Abstract

We utilize and extend the method of *shallow semantic embeddings* (SSEs) in classical higher-order logic (HOL) to construct a custom theorem proving environment for *abstract objects theory* (AOT) on the basis of Isabelle/HOL.

SSEs are a means for universal logical reasoning by translating a target logic to HOL using a representation of its semantics. AOT is a foundational metaphysical theory, developed by Edward Zalta, that explains the objects presupposed by the sciences as *abstract objects* that reify property patterns. In particular, AOT aspires to provide a philosphically grounded basis for the construction and analysis of the objects of mathematics.

We can support this claim by verifying Uri Nodelman's and Edward Zalta's reconstruction of Frege's theorem: we can confirm that the Dedekind-Peano postulates for natural numbers are consistently derivable in AOT using Frege's method. Furthermore, we can suggest and discuss generalizations and variants of the construction and can thereby provide theoretical insights into, and contribute to the philosophical justification of, the construction.

In the process, we can demonstrate that our method allows for a nearly transparent exchange of results between traditional pen-and-paper-based reasoning and the computerized implementation, which in turn can retain the automation mechanisms available for Isabelle/HOL.

During our work, we could significantly contribute to the evolution of our target theory itself, while simultaneously solving the technical challenge of using an SSE to implement a theory that is based on logical foundations that significantly differ from the meta-logic HOL.

In general, our results demonstrate the fruitfulness of the practice of Computational Metaphysics, i.e. the application of computational methods to metaphysical questions and theories.

A full description of this formalization including references can be found at http://dx.doi.org/10.17169/refubium-35141.

The version of Principia Logico-Metaphysica (PLM) implemented in this formalization can be found at http://mally.stanford.edu/principia-2021-10-13.pdf, while the latest version of PLM is available at http://mally.stanford.edu/principia.pdf.

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1 References

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2 Model for the Logic of AOT

We introduce a primitive type for hyperintensional propositions.

typedecl o

To be able to model modal operators following Kripke semantics, we introduce a primitive type for possible worlds and assert, by axiom, that there is a surjective function mapping propositions to the boolean-valued functions acting on possible worlds. We call the result of applying this function to a proposition the Montague intension of the proposition.

typedecl *w* — The primtive type of possible worlds.
axiomatization *AOT-model-d*o :: ‹o⇒(*w*⇒*bool*)› **where**
  *d*o*-surj*: ‹*surj AOT-model-d*o›

The axioms of PLM require the existence of a non-actual world.

consts w_0 :: *w* — The designated actual world.
axiomatization where *AOT-model-nonactual-world*: ‹$\exists\, w\ .\ w \neq w_0$›

Validity of a proposition in a given world can now be modelled as the result of applying that world to the Montague intension of the proposition.

definition *AOT-model-valid-in* :: ‹*w*⇒o⇒*bool*› **where**
  ‹*AOT-model-valid-in w* φ ≡ *AOT-model-d*o φ *w*›

By construction, we can choose a proposition for any given Montague intension, s.t. the proposition is valid in a possible world iff the Montague intension evaluates to true at that world.

definition *AOT-model-proposition-choice* :: ‹(*w*⇒*bool*) ⇒ o› (**binder** ‹ε_o › *8*)
  where ‹ε_o *w*. φ *w* ≡ (*inv AOT-model-d*o) φ›
lemma *AOT-model-proposition-choice-simp*: ‹*AOT-model-valid-in w* (ε_o *w*. φ *w*) = φ *w*›
  by (*simp add*: *surj-f-inv-f*[*OF d*o*-surj*] *AOT-model-valid-in-def*
        AOT-model-proposition-choice-def)

Nitpick can trivially show that there are models for the axioms above.

lemma ‹*True*› **nitpick**[*satisfy*, *user-axioms*, *expect* = *genuine*] **..**

typedecl ω — The primtive type of ordinary objects/urelements.

Validating extended relation comprehension requires a large set of special urelements. For simple models that do not validate extended relation comprehension (and consequently the predecessor axiom in the theory of natural numbers), it suffices to use a primitive type as σ, i.e. **typedecl** σ.

typedecl σ'
typedef σ = ‹*UNIV*::((ω ⇒ *w* ⇒ *bool*) *set* × (ω ⇒ *w* ⇒ *bool*) *set* × σ') *set*› **..**

typedecl *null* — Null-urelements representing non-denoting terms.

datatype υ = $\omega\upsilon$ ω | $\sigma\upsilon$ σ | *is-nullv*: *nullv null* — Type of urelements

Urrelations are proposition-valued functions on urelements. Urrelations are required to evaluate to necessarily false propositions for null-urelements (note that there may be several distinct necessarily false propositions).

typedef *urrel* = ‹{ φ . ∀ *x w* . ¬*AOT-model-valid-in w* (φ (*nullv x*)) }›
  by (*rule exI*[**where** *x*=‹λ *x* . (ε_o *w* . ¬*is-nullv x*)›])

(*auto simp*: *AOT-model-proposition-choice-simp*)

Abstract objects will be modelled as sets of urrelations and will have to be mapped surjectively into the set of special urelements. We show that any mapping from abstract objects to special urelements has to involve at least one large set of collapsed abstract objects. We will use this fact to extend arbitrary mappings from abstract objects to special urelements to surjective mappings.

lemma *ασ-pigeonhole*:
— For any arbitrary mapping *ασ* from sets of urrelations to special urelements, there exists an abstract object x, s.t. the cardinal of the set of special urelements is strictly smaller than the cardinal of the set of abstract objects that are mapped to the same urelement as x under *ασ*.
‹∃ *x* . |*UNIV*::*σ set*| <*o* |{*y* . *ασ x* = *ασ y*}|›
for *ασ* :: ‹*urrel set* ⇒ *σ*›
proof(*rule ccontr*)
have *card-σ-set-set-bound*: ‹|*UNIV*::*σ set set*| ≤*o* |*UNIV*::*urrel set*|›
proof −
let *?pick* = ‹λ*u s* . ε_o *w* . *case u of* (*συ s*′) ⇒ *s*′ ∈ *s* | - ⇒ *False*›
have ‹∃ *f* :: *σ set* ⇒ *urrel* . *inj f*›
proof
show ‹*inj* (λ*s* . *Abs-urrel* (λ*u* . *?pick u s*))›
proof(*rule injI*)
fix *x y*
assume ‹*Abs-urrel* (λ*u*. *?pick u x*) = *Abs-urrel* (λ*u*. *?pick u y*)›
hence ‹(λ*u*. *?pick u x*) = (λ*u*. *?pick u y*)›
by (*auto intro*!: *Abs-urrel-inject*[*THEN iffD1*]
simp: *AOT-model-proposition-choice-simp*)
hence ‹*AOT-model-valid-in* w_0 (*?pick* (*συ s*) *x*) =
AOT-model-valid-in w_0 (*?pick* (*συ s*) *y*)›
for *s* **by** *metis*
hence ‹(*s* ∈ *x*) = (*s* ∈ *y*)› **for** *s*
by (*auto simp*: *AOT-model-proposition-choice-simp*)
thus ‹*x* = *y*›
by *blast*
qed
qed
thus *?thesis*
by (*metis card-of-image inj-imp-surj-inv*)
qed

Assume, for a proof by contradiction, that there is no large collapsed set.

assume ‹∄ *x* . |*UNIV*::*σ set*| <*o* |{*y* . *ασ x* = *ασ y*}|›
hence *A*: ‹∀ *x* . |{*y* . *ασ x* = *ασ y*}| ≤*o* |*UNIV*::*σ set*|›
by *auto*
have *union-univ*: ‹(⋃ *x* ∈ *range*(*inv ασ*) . {*y* . *ασ x* = *ασ y*}) = *UNIV*›
by *auto* (*meson f-inv-into-f range-eqI*)

We refute by case distinction: there is either finitely many or infinitely many special urelements and in both cases we can derive a contradiction from the assumption above.

{

Finite case.

assume *finite-σ-set*: ‹*finite* (*UNIV*::*σ set*)›
hence *finite-collapsed*: ‹*finite* {*y* . *ασ x* = *ασ y*}› **for** *x*
using *A card-of-ordLeq-infinite* **by** *blast*
hence *0*: ‹∀ *x* . *card* {*y* . *ασ x* = *ασ y*} ≤ *card* (*UNIV*::*σ set*)›
by (*metis A finite-σ-set card-of-ordLeq inj-on-iff-card-le*)
have *1*: ‹*card* (*range* (*inv ασ*)) ≤ *card* (*UNIV*::*σ set*)›
using *finite-σ-set card-image-le* **by** *blast*
hence *2*: ‹*finite* (*range* (*inv ασ*))›
using *finite-σ-set* **by** *blast*

define *n* **where** ‹*n* = *card* (*UNIV*::*urrel set set*)›
define *m* **where** ‹*m* = *card* (*UNIV*::*σ set*)›

have ‹$n = card (\bigcup x \in range(inv\ \alpha\sigma)\ .\ \{y\ .\ \alpha\sigma\ x = \alpha\sigma\ y\})$›
unfolding *n-def* **using** *union-univ* **by** *argo*
also have ‹$\ldots \le (\sum i{\in}range\ (inv\ \alpha\sigma).\ card\ \{y.\ \alpha\sigma\ i = \alpha\sigma\ y\})$›
using *card-UN-le 2* **by** *blast*
also have ‹$\ldots \le (\sum i{\in}range\ (inv\ \alpha\sigma).\ card\ (UNIV{::}\sigma\ set))$›
by (*metis* (*no-types*, *lifting*) *0 sum-mono*)
also have ‹$\ldots \le card\ (range\ (inv\ \alpha\sigma)) * card\ (UNIV{::}\sigma\ set)$›
using *sum-bounded-above* **by** *auto*
also have ‹$\ldots \le card\ (UNIV{::}\sigma\ set) * card\ (UNIV{::}\sigma\ set)$›
using *1* **by** *force*
also have ‹$\ldots = m{*}m$›
unfolding *m-def* **by** *blast*
finally have *n-upper*: ‹$n \le m{*}m$›**.**

have ‹$finite\ (\bigcup x \in range(inv\ \alpha\sigma)\ .\ \{y\ .\ \alpha\sigma\ x = \alpha\sigma\ y\})$›
using *2 finite-collapsed* **by** *blast*
hence *finite-αset*: ‹$finite\ (UNIV{::}urrel\ set\ set)$›
using *union-univ* **by** *argo*

have ‹$2\hat{\ }2\hat{\ }m = (2{::}nat)\hat{\ }(card\ (UNIV{::}\sigma\ set\ set))$›
by (*metis Pow-UNIV card-Pow finite-σ-set m-def*)
moreover have ‹$card\ (UNIV{::}\sigma\ set\ set) \le (card\ (UNIV{::}urrel\ set))$›
using *card-σ-set-set-bound*
by (*meson Finite-Set.finite-set card-of-ordLeq finite-αset*
finite-σ-set inj-on-iff-card-le)
ultimately have ‹$2\hat{\ }2\hat{\ }m \le (2{::}nat)\hat{\ }(card\ (UNIV{::}\ urrel\ set))$›
by *simp*
also have ‹$\ldots = n$›
unfolding *n-def*
by (*metis Finite-Set.finite-set Pow-UNIV card-Pow finite-αset*)
finally have ‹$2\hat{\ }2\hat{\ }m \le n$› **by** *blast*
hence ‹$2\hat{\ }2\hat{\ }m \le m{*}m$› **using** *n-upper* **by** *linarith*
moreover {
have ‹$(2{::}nat)\hat{\ }(2\hat{\ }m) \ge (2\hat{\ }(m + 1))$›
by (*metis Suc-eq-plus1 Suc-leI less-exp one-le-numeral power-increasing*)
also have ‹$(2\hat{\ }(m + 1)) = (2{::}nat) * 2\hat{\ }m$›
by *auto*
have ‹$m < 2\hat{\ }m$›
by (*simp add*: *less-exp*)
hence ‹$m{*}m < (2\hat{\ }m){*}(2\hat{\ }m)$›
by (*simp add*: *mult-strict-mono*)
moreover have ‹$\ldots = 2\hat{\ }(m{+}m)$›
by (*simp add*: *power-add*)
ultimately have ‹$m{*}m < 2\ \hat{\ }\ (m + m)$› **by** *presburger*
moreover have ‹$m{+}m \le 2\hat{\ }m$›
proof (*induct m*)
case *0*
thus *?case* **by** *auto*
next
case (*Suc m*)
thus *?case*
by (*metis Suc-leI less-exp mult-2 mult-le-mono2 power-Suc*)
qed
ultimately have ‹$m{*}m < 2\hat{\ }2\hat{\ }m$›
by (*meson less-le-trans one-le-numeral power-increasing*)
}
ultimately have *False* **by** *auto*
}
moreover {

Infinite case.

