Abel's Limit Theorem in Isabelle/HOL

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Abstract

This theory proves the Abel's limit theorem on power series of real numbers, and then an example is shown to use the theorem to cover the boundary cases of binomial series.

Contents

1 Abel's limit theorem on real power series

theory *Abel-Limit-Theorem*
  imports *HOL−Analysis.Generalised-Binomial-Theorem*
begin

Abel's theorem or Abel's limit theorem [3] provides a crucial link between the behavior of a power series inside its interval of convergence (such as $(-1, 1)$) and its value at the boundary such as -1 or 1.

This section presents the proof of Abel's limit theorem, which relates a limit of a power series to the sum of its real coefficients, as shown below:

$$\lim_{x \to 1^-} f(x) = f(1) = \sum_{k=0}^{\infty} a_k \qquad \text{where } f(x) = \sum_{k=0}^{\infty} a_k x^k$$

if the power series has its radius of convergence equal to 1 and $\sum_{k=0}^{\infty} a_k$ converges, where a_k is the coefficient of the k-th term.

That is, $f(x)$ is continuous from the left at 1.

The proof of continuity or the limit of $f(x)$ is based on the ε-δ definition. This proof uses summation by parts or Abel transformation to express the power series $f(x)$ as a power series whose coefficients are the partial sums ($\sum_{k=0}^{n} a_k$) of the coefficients of $f(x)$, instead of a_k. Then the new power series is split into two parts. The goal is to show that each part contributes to $\varepsilon/2$ for any x satisfying $(1-x) < \delta$.

Several references [3, 1, 2] are used to construct this proof.

theorem *Abel-limit-theorem*:
  fixes *a* :: *nat* ⇒ *real*
  defines *f1* ≡ (λ(*x*::*real*) *n*. *a n* ∗ *x* ^ *n*)
  defines *f* ≡ (λ(*x*::*real*). ∑ *n*. *f1 x n*)
  assumes *summable-a*: *summable a* **and**
    conv-radius-1: *conv-radius a* = *1*
  shows (*f* ⟶ (∑ *n*. *a n*)) (*at-left 1*)
⟨*proof*⟩

lemma *filterlim-at-right-at-left-eq*:
  shows ((λ*x*. *f* (−*x*)) ⟶ *l*) (*at-right* (−*1*)) ⟷ ((λ*x*. *f* (*x*)) ⟶ *l*) (*at-left* (*1*::*real*))
  ⟨*proof*⟩

Abel's limit theorem is also suitable for continuous from the right at -1.

corollary *Abel-limit-theorem′*:
  fixes *a* :: *nat* ⇒ *real*
  defines *f1* ≡ (λ(*x*::*real*) *n*. *a n* ∗ *x* ^ *n*)
  defines *f* ≡ (λ(*x*::*real*). ∑ *n*. *f1 x n*)
  assumes *summable-a*: *summable a* **and**
    conv-radius-1: *conv-radius a* = *1*
  shows ((λ*x*. *f* (−*x*)) ⟶ (∑ *n*. *a n*)) (*at-right* (−*1*))
  ⟨*proof*⟩

end

2 Example application: boundary cases of binomial theorem

theory *Binomial-Sqrt-Series-Boundary*
  imports
    Abel-Limit-Theorem
    Catalan-Numbers.Catalan-Numbers
    HOL−Real-Asymp.Real-Asymp
begin

Newton's generalized binomial theorem is applicable to $|x| < 1$ as seen from this $|?z| < 1 \Longrightarrow (\lambda n.\ (1\ /\ 2\ gchoose\ n) * ?z^n)\ sums\ sqrt\ (1 + ?z)$.

However, it doesn't apply to the boundary cases where $|x| = 1$ or $|x| = -1$. Here, Abel's limit theorem is applied to establish the binomial theorem for the boundary cases.

