)) *j*
proof −
have *ack i1* (*ack i2 j*) < *ack* (*i1* + *i2*) (*ack* (*Suc* (*i1* + *i2*)) *j*)
by (*meson ack-le-mono1 ack-less-mono1 ack-less-mono2 le-add1 le-trans less-add-Suc2 not-less*)
also have ... = *ack* (*Suc* (*i1* + *i2*)) (*Suc j*)
by *simp*
also have ... ≤ *ack* (*2* + (*i1* + *i2*)) *j*
using *ack2-le-ack1 add-2-eq-Suc* **by** *presburger*
finally show *?thesis* **.**
qed

PROPERTY A 11

lemma *ack-add-bound*: *ack i1 j + ack i2 j < ack (4 + (i1 + i2)) j*
proof −
  have *ack i1 j ≤ ack (i1 + i2) j ack i2 j ≤ ack (i1 + i2) j*
    by (*simp-all add*: *ack-le-mono1*)
  then have *ack i1 j + ack i2 j < ack (Suc (Suc 0)) (ack (i1 + i2) j)*
    by *simp*
  also have *... < ack (4 + (i1 + i2)) j*
    by (*metis ack-nest-bound add.assoc numeral-2-eq-2 numeral-Bit0*)
  finally show *?thesis* **.**
qed

PROPERTY A 12. Article uses existential quantifier but the ALF proof used $k + 4$. Quantified version must be nested $\exists\, k'.\ \forall\, i\ j.\ \ldots$

lemma *ack-add-bound2*:
  assumes *i < ack k j* **shows** *i + j < ack (4 + k) j*
proof −
  have *i + j < ack k j + ack 0 j*
    using *assms* **by** *auto*
  also have *... < ack (4 + k) j*
    by (*metis ack-add-bound add.right-neutral*)
  finally show *?thesis* **.**
qed

1.2 Primitive Recursive Functions

primrec *hd0* :: *nat list ⇒ nat* **where**
  hd0 [] = 0
| *hd0 (m # ms) = m*

Inductive definition of the set of primitive recursive functions of type *nat list ⇒ nat*.

definition *SC* :: *nat list ⇒ nat*
  where *SC l = Suc (hd0 l)*

definition *CONSTANT* :: *nat ⇒ nat list ⇒ nat*
  where *CONSTANT n l = n*

definition *PROJ* :: *nat ⇒ nat list ⇒ nat*
  where *PROJ i l = hd0 (drop i l)*

definition *COMP* :: [*nat list ⇒ nat*, (*nat list ⇒ nat*) *list*, *nat list*] ⇒ *nat*
  where *COMP g fs l = g (map (λf. f l) fs)*

fun *PREC* :: [*nat list ⇒ nat*, *nat list ⇒ nat*, *nat list*] ⇒ *nat*
  where
    PREC f g [] = 0
  | *PREC f g (x # l) = rec-nat (f l) (λy r. g (r # y # l)) x*
    — Note that *g* is applied first to *PREC f g y* and then to *y*!

inductive *PRIMREC* :: (*nat list* ⇒ *nat*) ⇒ *bool* **where**
  SC: *PRIMREC SC*
| *CONSTANT*: *PRIMREC* (*CONSTANT k*)
| *PROJ*: *PRIMREC* (*PROJ i*)
| *COMP*: *PRIMREC g* ⟹ *listsp PRIMREC fs* ⟹ *PRIMREC* (*COMP g fs*)
| *PREC*: *PRIMREC f* ⟹ *PRIMREC g* ⟹ *PRIMREC* (*PREC f g*)
  monos *listsp-mono*

1.3 Main Result: Ackermann's Function is not Primitive Recursive

lemma *SC-case*: *SC l* < *ack 1* (*sum-list l*)
  unfolding *SC-def*
  by (*induct l*) (*simp-all add*: *le-add1 le-imp-less-Suc*)

lemma *CONSTANT-case*: *CONSTANT n l* < *ack n* (*sum-list l*)
  by (*simp add*: *CONSTANT-def*)

lemma *PROJ-case*: *PROJ i l* < *ack 0* (*sum-list l*)
proof −
  have *hd0* (*drop i l*) ≤ *sum-list l*
    by (*induct l arbitrary*: *i*) (*auto simp*: *drop-Cons' trans-le-add2*)
  then show *?thesis*
    by (*simp add*: *PROJ-def*)
qed