assume ‹$infinite\ (UNIV{::}\sigma\ set)$›

hence Cinfσ: ‹Cinfinite |UNIV::σ set|›
by (simp add: cinfinite-def)
have 1: ‹|range (inv ασ)| ≤o |UNIV::σ set|›
by auto
have 2: ‹∀ i∈range (inv ασ). |{y . ασ i = ασ y}| ≤o |UNIV::σ set|›
proof
fix i :: ‹urrel set›
assume ‹i ∈ range (inv ασ)›
show ‹|{y . ασ i = ασ y}| ≤o |UNIV::σ set|›
using A **by** blast
qed
have ‹|⋃ ((λi. {y. ασ i = ασ y}) ‘ (range (inv ασ)))| ≤o
|Sigma (range (inv ασ)) (λi. {y. ασ i = ασ y})|›
using card-of-UNION-Sigma **by** blast
hence ‹|UNIV::urrel set set| ≤o
|Sigma (range (inv ασ)) (λi. {y. ασ i = ασ y})|›
using union-univ **by** argo
moreover have ‹|Sigma (range (inv ασ)) (λi. {y. ασ i = ασ y})| ≤o |UNIV::σ set|›
using card-of-Sigma-ordLeq-Cinfinite[OF Cinfσ, OF 1, OF 2] **by** blast
ultimately have ‹|UNIV::urrel set set| ≤o |UNIV::σ set|›
using ordLeq-transitive **by** blast
moreover {
have ‹|UNIV::σ set| <o |UNIV::σ set set|›
by auto
moreover have ‹|UNIV::σ set set| ≤o |UNIV::urrel set|›
using card-σ-set-set-bound **by** blast
moreover have ‹|UNIV::urrel set| <o |UNIV::urrel set set|›
by auto
ultimately have ‹|UNIV::σ set| <o |UNIV::urrel set set|›
by (metis ordLess-imp-ordLeq ordLess-ordLeq-trans)
}
ultimately have False
using not-ordLeq-ordLess **by** blast
}
ultimately show False **by** blast
qed

We introduce a mapping from abstract objects (i.e. sets of urrelations) to special urelements ασ that is surjective and distinguishes all abstract objects that are distinguished by a (not necessarily surjective) mapping ασ′. ασ′ will be used to model extended relation comprehension.

consts ασ′ :: ‹urrel set ⇒ σ›
consts ασ :: ‹urrel set ⇒ σ›

specification(ασ)
ασ-surj: ‹surj ασ›
ασ-ασ′: ‹ασ x = ασ y ⟹ ασ′ x = ασ′ y›
proof −
obtain x **where** x-prop: ‹|UNIV::σ set| <o |{y. ασ′ x = ασ′ y}|›
using ασ-pigeonhole **by** blast
have ‹∃f :: urrel set ⇒ σ . f ‘ {y. ασ′ x = ασ′ y} = UNIV ∧ f x = ασ′ x›
proof −
have ‹∃f :: urrel set ⇒ σ . f ‘ {y. ασ′ x = ασ′ y} = UNIV›
by (simp add: x-prop card-of-ordLeq2 ordLess-imp-ordLeq)
then obtain f :: ‹urrel set ⇒ σ› **where** ‹f ‘ {y. ασ′ x = ασ′ y} = UNIV›
by presburger
moreover obtain a **where** ‹f a = ασ′ x› **and** ‹ασ′ a = ασ′ x›
by (smt (verit, best) calculation UNIV-I image-iff mem-Collect-eq)
ultimately have ‹(f (a := f x, x := f a)) ‘ {y. ασ′ x = ασ′ y} = UNIV ∧
(f (a := f x, x := f a)) x = ασ′ x›
by (auto simp: image-def)
thus ?thesis **by** blast
qed
then obtain f **where** fimage: ‹f ‘ {y. ασ′ x = ασ′ y} = UNIV›

```
          and fx: ‹f x = ασ' x›
    by blast

  define ασ :: ‹urrel set ⇒ σ› where
    ‹ασ ≡ λ urrels . if ασ' urrels = ασ' x ∧ f urrels ∉ range ασ'
                      then f urrels
                      else ασ' urrels›
  have ‹surj ασ›
  proof –
    {
    fix s :: σ
    {
      assume ‹s ∈ range ασ'›
      hence 0: ‹ασ' (inv ασ' s) = s›
        by (meson f-inv-into-f)
      {
        assume ‹s = ασ' x›
        hence ‹ασ x = s›
          using ασ-def fx by presburger
        hence ‹∃ f . ασ (f s) = s›
          by auto
      }
      moreover {
        assume ‹s ≠ ασ' x›
        hence ‹ασ (inv ασ' s) = s›
          unfolding ασ-def 0 by presburger
        hence ‹∃ f . ασ (f s) = s›
          by blast
      }
      ultimately have ‹∃ f . ασ (f s) = s›
        by blast
    }
    moreover {
      assume ‹s ∉ range ασ'›
      moreover obtain urrels where ‹f urrels = s› and ‹ασ' x = ασ' urrels›
        by (smt (verit, best) UNIV-I fimage image-iff mem-Collect-eq)
      ultimately have ‹ασ urrels = s›
        using ασ-def by presburger
      hence ‹∃ f . ασ (f s) = s›
        by (meson f-inv-into-f range-eqI)
    }
    ultimately have ‹∃ f . ασ (f s) = s›
      by blast
    }
    thus ?thesis
      by (metis surj-def)
  qed
  moreover have ‹∀ x y. ασ x = ασ y ⟶ ασ' x = ασ' y›
    by (metis ασ-def rangeI)
  ultimately show ?thesis
    by blast
qed
```

For extended models that validate extended relation comprehension (and consequently the predecessor axiom), we specify which abstract objects are distinguished by $\alpha\sigma'$.

```
definition urrel-to-ωrel :: ‹urrel ⇒ (ω ⇒ w ⇒ bool)› where
  ‹urrel-to-ωrel ≡ λ r u w . AOT-model-valid-in w (Rep-urrel r (ωυ u))›
definition ωrel-to-urrel :: ‹(ω ⇒ w ⇒ bool) ⇒ urrel› where
  ‹ωrel-to-urrel ≡ λ φ . Abs-urrel
    (λ u . ε∘ w . case u of ωυ x ⇒ φ x w | - ⇒ False)›

definition AOT-urrel-ωequiv :: ‹urrel ⇒ urrel ⇒ bool› where
  ‹AOT-urrel-ωequiv ≡ λ r s . ∀ u v . AOT-model-valid-in v (Rep-urrel r (ωυ u)) =
```

AOT-model-valid-in v (*Rep-urrel s* (*ωυ u*))›

lemma *urrel-ωrel-quot*: ‹*Quotient3 AOT-urrel-ωequiv urrel-to-ωrel ωrel-to-urrel*›
proof(*rule Quotient3I*)
  show ‹*urrel-to-ωrel* (*ωrel-to-urrel a*) = *a*› **for** *a*
    unfolding *ωrel-to-urrel-def urrel-to-ωrel-def*
    apply (*rule ext*)
    apply (*subst Abs-urrel-inverse*)
    by (*auto simp*: *AOT-model-proposition-choice-simp*)
next
  show ‹*AOT-urrel-ωequiv* (*ωrel-to-urrel a*) (*ωrel-to-urrel a*)› **for** *a*
    unfolding *ωrel-to-urrel-def AOT-urrel-ωequiv-def*
    apply (*subst* (*1 2*) *Abs-urrel-inverse*)
    by (*auto simp*: *AOT-model-proposition-choice-simp*)
next
  show ‹*AOT-urrel-ωequiv r s* = (*AOT-urrel-ωequiv r r* ∧ *AOT-urrel-ωequiv s s* ∧
                    urrel-to-ωrel r = *urrel-to-ωrel s*)› **for** *r s*
  proof
    assume ‹*AOT-urrel-ωequiv r s*›
    hence ‹*AOT-model-valid-in v* (*Rep-urrel r* (*ωυ u*)) =
          AOT-model-valid-in v (*Rep-urrel s* (*ωυ u*))› **for** *u v*
      using *AOT-urrel-ωequiv-def* **by** *metis*
    hence ‹*urrel-to-ωrel r* = *urrel-to-ωrel s*›
      unfolding *urrel-to-ωrel-def*
      by *simp*
    thus ‹*AOT-urrel-ωequiv r r* ∧ *AOT-urrel-ωequiv s s* ∧
        urrel-to-ωrel r = *urrel-to-ωrel s*›
      unfolding *AOT-urrel-ωequiv-def*
      by *auto*
  next
    assume ‹*AOT-urrel-ωequiv r r* ∧ *AOT-urrel-ωequiv s s* ∧
          urrel-to-ωrel r = *urrel-to-ωrel s*›
    hence ‹*AOT-model-valid-in v* (*Rep-urrel r* (*ωυ u*)) =
          AOT-model-valid-in v (*Rep-urrel s* (*ωυ u*))› **for** *u v*
      by (*metis urrel-to-ωrel-def*)
    thus ‹*AOT-urrel-ωequiv r s*›
      using *AOT-urrel-ωequiv-def* **by** *presburger*
  qed
qed

specification (*ασ′*)
  ασ-eq-ord-exts-all:
    ‹*ασ′ a* = *ασ′ b* ⟹ (⋀*s* . *urrel-to-ωrel s* = *urrel-to-ωrel r* ⟹ *s* ∈ *a*)
      ⟹ (⋀ *s* . *urrel-to-ωrel s* = *urrel-to-ωrel r* ⟹ *s* ∈ *b*)›
  ασ-eq-ord-exts-ex:
    ‹*ασ′ a* = *ασ′ b* ⟹ (∃ *s* . *s* ∈ *a* ∧ *urrel-to-ωrel s* = *urrel-to-ωrel r*)
      ⟹ (∃ *s* . *s* ∈ *b* ∧ *urrel-to-ωrel s* = *urrel-to-ωrel r*)›
proof −
  define *ασ-wit-intersection* **where**
      ‹*ασ-wit-intersection* ≡ λ *urrels* .
        {*ordext* . ∀ *urrel* . *urrel-to-ωrel urrel* = *ordext* ⟶ *urrel* ∈ *urrels*}›
  define *ασ-wit-union* **where**
      ‹*ασ-wit-union* ≡ λ *urrels* .
        {*ordext* . ∃ *urrel*∈*urrels* . *urrel-to-ωrel urrel* = *ordext*}›

  let *?ασ-wit* = ‹λ *urrels* .
    let ordexts = *ασ-wit-intersection urrels in*
    let ordexts′ = *ασ-wit-union urrels in*
    (*ordexts*, *ordexts′*, *undefined*)›
  define *ασ-wit* :: ‹*urrel set* ⇒ *σ*› **where**
    ‹*ασ-wit* ≡ λ *urrels* . *Abs-σ* (*?ασ-wit urrels*)›
  {
    fix *a b* :: ‹*urrel set*› **and** *r s*

```
    assume ‹ασ-wit a = ασ-wit b›
    hence 0: ‹{ordext. ∀ urrel. urrel-to-ωrel urrel = ordext ⟶ urrel ∈ a} =
             {ordext. ∀ urrel. urrel-to-ωrel urrel = ordext ⟶ urrel ∈ b}›
      unfolding ασ-wit-def Let-def
      apply (subst (asm) Abs-σ-inject)
      by (auto simp: ασ-wit-intersection-def ασ-wit-union-def)
    assume ‹urrel-to-ωrel s = urrel-to-ωrel r ⟹ s ∈ a› for s
    hence ‹urrel-to-ωrel r ∈
          {ordext. ∀ urrel. urrel-to-ωrel urrel = ordext ⟶ urrel ∈ a}›
      by auto
    hence ‹urrel-to-ωrel r ∈
          {ordext. ∀ urrel. urrel-to-ωrel urrel = ordext ⟶ urrel ∈ b}›
      using 0 by blast
    moreover assume ‹urrel-to-ωrel s = urrel-to-ωrel r›
    ultimately have ‹s ∈ b›
      by blast
  }
  moreover {
    fix a b :: ‹urrel set› and s r
    assume ‹ασ-wit a = ασ-wit b›
    hence 0: ‹{ordext. ∃ urrel ∈ a. urrel-to-ωrel urrel = ordext} =
             {ordext. ∃ urrel ∈ b. urrel-to-ωrel urrel = ordext}›
      unfolding ασ-wit-def
      using Abs-σ-inject ασ-wit-union-def by auto
    assume ‹s ∈ a›
    hence ‹urrel-to-ωrel s ∈ {ordext. ∃ urrel ∈ a. urrel-to-ωrel urrel = ordext}›
      by blast
    moreover assume ‹urrel-to-ωrel s = urrel-to-ωrel r›
    ultimately have ‹urrel-to-ωrel r ∈
                    {ordext. ∃ urrel ∈ b. urrel-to-ωrel urrel = ordext}›
      using 0 by argo
    hence ‹∃ s. s ∈ b ∧ urrel-to-ωrel s = urrel-to-ωrel r›
      by blast
  }
  ultimately show ?thesis
    by (safe intro!: exI[where x=ασ-wit]; metis)
qed
```

We enable the extended model version.

```
abbreviation (input) AOT-ExtendedModel where ‹AOT-ExtendedModel ≡ True›
```

Individual terms are either ordinary objects, represented by ordinary urelements, abstract objects, modelled as sets of urrelations, or null objects, used to represent non-denoting definite descriptions.

```
datatype κ = ωκ ω | ακ ‹urrel set› | is-nullκ: nullκ null
```

The mapping from abstract objects to urelements can be naturally lifted to a surjective mapping from individual terms to urelements.

```
primrec κυ :: ‹κ⇒υ› where
  ‹κυ (ωκ x) = ωυ x›
| ‹κυ (ακ x) = συ (ασ x)›
| ‹κυ (nullκ x) = nullυ x›

lemma κυ-surj: ‹surj κυ›
  using ασ-surj by (metis κυ.simps(1) κυ.simps(2) κυ.simps(3) υ.exhaust surj-def)
```