2.1 Binomial series

lemma *binomial-sqrt-series*:
fixes $x :: real$
assumes $|x| < 1$
shows $suminf\ (\lambda n.\ ((1/2)\ gchoose\ n) * x\ \hat{}\ n) = sqrt\ (1 + x)$
⟨*proof*⟩

The generalized binomial coefficient *a gchoose n* where $a = \frac{1}{2}$ can also be rewritten as an expression including a Catalan numbers. This is used to prove its summability using the property of Catalan numbers.

lemma *gbinomial-1-2-catalan*: $((1/2)\ gchoose\ (Suc\ n)) = ((-1)\hat{}(n)/(2\hat{}(2*n+1))) * real\ (catalan\ n)$
⟨*proof*⟩

lemma *gbinomial-1-2-catalan′*: $((1/2)\ gchoose\ (Suc\ n)) = ((-1)\hat{}n/2) * (1/4\hat{}n) * real\ (catalan\ n)$
⟨*proof*⟩

Rewrite the generalized binomial coefficient *a gchoose n* where $a = \frac{1}{2}$ as a binomial coefficient.

lemma *gbinomial-1-2-simp*:
$((1/2)\ gchoose\ (Suc\ n)) = ((-1)\hat{}n\ /\ real\ (2\hat{}(2*n+1) * (Suc\ n))) * ((2*n)\ choose\ n)$
⟨*proof*⟩

lemma *summable-real-powr-iff′*: $summable\ (\lambda n.\ 1\ /\ of\text{-}nat\ n\ powr\ s :: real) \longleftrightarrow s > 1$
⟨*proof*⟩

lemma *summable-1-2-gchoose*: $summable\ (\lambda n.\ ((1::real)/2)\ gchoose\ n)$
⟨*proof*⟩

lemma *gbinomial-1-2-gchoose-sum-sqrt-2*:
shows $(\sum n.\ (((1::real)\ /\ (2::real)\ gchoose\ n))) = sqrt\ 2$ (**is** $(\sum n.\ ?f\text{-}1\ n) = -$)
⟨*proof*⟩

2.2 Alternating series

lemma *gbinomial-ratio-limit′*:
fixes $a :: 'a :: real\text{-}normed\text{-}field$
assumes $a \notin \mathbb{N}$

shows $(\lambda n.\ ((a\ gchoose\ n) * (-1)\ \hat{}\ n)\ /\ ((a\ gchoose\ Suc\ n) * (-1)\ \hat{}\ (Suc\ n)))$
$\longrightarrow 1$
⟨*proof*⟩

lemma *conv-radius-gchoose-alternating*:
fixes $a :: 'a :: \{real\text{-}normed\text{-}field, banach\}$
assumes $a \notin \mathbb{N}$
shows $conv\text{-}radius\ (\lambda n{::}nat.\ (a\ gchoose\ n) * (-1)\ \hat{}\ n) = (1{::}ereal)$
⟨*proof*⟩

lemma *summable-1-2-gchoose-alternating*:
$summable\ (\lambda n{::}nat.\ (1\ /\ 2\ gchoose\ n) * (-1)\ \hat{}\ n :: real)$ (**is** *summable ?f*)
⟨*proof*⟩

lemma *gbinomial-1-2-gchoose-alternating-sum-0*:
shows $(\sum n.\ ((1/2\ gchoose\ n) * (-\ (1{::}real))\ \hat{}\ n)) = 0$ (**is** $(\sum n.\ ?f\text{-}1\ n) = 0$)
⟨*proof*⟩

2.3 Binomial sqrt series with the boundary cases

This lemma incorporates the boundary values where $x = 1$ and $x = -1$.

theorem *binomial-sqrt-series'*:
assumes $|x| \le (1 :: real)$
shows $suminf\ (\lambda n.\ ((1/2)\ gchoose\ n) * x\ \hat{}\ n) = sqrt\ (1 + x)$
⟨*proof*⟩

end

References

[1] Proof of Abel's limit theorem — planetmath.org. https://planetmath.org/proofofabelslimittheorem. [Accessed 11-11-2025].

[2] F. Holland. Abel's limit theorem, its converse, and multiplication formulae for $\Gamma(x)$. *Irish Math. Soc. Bull.*, 0089:57–64, 2022.

[3] Wikipedia contributors. Abel's theorem. [Accessed 11-11-2025]. URL: https://en.wikipedia.org/wiki/Abel%27s_theorem.