COMP case

lemma *COMP-map-aux*: ∀ *f* ∈ *set fs*. ∃ *kf*. ∀ *l*. *f l* < *ack kf* (*sum-list l*)
      ⟹ ∃ *k*. ∀ *l*. *sum-list* (*map* (λ*f*. *f l*) *fs*) < *ack k* (*sum-list l*)
proof (*induct fs*)
  case *Nil*
  then show *?case*
    by *auto*
next
  case (*Cons a fs*)
  then show *?case*
    by *simp* (*blast intro*: *add-less-mono ack-add-bound less-trans*)
qed

lemma *COMP-case*:
  assumes *1*: ∀ *l*. *g l* < *ack kg* (*sum-list l*)
    and *2*: ∀ *f* ∈ *set fs*. ∃ *kf*. ∀ *l*. *f l* < *ack kf* (*sum-list l*)
  shows ∃ *k*. ∀ *l*. *COMP g fs l* < *ack k* (*sum-list l*)
  unfolding *COMP-def*
 using *1 COMP-map-aux* [*OF 2*] **by** (*meson ack-less-mono2 ack-nest-bound less-trans*)

PREC case

lemma *PREC-case-aux*:
  assumes *f*: ⋀*l*. *f l* + *sum-list l* < *ack kf* (*sum-list l*)

and g: $\bigwedge l.\ g\ l + sum\text{-}list\ l < ack\ kg\ (sum\text{-}list\ l)$
shows $PREC\ f\ g\ (m\#l) + sum\text{-}list\ (m\#l) < ack\ (Suc\ (kf + kg))\ (sum\text{-}list\ (m\#l))$
proof (*induct m*)
case *0*
then show *?case*
using *ack-less-mono1-aux f less-trans* **by** *fastforce*
next
case (*Suc m*)
let $?r = PREC\ f\ g\ (m\#l)$
have $\neg\ g\ (?r\ \#\ m\ \#\ l) + sum\text{-}list\ (?r\ \#\ m\ \#\ l) < g\ (?r\ \#\ m\ \#\ l) + (m + sum\text{-}list\ l)$
by *force*
then have $g\ (?r\ \#\ m\ \#\ l) + (m + sum\text{-}list\ l) < ack\ kg\ (sum\text{-}list\ (?r\ \#\ m\ \#\ l))$
by (*meson g leI less-le-trans*)
moreover
have $\ldots < ack\ (kf + kg)\ (ack\ (Suc\ (kf + kg))\ (m + sum\text{-}list\ l))$
using *Suc.hyps* **by** *simp* (*meson ack-le-mono1 ack-less-mono2 le-add2 le-less-trans*)
ultimately show *?case*
by *auto*
qed

lemma *PREC-case-aux'*:
assumes f: $\bigwedge l.\ f\ l + sum\text{-}list\ l < ack\ kf\ (sum\text{-}list\ l)$
and g: $\bigwedge l.\ g\ l + sum\text{-}list\ l < ack\ kg\ (sum\text{-}list\ l)$
shows $PREC\ f\ g\ l + sum\text{-}list\ l < ack\ (Suc\ (kf + kg))\ (sum\text{-}list\ l)$
by (*smt* (*verit*, *best*) *PREC.elims PREC-case-aux add.commute add.right-neutral f g less-ack2*)

proposition *PREC-case*:
$[\![\bigwedge l.\ f\ l < ack\ kf\ (sum\text{-}list\ l);\ \bigwedge l.\ g\ l < ack\ kg\ (sum\text{-}list\ l)]\!]$
$\Longrightarrow \exists k.\ \forall l.\ PREC\ f\ g\ l < ack\ k\ (sum\text{-}list\ l)$
by (*metis le-less-trans* [*OF le-add1 PREC-case-aux'*] *ack-add-bound2*)

lemma *ack-bounds-PRIMREC*: $PRIMREC\ f \Longrightarrow \exists k.\ \forall l.\ f\ l < ack\ k\ (sum\text{-}list\ l)$
by (*erule PRIMREC.induct*) (*blast intro*: *SC-case CONSTANT-case PROJ-case COMP-case PREC-case*)+

theorem *ack-not-PRIMREC*:
$\neg\ PRIMREC\ (\lambda l.\ ack\ (hd0\ l)\ (hd0\ l))$
proof
assume $*$: $PRIMREC\ (\lambda l.\ ack\ (hd0\ l)\ (hd0\ l))$
then obtain m **where** m: $\bigwedge l.\ ack\ (hd0\ l)\ (hd0\ l) < ack\ m\ (sum\text{-}list\ l)$
using *ack-bounds-PRIMREC* **by** *blast*
show *False*
using m [*of* [m]] **by** *simp*
qed

end

References

[1] N. Szasz. A machine checked proof that Ackermann's function is not primitive recursive. In G. Huet and G. Plotkin, editors, *Logical Environments*, pages 317–338. Cambridge University Press, 1993.