By construction if the urelement of an individual term is exemplified by an urrelation, it cannot be a null-object.

```
lemma urrel-null-false:
  assumes ‹AOT-model-valid-in w (Rep-urrel f (κυ x))›
  shows ‹¬is-nullκ x›
  by (metis (mono-tags, lifting) assms Rep-urrel κ.collapse(3) κυ.simps(3)
      mem-Collect-eq)
```

AOT requires any ordinary object to be *possibly concrete* and that there is an object that is not actually, but possibly concrete.

```
consts AOT-model-concreteω :: ‹ω ⇒ w ⇒ bool›
specification (AOT-model-concreteω)
  AOT-model-ω-concrete-in-some-world:
  ‹∃ w . AOT-model-concreteω x w›
  AOT-model-contingent-object:
  ‹∃ x w . AOT-model-concreteω x w ∧ ¬AOT-model-concreteω x w0›
  by (rule exI[where x=‹λ- w. w ≠ w0›]) (auto simp: AOT-model-nonactual-world)
```

We define a type class for AOT's terms specifying the conditions under which objects of that type denote and require the set of denoting terms to be non-empty.

```
class AOT-Term =
  fixes AOT-model-denotes :: ‹'a ⇒ bool›
  assumes AOT-model-denoting-ex: ‹∃ x . AOT-model-denotes x›
```

All types except the type of propositions involve non-denoting terms. We define a refined type class for those.

```
class AOT-IncompleteTerm = AOT-Term +
  assumes AOT-model-nondenoting-ex: ‹∃ x . ¬AOT-model-denotes x›
```

Generic non-denoting term.

```
definition AOT-model-nondenoting :: ‹'a::AOT-IncompleteTerm› where
  ‹AOT-model-nondenoting ≡ SOME τ . ¬AOT-model-denotes τ›
lemma AOT-model-nondenoing: ‹¬AOT-model-denotes (AOT-model-nondenoting)›
  using someI-ex[OF AOT-model-nondenoting-ex]
  unfolding AOT-model-nondenoting-def by blast
```

AOT-model-denotes can trivially be extended to products of types.

```
instantiation prod :: (AOT-Term, AOT-Term) AOT-Term
begin
definition AOT-model-denotes-prod :: ‹'a×'b ⇒ bool› where
  ‹AOT-model-denotes-prod ≡ λ(x,y) . AOT-model-denotes x ∧ AOT-model-denotes y›
instance proof
  show ‹∃x::'a×'b. AOT-model-denotes x›
    by (simp add: AOT-model-denotes-prod-def AOT-model-denoting-ex)
qed
end
```

We specify a transformation of proposition-valued functions on terms, s.t. the result is fully determined by *regular* terms. This will be required for modelling n-ary relations as functions on tuples while preserving AOT's definition of n-ary relation identity.

```
locale AOT-model-irregular-spec =
  fixes AOT-model-irregular :: ‹('a ⇒ o) ⇒ 'a ⇒ o›
    and AOT-model-regular :: ‹'a ⇒ bool›
    and AOT-model-term-equiv :: ‹'a ⇒ 'a ⇒ bool›
  assumes AOT-model-irregular-false:
    ‹¬AOT-model-valid-in w (AOT-model-irregular φ x)›
  assumes AOT-model-irregular-equiv:
    ‹AOT-model-term-equiv x y ⟹
     AOT-model-irregular φ x = AOT-model-irregular φ y›
  assumes AOT-model-irregular-eqI:
    ‹(⋀ x . AOT-model-regular x ⟹ φ x = ψ x) ⟹
     AOT-model-irregular φ x = AOT-model-irregular ψ x›
```

We introduce a type class for individual terms that specifies being regular, being equivalent (i.e. conceptually *sharing urelements*) and the transformation on proposition-valued functions as specified above.

```
class AOT-IndividualTerm = AOT-IncompleteTerm +
  fixes AOT-model-regular :: ‹'a ⇒ bool›
  fixes AOT-model-term-equiv :: ‹'a ⇒ 'a ⇒ bool›
  fixes AOT-model-irregular :: ‹('a ⇒ o) ⇒ 'a ⇒ o›
```

assumes *AOT-model-irregular-nondenoting*:
‹¬*AOT-model-regular x* ⟹ ¬*AOT-model-denotes x*›
assumes *AOT-model-term-equiv-part-equivp*:
‹*equivp AOT-model-term-equiv*›
assumes *AOT-model-term-equiv-denotes*:
‹*AOT-model-term-equiv x y* ⟹ (*AOT-model-denotes x* = *AOT-model-denotes y*)›
assumes *AOT-model-term-equiv-regular*:
‹*AOT-model-term-equiv x y* ⟹ (*AOT-model-regular x* = *AOT-model-regular y*)›
assumes *AOT-model-irregular*:
‹*AOT-model-irregular-spec AOT-model-irregular AOT-model-regular*
AOT-model-term-equiv›

interpretation *AOT-model-irregular-spec AOT-model-irregular AOT-model-regular*
AOT-model-term-equiv
using *AOT-model-irregular* **.**

Our concrete type for individual terms satisfies the type class of individual terms. Note that all unary individuals are regular. In general, an individual term may be a tuple and is regular, if at most one tuple element does not denote.

instantiation κ :: *AOT-IndividualTerm*
begin
definition *AOT-model-term-equiv-κ* :: ‹κ ⇒ κ ⇒ *bool*› **where**
‹*AOT-model-term-equiv-κ* ≡ λ *x y* . κυ *x* = κυ *y*›
definition *AOT-model-denotes-κ* :: ‹κ ⇒ *bool*› **where**
‹*AOT-model-denotes-κ* ≡ λ *x* . ¬*is-null*κ *x*›
definition *AOT-model-regular-κ* :: ‹κ ⇒ *bool*› **where**
‹*AOT-model-regular-κ* ≡ λ *x* . *True*›
definition *AOT-model-irregular-κ* :: ‹(κ ⇒ o) ⇒ κ ⇒ o› **where**
‹*AOT-model-irregular-κ* ≡ *SOME* φ . *AOT-model-irregular-spec* φ
AOT-model-regular AOT-model-term-equiv›
instance proof
show ‹∃ *x* :: κ. *AOT-model-denotes x*›
by (*rule exI*[**where** *x*=‹ωκ *undefined*›])
(*simp add*: *AOT-model-denotes-κ-def*)
next
show ‹∃ *x* :: κ. ¬*AOT-model-denotes x*›
by (*rule exI*[**where** *x*=‹*null*κ *undefined*›])
(*simp add*: *AOT-model-denotes-κ-def AOT-model-regular-κ-def*)
next
show ¬*AOT-model-regular x* ⟹ ¬ *AOT-model-denotes x* **for** *x* :: κ
by (*simp add*: *AOT-model-regular-κ-def*)
next
show ‹*equivp* (*AOT-model-term-equiv* :: κ ⇒ κ ⇒ *bool*)›
by (*rule equivpI*; *rule reflpI exI sympI transpI*)
(*simp-all add*: *AOT-model-term-equiv-κ-def*)
next
fix *x y* :: κ
show ‹*AOT-model-term-equiv x y* ⟹ *AOT-model-denotes x* = *AOT-model-denotes y*›
by (*metis AOT-model-denotes-κ-def AOT-model-term-equiv-κ-def κ.exhaust-disc*
κυ.simps υ.disc(*1*,*3*,*5*,*6*) *is-ακ-def is-ωκ-def is-nullκ-def*)
next
fix *x y* :: κ
show ‹*AOT-model-term-equiv x y* ⟹ *AOT-model-regular x* = *AOT-model-regular y*›
by (*simp add*: *AOT-model-regular-κ-def*)
next
have *AOT-model-irregular-spec* (λ φ (*x*::κ) . ε_o *w* . *False*)
AOT-model-regular AOT-model-term-equiv
by *standard* (*auto simp*: *AOT-model-proposition-choice-simp*)
thus ‹*AOT-model-irregular-spec* (*AOT-model-irregular*::(κ⇒o) ⇒ κ ⇒ o)
AOT-model-regular AOT-model-term-equiv›
unfolding *AOT-model-irregular-κ-def* **by** (*metis* (*no-types*, *lifting*) *someI-ex*)
qed
end

We define relations among individuals as proposition valued functions. *Denoting* unary relations (among κ) will match the urrelations introduced above.

```
typedef 'a rel (‹<->›) = ‹UNIV::('a::AOT-IndividualTerm ⇒ o) set› ..
setup-lifting type-definition-rel
```

We will use the transformation specified above to "fix" the behaviour of functions on irregular terms when defining λ-expressions.

```
definition fix-irregular :: ‹('a::AOT-IndividualTerm ⇒ o) ⇒ ('a ⇒ o)› where
  ‹fix-irregular ≡ λ φ x . if AOT-model-regular x
                          then φ x else AOT-model-irregular φ x›
lemma fix-irregular-denoting:
  ‹AOT-model-denotes x ⟹ fix-irregular φ x = φ x›
  by (meson AOT-model-irregular-nondenoting fix-irregular-def)
lemma fix-irregular-regular:
  ‹AOT-model-regular x ⟹ fix-irregular φ x = φ x›
  by (meson AOT-model-irregular-nondenoting fix-irregular-def)
lemma fix-irregular-irregular:
  ‹¬AOT-model-regular x ⟹ fix-irregular φ x = AOT-model-irregular φ x›
  by (simp add: fix-irregular-def)
```

Relations among individual terms are (potentially non-denoting) terms. A relation denotes, if it agrees on all equivalent terms (i.e. terms sharing urelements), is necessarily false on all non-denoting terms and is well-behaved on irregular terms.

```
instantiation rel :: (AOT-IndividualTerm) AOT-IncompleteTerm
begin

lift-definition AOT-model-denotes-rel :: ‹<'a> ⇒ bool› is
  ‹λ φ . (∀ x y . AOT-model-term-equiv x y ⟶ φ x = φ y) ∧
      (∀ w x . AOT-model-valid-in w (φ x) ⟶ AOT-model-denotes x) ∧
      (∀ x . ¬AOT-model-regular x ⟶ φ x = AOT-model-irregular φ x)› .
instance proof
  have ‹AOT-model-irregular (fix-irregular φ) x = AOT-model-irregular φ x›
    for φ and x :: 'a
    by (rule AOT-model-irregular-eqI) (simp add: fix-irregular-def)
  thus ‹∃ x :: <'a> . AOT-model-denotes x›
    by (safe intro!: exI[where x=‹Abs-rel (fix-irregular (λx. εo w . False))›])
      (transfer; auto simp: AOT-model-proposition-choice-simp fix-irregular-def
                      AOT-model-irregular-equiv AOT-model-term-equiv-regular
                      AOT-model-irregular-false)
next
  show ‹∃f :: <'a> . ¬AOT-model-denotes f›
    by (rule exI[where x=‹Abs-rel (λx. εo w . True)›];
        auto simp: AOT-model-denotes-rel.abs-eq AOT-model-nondenoting-ex
                   AOT-model-proposition-choice-simp)
qed
end
```

Auxiliary lemmata.

```
lemma AOT-model-term-equiv-eps:
  shows ‹AOT-model-term-equiv (Eps (AOT-model-term-equiv κ)) κ›
    and ‹AOT-model-term-equiv κ (Eps (AOT-model-term-equiv κ))›
    and ‹AOT-model-term-equiv κ κ' ⟹
        (Eps (AOT-model-term-equiv κ)) = (Eps (AOT-model-term-equiv κ'))›
  apply (metis AOT-model-term-equiv-part-equivp equivp-def someI-ex)
  apply (metis AOT-model-term-equiv-part-equivp equivp-def someI-ex)
  by (metis AOT-model-term-equiv-part-equivp equivp-def)

lemma AOT-model-denotes-Abs-rel-fix-irregularI:
  assumes ‹⋀ x y . AOT-model-term-equiv x y ⟹ φ x = φ y›
     and ‹⋀ w x . AOT-model-valid-in w (φ x) ⟹ AOT-model-denotes x›
   shows ‹AOT-model-denotes (Abs-rel (fix-irregular φ))›
proof –
```

have ‹*AOT-model-irregular* φ *x* = *AOT-model-irregular*
(λ*x*. *if AOT-model-regular x then* φ *x else AOT-model-irregular* φ *x*) *x*›
if ‹¬ *AOT-model-regular x*›
for *x*
by (*rule AOT-model-irregular-eqI*) *auto*
thus *?thesis*
unfolding *AOT-model-denotes-rel.rep-eq*
using *assms* **by** (*auto simp*: *AOT-model-irregular-false Abs-rel-inverse*
AOT-model-irregular-equiv fix-irregular-def
AOT-model-term-equiv-regular)
qed

lemma *AOT-model-term-equiv-rel-equiv*:
assumes ‹*AOT-model-denotes x*›
and ‹*AOT-model-denotes y*›
shows ‹*AOT-model-term-equiv x y* = (∀ Π *w* . *AOT-model-denotes* Π ⟶
AOT-model-valid-in w (*Rep-rel* Π *x*) = *AOT-model-valid-in w* (*Rep-rel* Π *y*))›
proof
assume ‹*AOT-model-term-equiv x y*›
thus ‹∀ Π *w* . *AOT-model-denotes* Π ⟶ *AOT-model-valid-in w* (*Rep-rel* Π *x*) =
AOT-model-valid-in w (*Rep-rel* Π *y*)›
by (*simp add*: *AOT-model-denotes-rel.rep-eq*)
next
have *0*: ‹(*AOT-model-denotes x′* ∧ *AOT-model-term-equiv x′ y*) =
(*AOT-model-denotes y′* ∧ *AOT-model-term-equiv y′ y*)›
if ‹*AOT-model-term-equiv x′ y′*› **for** *x′ y′*
by (*metis that AOT-model-term-equiv-denotes AOT-model-term-equiv-part-equivp*
equivp-def)
assume ‹∀ Π *w* . *AOT-model-denotes* Π ⟶ *AOT-model-valid-in w* (*Rep-rel* Π *x*) =
AOT-model-valid-in w (*Rep-rel* Π *y*)›
moreover have ‹*AOT-model-denotes* (*Abs-rel* (*fix-irregular*
(λ *x* . $\varepsilon_\circ$ *w* . *AOT-model-denotes x* ∧ *AOT-model-term-equiv x y*)))›
(**is** *AOT-model-denotes ?r*)
by (*rule AOT-model-denotes-Abs-rel-fix-irregularI*)
(*auto simp*: *0 AOT-model-denotes-rel.rep-eq Abs-rel-inverse fix-irregular-def*
AOT-model-proposition-choice-simp AOT-model-irregular-false)
ultimately have ‹*AOT-model-valid-in w* (*Rep-rel ?r x*) =
AOT-model-valid-in w (*Rep-rel ?r y*)› **for** *w*
by *blast*
thus ‹*AOT-model-term-equiv x y*›
by (*simp add*: *Abs-rel-inverse AOT-model-proposition-choice-simp*
fix-irregular-denoting[*OF assms*(*1*)] *AOT-model-term-equiv-part-equivp*
fix-irregular-denoting[*OF assms*(*2*)] *assms equivp-reflp*)
qed

Denoting relations among terms of type κ correspond to urrelations.

definition *rel-to-urrel* :: ‹<κ> ⇒ *urrel*› **where**
‹*rel-to-urrel* ≡ λ Π . *Abs-urrel* (λ *u* . *Rep-rel* Π (*SOME x* . κυ *x* = *u*))›
definition *urrel-to-rel* :: ‹*urrel* ⇒ <κ>› **where**
‹*urrel-to-rel* ≡ λ φ . *Abs-rel* (λ *x* . *Rep-urrel* φ (κυ *x*))›
definition *AOT-rel-equiv* :: ‹<′*a*::*AOT-IndividualTerm*> ⇒ <′*a*> ⇒ *bool*› **where**
‹*AOT-rel-equiv* ≡ λ *f g* . *AOT-model-denotes f* ∧ *AOT-model-denotes g* ∧ *f* = *g*›

lemma *urrel-quotient3*: ‹*Quotient3 AOT-rel-equiv rel-to-urrel urrel-to-rel*›
proof (*rule Quotient3I*)
have ‹(λ*u*. *Rep-urrel a* (κυ (*SOME x*. κυ *x* = *u*))) = (λ*u*. *Rep-urrel a u*)› **for** *a*
by (*rule ext*) (*metis* (*mono-tags*, *lifting*) κυ*-surj surj-f-inv-f verit-sko-ex′*)
thus ‹*rel-to-urrel* (*urrel-to-rel a*) = *a*› **for** *a*
by (*simp add*: *Abs-rel-inverse rel-to-urrel-def urrel-to-rel-def*
Rep-urrel-inverse)
next
show ‹*AOT-rel-equiv* (*urrel-to-rel a*) (*urrel-to-rel a*)› **for** *a*
unfolding *AOT-rel-equiv-def urrel-to-rel-def*

```
    by transfer (simp add: AOT-model-regular-κ-def AOT-model-denotes-κ-def
                           AOT-model-term-equiv-κ-def urrel-null-false)
next
  {
    fix a
    assume ‹∀ w x. AOT-model-valid-in w (a x) ⟶ ¬ is-nullκ x›
    hence ‹(λu. a (SOME x. κυ x = u)) ∈
          {φ. ∀ x w. ¬ AOT-model-valid-in w (φ (nullυ x))}›
      by (simp; metis (mono-tags, lifting) κ.exhaust-disc κυ.simps υ.disc(1,3,5)
                                       υ.disc(6) is-ακ-def is-ωκ-def someI-ex)
  } note 1 = this
  {
    fix r s :: ‹κ ⇒ o›
    assume A: ‹∀ x y. AOT-model-term-equiv x y ⟶ r x = r y›
    assume ‹∀ w x. AOT-model-valid-in w (r x) ⟶ AOT-model-denotes x›
    hence 2: ‹(λu. r (SOME x. κυ x = u)) ∈
             {φ. ∀ x w. ¬ AOT-model-valid-in w (φ (nullυ x))}›
      using 1 AOT-model-denotes-κ-def by meson
    assume B: ‹∀ x y. AOT-model-term-equiv x y ⟶ s x = s y›
    assume ‹∀ w x. AOT-model-valid-in w (s x) ⟶ AOT-model-denotes x›
    hence 3: ‹(λu. s (SOME x. κυ x = u)) ∈
             {φ. ∀ x w. ¬ AOT-model-valid-in w (φ (nullυ x))}›
      using 1 AOT-model-denotes-κ-def by meson
    assume ‹Abs-urrel (λu. r (SOME x. κυ x = u)) =
           Abs-urrel (λu. s (SOME x. κυ x = u))›
    hence 4: ‹r (SOME x. κυ x = u) = s (SOME x::κ. κυ x = u)› for u
      unfolding Abs-urrel-inject[OF 2 3] by metis
    have ‹r x = s x› for x
      using 4[of ‹κυ x›]
      by (metis (mono-tags, lifting) A B AOT-model-term-equiv-κ-def someI-ex)
    hence ‹r = s› by auto
  }
  thus ‹AOT-rel-equiv r s = (AOT-rel-equiv r r ∧ AOT-rel-equiv s s ∧
                            rel-to-urrel r = rel-to-urrel s)› for r s
    unfolding AOT-rel-equiv-def rel-to-urrel-def
    by transfer auto
qed

lemma urrel-quotient:
  ‹Quotient AOT-rel-equiv rel-to-urrel urrel-to-rel
           (λx y. AOT-rel-equiv x x ∧ rel-to-urrel x = y)›
  using Quotient3-to-Quotient[OF urrel-quotient3] by auto
```

Unary individual terms are always regular and equipped with encoding and concreteness. The specification of the type class anticipates the required properties for deriving the axiom system.

```
class AOT-UnaryIndividualTerm =
  fixes AOT-model-enc :: ‹'a ⇒ <'a::AOT-IndividualTerm> ⇒ bool›
    and AOT-model-concrete :: ‹w ⇒ 'a ⇒ bool›
  assumes AOT-model-unary-regular:
      ‹AOT-model-regular x› — All unary individual terms are regular.
      and AOT-model-enc-relid:
        ‹AOT-model-denotes F ⟹
         AOT-model-denotes G ⟹
         (⋀ x . AOT-model-enc x F ⟷ AOT-model-enc x G)
         ⟹ F = G›
      and AOT-model-A-objects:
        ‹∃ x . AOT-model-denotes x ∧
              (∀ w. ¬ AOT-model-concrete w x) ∧
              (∀ F. AOT-model-denotes F ⟶ AOT-model-enc x F = φ F)›
      and AOT-model-contingent:
        ‹∃ x w. AOT-model-concrete w x ∧ ¬ AOT-model-concrete w₀ x›
      and AOT-model-nocoder:
        ‹AOT-model-concrete w x ⟹ ¬AOT-model-enc x F›
```

```
    and AOT-model-concrete-equiv:
      ‹AOT-model-term-equiv x y ⟹
        AOT-model-concrete w x = AOT-model-concrete w y›
    and AOT-model-concrete-denotes:
      ‹AOT-model-concrete w x ⟹ AOT-model-denotes x›
    — The following are properties that will only hold in the extended models.
    and AOT-model-enc-indistinguishable-all:
     ‹AOT-ExtendedModel ⟹
      AOT-model-denotes a ⟹ ¬(∃ w . AOT-model-concrete w a) ⟹
      AOT-model-denotes b ⟹ ¬(∃ w . AOT-model-concrete w b) ⟹
      AOT-model-denotes Π ⟹
      (⋀ Π' . AOT-model-denotes Π' ⟹
        (⋀ v . AOT-model-valid-in v (Rep-rel Π' a) =
               AOT-model-valid-in v (Rep-rel Π' b))) ⟹
      (⋀ Π' . AOT-model-denotes Π' ⟹
          (⋀ v x . ∃ w . AOT-model-concrete w x ⟹
              AOT-model-valid-in v (Rep-rel Π' x) =
              AOT-model-valid-in v (Rep-rel Π x)) ⟹
          AOT-model-enc a Π') ⟹
      (⋀ Π' . AOT-model-denotes Π' ⟹
          (⋀ v x . ∃ w . AOT-model-concrete w x ⟹
              AOT-model-valid-in v (Rep-rel Π' x) =
              AOT-model-valid-in v (Rep-rel Π x)) ⟹
          AOT-model-enc b Π')›
    and AOT-model-enc-indistinguishable-ex:
     ‹AOT-ExtendedModel ⟹
      AOT-model-denotes a ⟹ ¬(∃ w . AOT-model-concrete w a) ⟹
      AOT-model-denotes b ⟹ ¬(∃ w . AOT-model-concrete w b) ⟹
      AOT-model-denotes Π ⟹
      (⋀ Π' . AOT-model-denotes Π' ⟹
        (⋀ v . AOT-model-valid-in v (Rep-rel Π' a) =
               AOT-model-valid-in v (Rep-rel Π' b))) ⟹
      (∃ Π' . AOT-model-denotes Π' ∧ AOT-model-enc a Π' ∧
          (∀ v x . (∃ w . AOT-model-concrete w x) ⟶
              AOT-model-valid-in v (Rep-rel Π' x) =
              AOT-model-valid-in v (Rep-rel Π x))) ⟹
      (∃ Π' . AOT-model-denotes Π' ∧ AOT-model-enc b Π' ∧
          (∀ v x . (∃ w . AOT-model-concrete w x) ⟶
              AOT-model-valid-in v (Rep-rel Π' x) =
              AOT-model-valid-in v (Rep-rel Π x)))›
```

Instantiate the class of unary individual terms for our concrete type of individual terms κ.

```
instantiation κ :: AOT-UnaryIndividualTerm
begin

definition AOT-model-enc-κ :: ‹κ ⇒ <κ> ⇒ bool› where
  ‹AOT-model-enc-κ ≡ λ x F .
      case x of ακ a ⇒ AOT-model-denotes F ∧ rel-to-urrel F ∈ a
              | - ⇒ False›
primrec AOT-model-concrete-κ :: ‹w ⇒ κ ⇒ bool› where
  ‹AOT-model-concrete-κ w (ωκ x) = AOT-model-concreteω x w›
| ‹AOT-model-concrete-κ w (ακ x) = False›
| ‹AOT-model-concrete-κ w (nullκ x) = False›

lemma AOT-meta-A-objects-κ:
  ‹∃x :: κ. AOT-model-denotes x ∧
            (∀w. ¬ AOT-model-concrete w x) ∧
            (∀F. AOT-model-denotes F ⟶ AOT-model-enc x F = φ F)› for φ
  apply (rule exI[where x=‹ακ {f . φ (urrel-to-rel f)}›])
  apply (simp add: AOT-model-enc-κ-def AOT-model-denotes-κ-def)
  by (metis (no-types, lifting) AOT-rel-equiv-def urrel-quotient
                                Quotient-rep-abs-fold-unmap)
```

```
instance proof
  show ‹AOT-model-regular x› for x :: κ
    by (simp add: AOT-model-regular-κ-def)
next
  fix F G :: ‹<κ>›
  assume ‹AOT-model-denotes F›
  moreover assume ‹AOT-model-denotes G›
  moreover assume ‹⋀x. AOT-model-enc x F = AOT-model-enc x G›
  moreover obtain x where ‹∀ G. AOT-model-denotes G ⟶ AOT-model-enc x G = (F = G)›
    using AOT-meta-A-objects-κ by blast
  ultimately show ‹F = G› by blast
next
  show ‹∃ x :: κ. AOT-model-denotes x ∧
                (∀ w. ¬ AOT-model-concrete w x) ∧
                (∀ F. AOT-model-denotes F ⟶ AOT-model-enc x F = φ F)› for φ
    using AOT-meta-A-objects-κ .
next
  show ‹∃ (x::κ) w. AOT-model-concrete w x ∧ ¬ AOT-model-concrete w0 x›
    using AOT-model-concrete-κ.simps(1) AOT-model-contingent-object by blast
next
  show ‹AOT-model-concrete w x ⟹ ¬ AOT-model-enc x F› for w and x :: κ and F
    by (metis AOT-model-concrete-κ.simps(2) AOT-model-enc-κ-def κ.case-eq-if
          κ.collapse(2))
next
  show ‹AOT-model-concrete w x = AOT-model-concrete w y›
    if ‹AOT-model-term-equiv x y›
    for x y :: κ and w
    using that by (induct x; induct y; auto simp: AOT-model-term-equiv-κ-def)
next
  show ‹AOT-model-concrete w x ⟹ AOT-model-denotes x› for w and x :: κ
    by (metis AOT-model-concrete-κ.simps(3) AOT-model-denotes-κ-def κ.collapse(3))

next
  fix κ κ' :: κ and Π Π' :: ‹<κ>› and w :: w
  assume ext: ‹AOT-ExtendedModel›
  assume ‹AOT-model-denotes κ›
  moreover assume ‹∄ w. AOT-model-concrete w κ›
  ultimately obtain a where a-def: ‹ακ a = κ›
    by (metis AOT-model-ω-concrete-in-some-world AOT-model-concrete-κ.simps(1)
          AOT-model-denotes-κ-def κ.discI(3) κ.exhaust-sel)
  assume ‹AOT-model-denotes κ'›
  moreover assume ‹∄ w. AOT-model-concrete w κ'›
  ultimately obtain b where b-def: ‹ακ b = κ'›
    by (metis AOT-model-ω-concrete-in-some-world AOT-model-concrete-κ.simps(1)
          AOT-model-denotes-κ-def κ.discI(3) κ.exhaust-sel)
  assume ‹AOT-model-denotes Π' ⟹ AOT-model-valid-in w (Rep-rel Π' κ) =
                              AOT-model-valid-in w (Rep-rel Π' κ')› for Π' w
  hence ‹AOT-model-valid-in w (Rep-urrel r (κυ κ)) =
       AOT-model-valid-in w (Rep-urrel r (κυ κ'))› for r
    by (metis AOT-rel-equiv-def Abs-rel-inverse Quotient3-rel-rep
          iso-tuple-UNIV-I urrel-quotient3 urrel-to-rel-def)
  hence ‹let r = (Abs-urrel (λ u . εo w . u = κυ κ)) in
       AOT-model-valid-in w (Rep-urrel r (κυ κ)) =
       AOT-model-valid-in w (Rep-urrel r (κυ κ'))›
    by presburger
  hence ασ-eq: ‹ασ a = ασ b›
    unfolding Let-def
    apply (subst (asm) (1 2) Abs-urrel-inverse)
    using AOT-model-proposition-choice-simp a-def b-def by force+
  assume Π-den: ‹AOT-model-denotes Π›
  have ‹¬AOT-model-valid-in w (Rep-rel Π (SOME xa. κυ xa = nullυ x))› for x w
    by (metis (mono-tags, lifting) AOT-model-denotes-κ-def
          AOT-model-denotes-rel.rep-eq κ.exhaust-disc κυ.simps(1,2,3))
```

‹AOT-model-denotes Π› υ.disc(8,9) υ.distinct(3)
is-ακ-def is-ωκ-def verit-sko-ex′)
moreover have ‹Rep-rel Π (ωκ x) = Rep-rel Π (SOME y. κυ y = ωυ x)› **for** x
by (metis (mono-tags, lifting) AOT-model-denotes-rel.rep-eq
AOT-model-term-equiv-κ-def κυ.simps(1) Π-den verit-sko-ex′)
ultimately have ‹Rep-rel Π (ωκ x) = Rep-urrel (rel-to-urrel Π) (ωυ x)› **for** x
unfolding rel-to-urrel-def
by (subst Abs-urrel-inverse) auto
hence ‹∃ r . ∀ x . Rep-rel Π (ωκ x) = Rep-urrel r (ωυ x)›
by (auto intro!: exI[**where** x=‹rel-to-urrel Π›])
then obtain r **where** r-prop: ‹Rep-rel Π (ωκ x) = Rep-urrel r (ωυ x)› **for** x
by blast
assume ‹AOT-model-denotes Π′ ⟹
(⋀v x. ∃ w. AOT-model-concrete w x ⟹
AOT-model-valid-in v (Rep-rel Π′ x) =
AOT-model-valid-in v (Rep-rel Π x)) ⟹ AOT-model-enc κ Π′› **for** Π′
hence ‹AOT-model-denotes Π′ ⟹
(⋀v x. AOT-model-valid-in v (Rep-rel Π′ (ωκ x)) =
AOT-model-valid-in v (Rep-rel Π (ωκ x))) ⟹ AOT-model-enc κ Π′› **for** Π′
by (metis AOT-model-concrete-κ.simps(2) AOT-model-concrete-κ.simps(3)
κ.exhaust-disc is-ακ-def is-ωκ-def is-nullκ-def)
hence ‹(⋀v x. AOT-model-valid-in v (Rep-urrel r (ωυ x)) =
AOT-model-valid-in v (Rep-rel Π (ωκ x))) ⟹ r ∈ a› **for** r
unfolding a-def[symmetric] AOT-model-enc-κ-def **apply** simp
by (smt (verit, best) AOT-rel-equiv-def Abs-rel-inverse Quotient3-def
κυ.simps(1) iso-tuple-UNIV-I urrel-quotient3 urrel-to-rel-def)
hence ‹(⋀v x. AOT-model-valid-in v (Rep-urrel r′ (ωυ x)) =
AOT-model-valid-in v (Rep-urrel r (ωυ x))) ⟹ r′ ∈ a› **for** r′
unfolding r-prop**.**
hence ‹⋀s. urrel-to-ωrel s = urrel-to-ωrel r ⟹ s ∈ a›
by (metis urrel-to-ωrel-def)
hence 0: ‹⋀s. urrel-to-ωrel s = urrel-to-ωrel r ⟹ s ∈ b›
using ασ-eq-ord-exts-all ασ-eq ext ασ-ασ′ **by** blast

assume Π′-den: ‹AOT-model-denotes Π′›
assume ‹∃ w. AOT-model-concrete w x ⟹ AOT-model-valid-in v (Rep-rel Π′ x) =
AOT-model-valid-in v (Rep-rel Π x)› **for** v x
hence ‹AOT-model-valid-in v (Rep-rel Π′ (ωκ x)) =
AOT-model-valid-in v (Rep-rel Π (ωκ x))› **for** v x
using AOT-model-ω-concrete-in-some-world AOT-model-concrete-κ.simps(1)
by presburger
hence ‹AOT-model-valid-in v (Rep-urrel (rel-to-urrel Π′) (ωυ x)) =
AOT-model-valid-in v (Rep-urrel r (ωυ x))› **for** v x
by (smt (verit, best) AOT-rel-equiv-def Abs-rel-inverse Quotient3-def
κυ.simps(1) iso-tuple-UNIV-I r-prop urrel-quotient3 urrel-to-rel-def Π′-den)
hence ‹urrel-to-ωrel (rel-to-urrel Π′) = urrel-to-ωrel r›
by (metis (full-types) AOT-urrel-ωequiv-def Quotient3-def urrel-ωrel-quot)
hence ‹rel-to-urrel Π′ ∈ b› **using** 0 **by** blast
thus ‹AOT-model-enc κ′ Π′›
unfolding b-def[symmetric] AOT-model-enc-κ-def **by** (auto simp: Π′-den)
next
fix κ κ′ :: κ **and** Π Π′ :: ‹<κ>› **and** w :: w
assume ext: ‹AOT-ExtendedModel›
assume ‹AOT-model-denotes κ›
moreover assume ‹∄ w. AOT-model-concrete w κ›
ultimately obtain a **where** a-def: ‹ακ a = κ›
by (metis AOT-model-ω-concrete-in-some-world AOT-model-concrete-κ.simps(1)
AOT-model-denotes-κ-def κ.discI(3) κ.exhaust-sel)
assume ‹AOT-model-denotes κ′›
moreover assume ‹∄ w. AOT-model-concrete w κ′›
ultimately obtain b **where** b-def: ‹ακ b = κ′›
by (metis AOT-model-ω-concrete-in-some-world AOT-model-concrete-κ.simps(1)
AOT-model-denotes-κ-def κ.discI(3) κ.exhaust-sel)

assume ‹*AOT-model-denotes* Π′ ⟹ *AOT-model-valid-in w* (*Rep-rel* Π′ κ) =
AOT-model-valid-in w (*Rep-rel* Π′ κ′)› **for** Π′ *w*
hence ‹*AOT-model-valid-in w* (*Rep-urrel r* (κυ κ)) =
AOT-model-valid-in w (*Rep-urrel r* (κυ κ′))› **for** *r*
by (*metis AOT-rel-equiv-def Abs-rel-inverse Quotient3-rel-rep*
iso-tuple-UNIV-I urrel-quotient3 urrel-to-rel-def)
hence ‹*let r* = (*Abs-urrel* (λ *u* . $\varepsilon_\circ$ *w* . *u* = κυ κ)) *in*
AOT-model-valid-in w (*Rep-urrel r* (κυ κ)) =
AOT-model-valid-in w (*Rep-urrel r* (κυ κ′))›
by *presburger*
hence *ασ-eq*: ‹ασ *a* = ασ *b*›
unfolding *Let-def*
apply (*subst* (*asm*) (*1 2*) *Abs-urrel-inverse*)
using *AOT-model-proposition-choice-simp a-def b-def* **by** *force+*
assume Π-*den*: ‹*AOT-model-denotes* Π›
have ‹¬*AOT-model-valid-in w* (*Rep-rel* Π (*SOME xa*. κυ *xa* = *nullv x*))› **for** *x w*
by (*metis* (*mono-tags*, *lifting*) *AOT-model-denotes-κ-def*
AOT-model-denotes-rel.rep-eq κ.exhaust-disc κυ.simps(*1*,*2*,*3*)
‹*AOT-model-denotes* Π› *υ.disc*(*8*) *υ.disc*(*9*) *υ.distinct*(*3*)
is-ακ-def is-ωκ-def verit-sko-ex′)
moreover have ‹*Rep-rel* Π (ωκ *x*) = *Rep-rel* Π (*SOME xa*. κυ *xa* = ωυ *x*)› **for** *x*
by (*metis* (*mono-tags*, *lifting*) *AOT-model-denotes-rel.rep-eq*
AOT-model-term-equiv-κ-def κυ.simps(*1*) Π-*den verit-sko-ex*′)
ultimately have ‹*Rep-rel* Π (ωκ *x*) = *Rep-urrel* (*rel-to-urrel* Π) (ωυ *x*)› **for** *x*
unfolding *rel-to-urrel-def*
by (*subst Abs-urrel-inverse*) *auto*
hence ‹∃ *r* . ∀ *x* . *Rep-rel* Π (ωκ *x*) = *Rep-urrel r* (ωυ *x*)›
by (*auto intro*!: *exI*[**where** *x*=‹*rel-to-urrel* Π›])
then obtain *r* **where** *r-prop*: ‹*Rep-rel* Π (ωκ *x*) = *Rep-urrel r* (ωυ *x*)› **for** *x*
by *blast*

assume ‹∃ Π′. *AOT-model-denotes* Π′ ∧
AOT-model-enc κ Π′ ∧
(∀ *v x*. (∃ *w*. *AOT-model-concrete w x*) ⟶ *AOT-model-valid-in v* (*Rep-rel* Π′ *x*) =
AOT-model-valid-in v (*Rep-rel* Π *x*))›
then obtain Π′ **where**
Π′-*den*: ‹*AOT-model-denotes* Π′› **and**
κ-*enc*-Π′: ‹*AOT-model-enc* κ Π′› **and**
Π′-*prop*: ‹∃ *w*. *AOT-model-concrete w x* ⟹
AOT-model-valid-in v (*Rep-rel* Π′ *x*) =
AOT-model-valid-in v (*Rep-rel* Π *x*)› **for** *v x*
by *blast*
have ‹*AOT-model-valid-in v* (*Rep-rel* Π′ (ωκ *x*)) =
AOT-model-valid-in v (*Rep-rel* Π (ωκ *x*))› **for** *x v*
by (*simp add*: *AOT-model-ω-concrete-in-some-world* Π′-*prop*)
hence *0*: ‹*AOT-urrel-ωequiv* (*rel-to-urrel* Π′) (*rel-to-urrel* Π)›
unfolding *AOT-urrel-ωequiv-def*
by (*smt* (*verit*) *AOT-rel-equiv-def Abs-rel-inverse Quotient3-def*
κυ.simps(*1*) *iso-tuple-UNIV-I urrel-quotient3 urrel-to-rel-def*
Π-*den* Π′-*den*)
have ‹*rel-to-urrel* Π′ ∈ *a*›
and ‹*urrel-to-ωrel* (*rel-to-urrel* Π′) = *urrel-to-ωrel* (*rel-to-urrel* Π)›
apply (*metis AOT-model-enc-κ-def κ.simps*(*11*) κ-*enc*-Π′ *a-def*)
by (*metis Quotient3-rel 0 urrel-ωrel-quot*)
hence ‹∃ *s*. *s* ∈ *b* ∧ *urrel-to-ωrel s* = *urrel-to-ωrel* (*rel-to-urrel* Π)›
using *ασ-eq-ord-exts-ex ασ-eq ext ασ-ασ*′ **by** *blast*
then obtain *s* **where**
s-prop: ‹*s* ∈ *b* ∧ *urrel-to-ωrel s* = *urrel-to-ωrel* (*rel-to-urrel* Π)›
by *blast*
then obtain Π′′ **where**
Π′′-*prop*: ‹*rel-to-urrel* Π′′ = *s*› **and** Π′′-*den*: ‹*AOT-model-denotes* Π′′›
by (*metis AOT-rel-equiv-def Quotient3-def urrel-quotient3*)
moreover have ‹*AOT-model-enc* κ′ Π′′›

```
    by (metis AOT-model-enc-κ-def Π''-den Π''-prop κ.simps(11) b-def s-prop)
  moreover have ‹AOT-model-valid-in v (Rep-rel Π'' x) =
                 AOT-model-valid-in v (Rep-rel Π x)›
            if ‹∃ w. AOT-model-concrete w x› for v x
  proof(insert that)
    assume ‹∃ w. AOT-model-concrete w x›
    then obtain u where x-def: ‹x = ωκ u›
      by (metis AOT-model-concrete-κ.simps(2,3) κ.exhaust)
    show ‹AOT-model-valid-in v (Rep-rel Π'' x) =
          AOT-model-valid-in v (Rep-rel Π x)›
      unfolding x-def
      by (smt (verit, best) AOT-rel-equiv-def Abs-rel-inverse Quotient3-def
            Π''-den Π''-prop Π-den κv.simps(1) iso-tuple-UNIV-I s-prop
            urrel-quotient3 urrel-to-ωrel-def urrel-to-rel-def)
  qed
  ultimately show ‹∃ Π'. AOT-model-denotes Π' ∧ AOT-model-enc κ' Π' ∧
    (∀ v x. (∃ w. AOT-model-concrete w x) ⟶ AOT-model-valid-in v (Rep-rel Π' x) =
                                        AOT-model-valid-in v (Rep-rel Π x))›
    apply (safe intro!: exI[where x=Π''])
    by auto
qed
end
```

Products of unary individual terms and individual terms are individual terms. A tuple is regular, if at most one element does not denote. I.e. a pair is regular, if the first (unary) element denotes and the second is regular (i.e. at most one of its recursive tuple elements does not denote), or the first does not denote, but the second denotes (i.e. all its recursive tuple elements denote).

```
instantiation prod :: (AOT-UnaryIndividualTerm, AOT-IndividualTerm) AOT-IndividualTerm
begin
definition AOT-model-regular-prod :: ‹'a×'b ⇒ bool› where
  ‹AOT-model-regular-prod ≡ λ (x,y) . AOT-model-denotes x ∧ AOT-model-regular y ∨
                                  ¬AOT-model-denotes x ∧ AOT-model-denotes y›
definition AOT-model-term-equiv-prod :: ‹'a×'b ⇒ 'a×'b ⇒ bool› where
  ‹AOT-model-term-equiv-prod ≡  λ (x1,y1) (x2,y2) .
    AOT-model-term-equiv x1 x2 ∧ AOT-model-term-equiv y1 y2›
function AOT-model-irregular-prod :: ‹('a×'b ⇒ o) ⇒ 'a×'b ⇒ o› where
  AOT-model-irregular-proj2: ‹AOT-model-denotes x ⟹
    AOT-model-irregular φ (x,y) =
    AOT-model-irregular (λy. φ (SOME x' . AOT-model-term-equiv x x', y)) y›
| AOT-model-irregular-proj1: ‹¬AOT-model-denotes x ∧ AOT-model-denotes y ⟹
    AOT-model-irregular φ (x,y) =
    AOT-model-irregular (λx. φ (x, SOME y' . AOT-model-term-equiv y y')) x›
| AOT-model-irregular-prod-generic: ‹¬AOT-model-denotes x ∧ ¬AOT-model-denotes y ⟹
    AOT-model-irregular φ (x,y) =
    (SOME Φ . AOT-model-irregular-spec Φ AOT-model-regular AOT-model-term-equiv)
      φ (x,y)›
  by auto blast
termination using termination by blast

instance proof
  obtain x :: 'a and y :: 'b where
    ‹¬AOT-model-denotes x› and ‹¬AOT-model-denotes y›
    by (meson AOT-model-nondenoting-ex AOT-model-denoting-ex)
  thus ‹∃ x::'a×'b. ¬AOT-model-denotes x›
    by (auto simp: AOT-model-denotes-prod-def AOT-model-regular-prod-def)
next
  show ‹equivp (AOT-model-term-equiv :: 'a×'b ⇒ 'a×'b ⇒ bool)›
    by (rule equivpI; rule reflpI sympI transpI;
        simp add: AOT-model-term-equiv-prod-def AOT-model-term-equiv-part-equivp
                  equivp-reflp prod.case-eq-if case-prod-unfold equivp-symp)
      (metis equivp-transp[OF AOT-model-term-equiv-part-equivp])
next
  show ‹¬AOT-model-regular x ⟹ ¬ AOT-model-denotes x› for x :: ‹'a×'b›
```

```
    by (metis (mono-tags, lifting) AOT-model-denotes-prod-def case-prod-unfold
          AOT-model-irregular-nondenoting AOT-model-regular-prod-def)
next
  fix x y :: ‹'a×'b›
  show ‹AOT-model-term-equiv x y ⟹ AOT-model-denotes x = AOT-model-denotes y›
    by (metis (mono-tags, lifting) AOT-model-denotes-prod-def case-prod-beta
        AOT-model-term-equiv-denotes AOT-model-term-equiv-prod-def )
next
  fix x y :: ‹'a×'b›
  show ‹AOT-model-term-equiv x y ⟹ AOT-model-regular x = AOT-model-regular y›
    by (induct x; induct y;
        simp add: AOT-model-term-equiv-prod-def AOT-model-regular-prod-def)
       (meson AOT-model-term-equiv-denotes AOT-model-term-equiv-regular)
next
  interpret sp: AOT-model-irregular-spec ‹λφ (x::'a×'b) . εo w . False›
                  AOT-model-regular AOT-model-term-equiv
    by (simp add: AOT-model-irregular-spec-def AOT-model-proposition-choice-simp)
  have ex-spec: ‹∃ φ :: ('a×'b ⇒ o) ⇒ 'a×'b ⇒ o .
    AOT-model-irregular-spec φ AOT-model-regular AOT-model-term-equiv›
    using sp.AOT-model-irregular-spec-axioms by blast
  have some-spec: ‹AOT-model-irregular-spec
    (SOME φ :: ('a×'b ⇒ o) ⇒ 'a×'b ⇒ o .
        AOT-model-irregular-spec φ AOT-model-regular AOT-model-term-equiv)
    AOT-model-regular AOT-model-term-equiv›
    using someI-ex[OF ex-spec] by argo
  interpret sp-some: AOT-model-irregular-spec
    ‹SOME φ :: ('a×'b ⇒ o) ⇒ 'a×'b ⇒ o .
        AOT-model-irregular-spec φ AOT-model-regular AOT-model-term-equiv›
    AOT-model-regular AOT-model-term-equiv
    using some-spec by blast
  show ‹AOT-model-irregular-spec (AOT-model-irregular :: ('a×'b ⇒ o) ⇒ 'a×'b ⇒ o)
          AOT-model-regular AOT-model-term-equiv›
  proof
    have ‹¬AOT-model-valid-in w (AOT-model-irregular φ (a, b))›
      for w φ and a :: 'a and b :: 'b
      by (induct arbitrary: φ rule: AOT-model-irregular-prod.induct)
         (auto simp: AOT-model-irregular-false sp-some.AOT-model-irregular-false)
    thus ¬AOT-model-valid-in w (AOT-model-irregular φ x) for w φ and x :: ‹'a×'b›
      by (induct x)
  next
    {
      fix x1 y1 :: 'a and x2 y2 :: 'b and φ :: ‹'a×'b⇒o›
      assume x1y1-equiv: ‹AOT-model-term-equiv x1 y1›
      moreover assume x2y2-equiv: ‹AOT-model-term-equiv x2 y2›
      ultimately have xy-equiv: ‹AOT-model-term-equiv (x1,x2) (y1,y2)›
        by (simp add: AOT-model-term-equiv-prod-def)
      {
        assume ‹AOT-model-denotes x1›
        moreover hence ‹AOT-model-denotes y1›
          using AOT-model-term-equiv-denotes AOT-model-term-equiv-regular
                x1y1-equiv x2y2-equiv by blast
        ultimately have ‹AOT-model-irregular φ (x1,x2) =
                         AOT-model-irregular φ (y1,y2)›
          using AOT-model-irregular-equiv AOT-model-term-equiv-eps(3)
                x1y1-equiv x2y2-equiv by fastforce
      }
      moreover {
        assume ‹~AOT-model-denotes x1 ∧ AOT-model-denotes x2›
        moreover hence ‹~AOT-model-denotes y1 ∧ AOT-model-denotes y2›
          by (meson AOT-model-term-equiv-denotes x1y1-equiv x2y2-equiv)
        ultimately have ‹AOT-model-irregular φ (x1,x2) =
                         AOT-model-irregular φ (y1,y2)›
          using AOT-model-irregular-equiv AOT-model-term-equiv-eps(3)
```

```
          x1y1-equiv x2y2-equiv by fastforce
    }
    moreover {
      assume denotes-x: ‹(¬AOT-model-denotes x1 ∧ ¬AOT-model-denotes x2)›
      hence denotes-y: ‹(¬AOT-model-denotes y1 ∧ ¬AOT-model-denotes y2)›
        by (meson AOT-model-term-equiv-denotes AOT-model-term-equiv-regular
              x1y1-equiv x2y2-equiv)
      have eps-eq: ‹Eps (AOT-model-term-equiv x1) = Eps (AOT-model-term-equiv y1)›
        by (simp add: AOT-model-term-equiv-eps(3) x1y1-equiv)
      have ‹AOT-model-irregular φ (x1,x2) = AOT-model-irregular φ (y1,y2)›
        using denotes-x denotes-y
        using sp-some.AOT-model-irregular-equiv xy-equiv by auto
    }
    moreover {
      assume denotes-x: ‹¬AOT-model-denotes x1 ∧ AOT-model-denotes x2›
      hence denotes-y: ‹¬AOT-model-denotes y1 ∧ AOT-model-denotes y2›
        by (meson AOT-model-term-equiv-denotes x1y1-equiv x2y2-equiv)
      have eps-eq: ‹Eps (AOT-model-term-equiv x2) = Eps (AOT-model-term-equiv y2)›
        by (simp add: AOT-model-term-equiv-eps(3) x2y2-equiv)
      have ‹AOT-model-irregular φ (x1,x2) = AOT-model-irregular φ (y1,y2)›
        using denotes-x denotes-y
        using AOT-model-irregular-nondenoting calculation(2) by blast
    }
    ultimately have ‹AOT-model-irregular φ (x1,x2) = AOT-model-irregular φ (y1,y2)›
      using AOT-model-term-equiv-denotes AOT-model-term-equiv-regular
          sp-some.AOT-model-irregular-equiv x1y1-equiv x2y2-equiv xy-equiv
      by blast
  } note 0 = this
  show ‹AOT-model-term-equiv x y ⟹
       AOT-model-irregular φ x = AOT-model-irregular φ y›
    for x y :: ‹'a×'b› and φ
    by (induct x; induct y; simp add: AOT-model-term-equiv-prod-def 0)
next
  fix φ ψ :: ‹'a×'b ⇒ o›
  assume ‹AOT-model-regular x ⟹ φ x = ψ x› for x
  hence ‹φ (x, y) = ψ (x, y)›
    if ‹AOT-model-denotes x ∧ AOT-model-regular y ∨
       ¬AOT-model-denotes x ∧ AOT-model-denotes y› for x y
    using that unfolding AOT-model-regular-prod-def by simp
  hence ‹AOT-model-irregular φ (x,y) = AOT-model-irregular ψ (x,y)›
    for x :: 'a and y :: 'b
  proof (induct arbitrary: ψ φ rule: AOT-model-irregular-prod.induct)
    case (1 x y φ)
    thus ?case
      apply simp
      by (meson AOT-model-irregular-eqI AOT-model-irregular-nondenoting
              AOT-model-term-equiv-denotes AOT-model-term-equiv-eps(1))
  next
    case (2 x y φ)
    thus ?case
      apply simp
      by (meson AOT-model-irregular-nondenoting AOT-model-term-equiv-denotes
              AOT-model-term-equiv-eps(1))
  next
    case (3 x y φ)
    thus ?case
      apply simp
      by (metis (mono-tags, lifting) AOT-model-regular-prod-def case-prod-conv
                           sp-some.AOT-model-irregular-eqI surj-pair)
  qed
  thus ‹AOT-model-irregular φ x = AOT-model-irregular ψ x› for  x :: ‹'a×'b›
    by (metis surjective-pairing)
qed
```

```
qed
end
```

Introduction rules for term equivalence on tuple terms.

```
lemma AOT-meta-prod-equivI:
  shows ⋀ (a::'a::AOT-UnaryIndividualTerm) x (y :: 'b::AOT-IndividualTerm) .
            AOT-model-term-equiv x y ⟹ AOT-model-term-equiv (a,x) (a,y)
    and ⋀ (x::'a::AOT-UnaryIndividualTerm) y (b :: 'b::AOT-IndividualTerm) .
            AOT-model-term-equiv x y ⟹ AOT-model-term-equiv (x,b) (y,b)
    unfolding AOT-model-term-equiv-prod-def
    by (simp add: AOT-model-term-equiv-part-equivp equivp-reflp)+
```

The type of propositions are trivial instances of terms.

```
instantiation o :: AOT-Term
begin
definition AOT-model-denotes-o :: ‹o ⇒ bool› where
  ‹AOT-model-denotes-o ≡ λ-. True›
instance proof
  show ‹∃ x::o. AOT-model-denotes x›
    by (simp add: AOT-model-denotes-o-def)
qed
end
```

AOT's variables are modelled by restricting the type of terms to those terms that denote.

```
typedef 'a AOT-var = ‹{ x :: 'a::AOT-Term . AOT-model-denotes x }›
  morphisms AOT-term-of-var AOT-var-of-term
  by (simp add: AOT-model-denoting-ex)
```

Simplify automatically generated theorems and rules.

```
declare AOT-var-of-term-induct[induct del]
        AOT-var-of-term-cases[cases del]
        AOT-term-of-var-induct[induct del]
        AOT-term-of-var-cases[cases del]
lemmas AOT-var-of-term-inverse = AOT-var-of-term-inverse[simplified]
   and AOT-var-of-term-inject = AOT-var-of-term-inject[simplified]
   and AOT-var-of-term-induct =
         AOT-var-of-term-induct[simplified, induct type: AOT-var]
   and AOT-var-of-term-cases =
         AOT-var-of-term-cases[simplified, cases type: AOT-var]
   and AOT-term-of-var = AOT-term-of-var[simplified]
   and AOT-term-of-var-cases =
         AOT-term-of-var-cases[simplified, induct pred: AOT-term-of-var]
   and AOT-term-of-var-induct =
         AOT-term-of-var-induct[simplified, induct pred: AOT-term-of-var]
   and AOT-term-of-var-inverse = AOT-term-of-var-inverse[simplified]
   and AOT-term-of-var-inject = AOT-term-of-var-inject[simplified]
```

Equivalence by definition is modelled as necessary equivalence.

```
consts AOT-model-equiv-def :: ‹o ⇒ o ⇒ bool›
specification(AOT-model-equiv-def)
  AOT-model-equiv-def: ‹AOT-model-equiv-def φ ψ = (∀ v . AOT-model-valid-in v φ =
                                                   AOT-model-valid-in v ψ)›
  by (rule exI[where x=‹λ φ ψ . ∀ v . AOT-model-valid-in v φ =
                                      AOT-model-valid-in v ψ›]) simp
```

Identity by definition is modelled as identity for denoting terms plus co-denoting.

```
consts AOT-model-id-def :: ‹('b ⇒ 'a::AOT-Term) ⇒ ('b ⇒ 'a) ⇒ bool›
specification(AOT-model-id-def)
  AOT-model-id-def: ‹(AOT-model-id-def τ σ) = (∀ α . if AOT-model-denotes (σ α)
                                                  then τ α = σ α
                                                  else ¬AOT-model-denotes (τ α))›
  by (rule exI[where x=λ τ σ . ∀ α . if AOT-model-denotes (σ α)
```

then $\tau\ \alpha = \sigma\ \alpha$
else ¬*AOT-model-denotes* ($\tau\ \alpha$)])
blast

To reduce definitions by identity without free variables to definitions by identity with free variables acting on the unit type, we give the unit type a trivial instantiation to *AOT-Term*.

instantiation *unit* :: *AOT-Term*
begin
definition *AOT-model-denotes-unit* :: ‹*unit* ⇒ *bool*› **where**
‹*AOT-model-denotes-unit* ≡ λ-. *True*›
instance proof qed(*simp add*: *AOT-model-denotes-unit-def*)
end

Modally-strict and modally-fragile axioms are as necessary, resp. actually valid propositions.

definition *AOT-model-axiom* **where**
‹*AOT-model-axiom* ≡ λ φ . ∀ v . *AOT-model-valid-in* v φ›
definition *AOT-model-act-axiom* **where**
‹*AOT-model-act-axiom* ≡ λ φ . *AOT-model-valid-in* w_0 φ›

lemma *AOT-model-axiomI*:
assumes ‹⋀v . *AOT-model-valid-in* v φ›
shows ‹*AOT-model-axiom* φ›
unfolding *AOT-model-axiom-def* **using** *assms* **..**

lemma *AOT-model-act-axiomI*:
assumes ‹*AOT-model-valid-in* w_0 φ›
shows ‹*AOT-model-act-axiom* φ›
unfolding *AOT-model-act-axiom-def* **using** *assms* **.**

3 Outer Syntax Commands

nonterminal *AOT-prop*
nonterminal φ
nonterminal φ′
nonterminal τ
nonterminal τ′
nonterminal *AOT-axiom*
nonterminal *AOT-act-axiom*
ML-file *AOT-keys.ML*
ML-file *AOT-commands.ML*
setup‹*AOT-Theorems.setup*›
setup‹*AOT-Definitions.setup*›
setup‹*AOT-no-atp.setup*›

4 Approximation of the Syntax of PLM

locale *AOT-meta-syntax*
begin
notation *AOT-model-valid-in* (‹[- ⊨ -]›)
notation *AOT-model-axiom* (‹□[-]›)
notation *AOT-model-act-axiom* (‹𝓐[-]›)
end
locale *AOT-no-meta-syntax*
begin
no-notation *AOT-model-valid-in* (‹[- ⊨ -]›)
no-notation *AOT-model-axiom* (‹□[-]›)
no-notation *AOT-model-act-axiom* (‹𝓐[-]›)
end

```
consts AOT-denotes :: ‹'a::AOT-Term ⇒ o›
       AOT-imp :: ‹[o, o] ⇒ o›
       AOT-not :: ‹o ⇒ o›
       AOT-box :: ‹o ⇒ o›
       AOT-act :: ‹o ⇒ o›
       AOT-forall :: ‹('a::AOT-Term ⇒ o) ⇒ o›
       AOT-eq :: ‹'a::AOT-Term ⇒ 'a::AOT-Term ⇒ o›
       AOT-desc :: ‹('a::AOT-UnaryIndividualTerm ⇒ o) ⇒ 'a›
       AOT-exe :: ‹<'a::AOT-IndividualTerm> ⇒ 'a ⇒ o›
       AOT-lambda :: ‹('a::AOT-IndividualTerm ⇒ o) ⇒ <'a>›
       AOT-lambda0 :: ‹o ⇒ o›
       AOT-concrete :: ‹<'a::AOT-UnaryIndividualTerm> AOT-var›

nonterminal κs and Π and Π0 and α and exe-arg and exe-args
        and lambda-args and desc and free-var and free-vars
        and AOT-props and AOT-premises and AOT-world-relative-prop

syntax -AOT-process-frees :: ‹φ ⇒ φ'› (‹-›)
       -AOT-verbatim :: ‹any ⇒ φ› (‹«-»›)
       -AOT-verbatim :: ‹any ⇒ τ› (‹«-»›)
       -AOT-quoted :: ‹φ' ⇒ any› (‹«-»›)
       -AOT-quoted :: ‹τ' ⇒ any› (‹«-»›)
        :: ‹φ ⇒ φ› (‹'(-')›)
       -AOT-process-frees :: ‹τ ⇒ τ'› (‹-›)
        :: ‹κs ⇒ τ› (‹-›)
        :: ‹Π ⇒ τ› (‹-›)
        :: ‹φ ⇒ τ› (‹'(-')›)
       -AOT-term-var :: ‹id-position ⇒ τ› (‹-›)
       -AOT-term-var :: ‹id-position ⇒ φ› (‹-›)
       -AOT-exe-vars :: ‹id-position ⇒ exe-arg› (‹-›)
       -AOT-lambda-vars :: ‹id-position ⇒ lambda-args› (‹-›)
       -AOT-var :: ‹id-position ⇒ α› (‹-›)
       -AOT-vars :: ‹id-position ⇒ any›
       -AOT-verbatim :: ‹any ⇒ α› (‹«-»›)
       -AOT-valid :: ‹w ⇒ φ' ⇒ bool› (‹[- ⊨ -]›)
       -AOT-denotes :: ‹τ ⇒ φ› (‹-↓›)
       -AOT-imp :: ‹[φ, φ] ⇒ φ› (infixl ‹→› 25)
       -AOT-not :: ‹φ ⇒ φ› (‹~-› [50] 50)
       -AOT-not :: ‹φ ⇒ φ› (‹¬-› [50] 50)
       -AOT-box :: ‹φ ⇒ φ› (‹□-› [49] 54)
       -AOT-act :: ‹φ ⇒ φ› (‹𝓐-› [49] 54)
       -AOT-all :: ‹α ⇒ φ ⇒ φ› (‹∀ - -› [1,40])
syntax (input)
       -AOT-all-ellipse
           :: ‹id-position ⇒ id-position ⇒ φ ⇒ φ› (‹∀ -...∀ - -› [1,40])
syntax (output)
       -AOT-all-ellipse
           :: ‹id-position ⇒ id-position ⇒ φ ⇒ φ› (‹∀ -...∀ -'(-')› [1,40])
syntax
       -AOT-eq :: ‹[τ, τ] ⇒ φ› (infixl ‹=› 50)
       -AOT-desc :: ‹α ⇒ φ ⇒ desc› (‹ι--› [1,1000])
        :: ‹desc ⇒ κs› (‹-›)
       -AOT-lambda :: ‹lambda-args ⇒ φ ⇒ Π› (‹[λ- -]›)
       -explicitRelation :: ‹τ ⇒ Π› (‹[-]›)
        :: ‹κs ⇒ exe-arg› (‹-›)
        :: ‹exe-arg ⇒ exe-args› (‹-›)
       -AOT-exe-args :: ‹exe-arg ⇒ exe-args ⇒ exe-args› (‹--›)
       -AOT-exe-arg-ellipse :: ‹id-position ⇒ id-position ⇒ exe-arg› (‹-...-›)
       -AOT-lambda-arg-ellipse
           :: ‹id-position ⇒ id-position ⇒ lambda-args› (‹-...-›)
       -AOT-term-ellipse :: ‹id-position ⇒ id-position ⇒ τ› (‹-...-›)
       -AOT-exe :: ‹Π ⇒ exe-args ⇒ φ› (‹--›)
       -AOT-enc :: ‹exe-args ⇒ Π ⇒ φ› (‹--›)
```

```
      -AOT-lambda0 :: ‹φ ⇒ Π0› (‹[λ -]›)
       :: ‹Π0 ⇒ φ› (‹-›)
       :: ‹Π0 ⇒ τ› (‹-›)
      -AOT-concrete :: ‹Π› (‹E!›)
       :: ‹any ⇒ exe-arg› (‹«-»›)
       :: ‹desc ⇒ free-var› (‹-›)
       :: ‹Π ⇒ free-var› (‹-›)
      -AOT-appl :: ‹id-position ⇒ free-vars ⇒ φ› (‹-'{-'}›)
      -AOT-appl :: ‹id-position ⇒ free-vars ⇒ τ› (‹-'{-'}›)
      -AOT-appl :: ‹id-position ⇒ free-vars ⇒ free-vars› (‹-'{-'}›)
      -AOT-appl :: ‹id-position ⇒ free-vars ⇒ free-vars› (‹-'{-'}›)
      -AOT-term-var :: ‹id-position ⇒ free-var› (‹-›)
       :: ‹any ⇒ free-var› (‹«-»›)
       :: ‹free-var ⇒ free-vars› (‹-›)
      -AOT-args :: ‹free-var ⇒ free-vars ⇒ free-vars› (‹-,-›)
      -AOT-free-var-ellipse :: ‹id-position ⇒ id-position ⇒ free-var› (‹-...-›)
syntax -AOT-premises
          :: ‹AOT-world-relative-prop ⇒ AOT-premises ⇒ AOT-premises› (infixr ‹,› 3)
      -AOT-world-relative-prop :: φ ⇒ AOT-world-relative-prop (‹-›)
       :: AOT-world-relative-prop ⇒ AOT-premises (‹-›)
      -AOT-prop :: ‹AOT-world-relative-prop ⇒ AOT-prop› (‹-›)
       :: ‹AOT-prop ⇒ AOT-props› (‹-›)
      -AOT-derivable :: AOT-premises ⇒ φ' ⇒ AOT-prop (infixl ‹⊢› 2)
      -AOT-nec-derivable :: AOT-premises ⇒ φ' ⇒ AOT-prop (infixl ‹⊢□› 2)
      -AOT-theorem :: φ' ⇒ AOT-prop (‹⊢ -›)
      -AOT-nec-theorem :: φ' ⇒ AOT-prop (‹⊢□ -›)
      -AOT-equiv-def :: ‹φ ⇒ φ ⇒ AOT-prop› (infixl ‹≡df› 3)
      -AOT-axiom :: φ' ⇒ AOT-axiom (‹-›)
      -AOT-act-axiom :: φ' ⇒ AOT-act-axiom (‹-›)
      -AOT-axiom :: φ' ⇒ AOT-prop (‹- ∈ Λ□›)
      -AOT-act-axiom :: φ' ⇒ AOT-prop (‹- ∈ Λ›)
      -AOT-id-def :: ‹τ ⇒ τ ⇒ AOT-prop› (infixl ‹=df› 3)
      -AOT-for-arbitrary
          :: ‹id-position ⇒ AOT-prop ⇒ AOT-prop› (‹for arbitrary -: -› [1000,1] 1)
syntax (output) -lambda-args :: ‹any ⇒ patterns ⇒ patterns› (‹--›)

translations
  [w ⊨ φ] => CONST AOT-model-valid-in w φ

AOT-syntax-print-translations
  [w ⊨ φ] <= CONST AOT-model-valid-in w φ

ML-file AOT-syntax.ML

AOT-register-type-constraints
  Individual: ‹-::AOT-UnaryIndividualTerm› ‹-::AOT-IndividualTerm› and
  Proposition: o and
  Relation: ‹<-::AOT-IndividualTerm>› and
  Term: ‹-::AOT-Term›

AOT-register-variable-names
  Individual: x y z ν μ a b c d and
  Proposition: p q r s and
  Relation: F G H P Q R S and
  Term: α β γ δ

AOT-register-metavariable-names
  Individual: κ and
  Proposition: φ ψ χ ϑ ζ ξ Θ and
  Relation: Π and
  Term: τ σ

AOT-register-premise-set-names Γ Δ Λ
```

parse-ast-translation‹[
  (***syntax-const*** ‹-*AOT-var*›, *K AOT-check-var*),
  (***syntax-const*** ‹-*AOT-exe-vars*›, *K AOT-split-exe-vars*),
  (***syntax-const*** ‹-*AOT-lambda-vars*›, *K AOT-split-lambda-args*)
]›

translations
  -AOT-denotes τ => *CONST AOT-denotes* τ
  -AOT-imp φ ψ => *CONST AOT-imp* φ ψ
  -AOT-not φ => *CONST AOT-not* φ
  -AOT-box φ => *CONST AOT-box* φ
  -AOT-act φ => *CONST AOT-act* φ
  -AOT-eq τ τ' => *CONST AOT-eq* τ τ'
  -AOT-lambda0 φ => *CONST AOT-lambda0* φ
  -AOT-concrete => *CONST AOT-term-of-var* (*CONST AOT-concrete*)
  -AOT-lambda α φ => *CONST AOT-lambda* (*-abs* α φ)
  -explicitRelation Π => Π

AOT-syntax-print-translations
  -AOT-lambda (*-lambda-args x y*) φ <= *CONST AOT-lambda* (*-abs* (*-pattern x y*) φ)
  -AOT-lambda (*-lambda-args x y*) φ <= *CONST AOT-lambda* (*-abs* (*-patterns x y*) φ)
  -AOT-lambda x φ <= *CONST AOT-lambda* (*-abs x* φ)
  -lambda-args x (*-lambda-args y z*) <= *-lambda-args x* (*-patterns y z*)
  -lambda-args (*x y z*) <= *-lambda-args* (*-tuple x* (*-tuple-arg* (*-tuple y z*)))

AOT-syntax-print-translations
  -AOT-imp φ ψ <= *CONST AOT-imp* φ ψ
  -AOT-not φ <= *CONST AOT-not* φ
  -AOT-box φ <= *CONST AOT-box* φ
  -AOT-act φ <= *CONST AOT-act* φ
  -AOT-all α φ <= *CONST AOT-forall* (*-abs* α φ)
  -AOT-all α φ <= *CONST AOT-forall* ($\lambda\alpha.$ φ)
  -AOT-eq τ τ' <= *CONST AOT-eq* τ τ'
  -AOT-desc x φ <= *CONST AOT-desc* (*-abs x* φ)
  -AOT-desc x φ <= *CONST AOT-desc* ($\lambda x.$ φ)
  -AOT-lambda0 φ <= *CONST AOT-lambda0* φ
  -AOT-concrete <= *CONST AOT-term-of-var* (*CONST AOT-concrete*)

translations
  -AOT-appl φ (*-AOT-args a b*) => *-AOT-appl* (φ *a*) *b*
  -AOT-appl φ *a* => φ *a*

parse-translation‹
[
  (***syntax-const*** ‹-*AOT-var*›, *parseVar true*),
  (***syntax-const*** ‹-*AOT-vars*›, *parseVar false*),
  (***syntax-const*** ‹-*AOT-valid*›, *fn ctxt* => *fn* [*w*,*x*] =>
    const ‹*AOT-model-valid-in*› $ *w* $ *x*),
  (***syntax-const*** ‹-*AOT-quoted*›, *fn ctxt* => *fn* [*x*] => *x*),
  (***syntax-const*** ‹-*AOT-process-frees*›, *fn ctxt* => *fn* [*x*] => *processFrees ctxt x*),
  (***syntax-const*** ‹-*AOT-world-relative-prop*›, *fn ctxt* => *fn* [*x*] => *let*
    val (*x*, *premises*) = *processFreesAndPremises ctxt x*
    val (*world*::*formulas*) = *Variable.variant-names* (*Variable.declare-names x ctxt*)
        ((*v*, *dummyT*)::(*map* (*fn* - => (φ, *dummyT*)) *premises*))
    val term = *HOLogic.mk-Trueprop*
        (@{*const AOT-model-valid-in*} $ *Free world* $ *processFrees ctxt x*)
    val term = *fold* (*fn* (*premise*,*form*) => *fn trm* =>
        @{*const Pure.imp*} $
        HOLogic.mk-Trueprop
          (*Const* (***const-name*** ‹*Set.member*›, *dummyT*) $ *Free form* $ *premise*) $

```
        (Term.absfree (Term.dest-Free (dropConstraints premise)) trm $ Free form)
    ) (ListPair.zipEq (premises,formulas)) term
    val term = fold (fn (form) => fn trm =>
         Const (const-name ‹Pure.all›, dummyT) $
        (Term.absfree form trm)
    ) formulas term
    val term = Term.absfree world term
    in term end),
  (syntax-const ‹-AOT-prop›, fn ctxt => fn [x] => let
    val world = case (AOT-ProofData.get ctxt) of SOME w => w
        | - => raise Fail Expected world to be stored in the proof state.
    in x $ world end),
  (syntax-const ‹-AOT-theorem›, fn ctxt => fn [x] =>
      HOLogic.mk-Trueprop (@{const AOT-model-valid-in} $ @{const w0} $ x)),
  (syntax-const ‹-AOT-axiom›, fn ctxt => fn [x] =>
      HOLogic.mk-Trueprop (@{const AOT-model-axiom} $ x)),
  (syntax-const ‹-AOT-act-axiom›, fn ctxt => fn [x] =>
      HOLogic.mk-Trueprop (@{const AOT-model-act-axiom} $ x)),
  (syntax-const ‹-AOT-nec-theorem›, fn ctxt => fn [trm] => let
    val world = singleton (Variable.variant-names (Variable.declare-names trm ctxt)) (v, @{typ w})
    val trm = HOLogic.mk-Trueprop (@{const AOT-model-valid-in} $ Free world $ trm)
    val trm = Term.absfree world trm
    val trm = Const (const-name ‹Pure.all›, dummyT) $ trm
    in trm end),
  (syntax-const ‹-AOT-derivable›, fn ctxt => fn [x,y] => let
    val world = case (AOT-ProofData.get ctxt) of SOME w => w
      | - => raise Fail Expected world to be stored in the proof state.
    in foldPremises world x y end),
  (syntax-const ‹-AOT-nec-derivable›, fn ctxt => fn [x,y] => let
    in Const (const-name ‹Pure.all›, dummyT) $
      Abs (v, dummyT, foldPremises (Bound 0) x y) end),
  (syntax-const ‹-AOT-for-arbitrary›, fn ctxt => fn [- $ var $ pos,trm] => let
    val trm = Const (const-name ‹Pure.all›, dummyT) $
        (Const (-constrainAbs, dummyT) $ Term.absfree (Term.dest-Free var) trm $ pos)
    in trm end),
  (syntax-const ‹-AOT-equiv-def›, parseEquivDef),
  (syntax-const ‹-AOT-exe›, parseExe),
  (syntax-const ‹-AOT-enc›, parseEnc)
]
›

parse-ast-translation‹
[
  (syntax-const ‹-AOT-exe-arg-ellipse›, parseEllipseList -AOT-term-vars),
  (syntax-const ‹-AOT-lambda-arg-ellipse›, parseEllipseList -AOT-vars),
  (syntax-const ‹-AOT-free-var-ellipse›, parseEllipseList -AOT-term-vars),
  (syntax-const ‹-AOT-term-ellipse›, parseEllipseList -AOT-term-vars),
  (syntax-const ‹-AOT-all-ellipse›, fn ctx => fn [a,b,c] =>
      Ast.mk-appl (Ast.Constant const-name ‹AOT-forall›) [
        Ast.mk-appl (Ast.Constant -abs) [parseEllipseList -AOT-vars ctx [a,b],c]
      ])
]
›

syntax (output)
  -AOT-individual-term :: ‹'a ⇒ tuple-args› (‹-›)
  -AOT-individual-terms :: ‹tuple-args ⇒ tuple-args ⇒ tuple-args› (‹--›)
  -AOT-relation-term :: ‹'a ⇒ Π›
  -AOT-any-term :: ‹'a ⇒ τ›


print-ast-translation‹AOT-syntax-print-ast-translations[
 (syntax-const ‹-AOT-individual-term›, AOT-print-individual-term),
```

(*syntax-const* ‹-*AOT-relation-term*›, *AOT-print-relation-term*),
(*syntax-const* ‹-*AOT-any-term*›, *AOT-print-generic-term*)
]›

AOT-syntax-print-translations
-*AOT-individual-terms* (-*AOT-individual-term* *x*) (-*AOT-individual-terms* (-*tuple* *y* *z*))
<= -*AOT-individual-terms* (-*tuple* *x* (-*tuple-args* *y* *z*))
-*AOT-individual-terms* (-*AOT-individual-term* *x*) (-*AOT-individual-term* *y*)
<= -*AOT-individual-terms* (-*tuple* *x* (-*tuple-arg* *y*))
-*AOT-individual-terms* (-*tuple* *x* *y*) <= -*AOT-individual-term* (-*tuple* *x* *y*)
-*AOT-exe* (-*AOT-relation-term* Π) (-*AOT-individual-term* κ) <= *CONST* *AOT-exe* Π κ
-*AOT-denotes* (-*AOT-any-term* κ) <= *CONST* *AOT-denotes* κ

AOT-define *AOT-conj* :: ‹[φ, φ] ⇒ φ› (**infixl** ‹&› *35*) ‹φ & ψ $\equiv_{df}$ ¬(φ → ¬ψ)›
declare *AOT-conj*[*AOT del*, *AOT-defs del*]
AOT-define *AOT-disj* :: ‹[φ, φ] ⇒ φ› (**infixl** ‹∨› *35*) ‹φ ∨ ψ $\equiv_{df}$ ¬φ → ψ›
declare *AOT-disj*[*AOT del*, *AOT-defs del*]
AOT-define *AOT-equiv* :: ‹[φ, φ] ⇒ φ› (**infix** ‹≡› *20*) ‹φ ≡ ψ $\equiv_{df}$ (φ → ψ) & (ψ → φ)›
declare *AOT-equiv*[*AOT del*, *AOT-defs del*]
AOT-define *AOT-dia* :: ‹φ ⇒ φ› (‹◇-› [*49*] *54*) ‹◇φ $\equiv_{df}$ ¬□¬φ›
declare *AOT-dia*[*AOT del*, *AOT-defs del*]

context *AOT-meta-syntax*
begin
notation *AOT-dia* (‹◇-› [*49*] *54*)
notation *AOT-conj* (**infixl** ‹**&**› *35*)
notation *AOT-disj* (**infixl** ‹∨› *35*)
notation *AOT-equiv* (**infixl** ‹≡› *20*)
end
context *AOT-no-meta-syntax*
begin
no-notation *AOT-dia* (‹◇-› [*49*] *54*)
no-notation *AOT-conj* (**infixl** ‹**&**› *35*)
no-notation *AOT-disj* (**infixl** ‹∨› *35*)
no-notation *AOT-equiv* (**infixl** ‹≡› *20*)
end

print-translation ‹
AOT-syntax-print-translations
[
AOT-preserve-binder-abs-tr′
const-syntax ‹*AOT-forall*›
syntax-const ‹-*AOT-all*›
(***syntax-const*** ‹-*AOT-all-ellipse*›, *true*)
const-name ‹*AOT-imp*›,
AOT-binder-trans @{*theory*} @{*binding AOT-forall-binder*} ***syntax-const*** ‹-*AOT-all*›,
(***const-syntax*** ‹*AOT-desc*›, *fn ctxt* => *Syntax-Trans.preserve-binder-abs-tr′* ***syntax-const*** ‹-*AOT-desc*› *ctxt dummyT*),
AOT-binder-trans @{*theory*} @{*binding AOT-desc-binder*} ***syntax-const*** ‹-*AOT-desc*›,
AOT-preserve-binder-abs-tr′
const-syntax ‹*AOT-lambda*›
syntax-const ‹-*AOT-lambda*›
(***syntax-const*** ‹-*AOT-lambda-arg-ellipse*›, *false*)
const-name ‹*undefined*›,
AOT-binder-trans
@{*theory*}
@{*binding AOT-lambda-binder*}
syntax-const ‹-*AOT-lambda*›
]
›

parse-translation‹

```
[(syntax-const ‹-AOT-id-def›, parseIdDef)]
›

parse-ast-translation‹[
 (syntax-const ‹-AOT-all›,
  AOT-restricted-binder const-name ‹AOT-forall› const-name ‹AOT-imp›),
 (syntax-const ‹-AOT-desc›,
  AOT-restricted-binder const-name ‹AOT-desc› const-name ‹AOT-conj›)
]›

AOT-define AOT-exists :: ‹α ⇒ φ ⇒ φ› ‹«AOT-exists φ» ≡df ¬∀ α ¬φ{α}›
declare AOT-exists[AOT del, AOT-defs del]
syntax -AOT-exists :: ‹α ⇒ φ ⇒ φ› (‹∃ - -› [1,40])

AOT-syntax-print-translations
  -AOT-exists α φ <= CONST AOT-exists (-abs α φ)
  -AOT-exists α φ <= CONST AOT-exists (λα. φ)

parse-ast-translation‹
[(syntax-const ‹-AOT-exists›,
  AOT-restricted-binder const-name ‹AOT-exists› const-name ‹AOT-conj›)]
›

context AOT-meta-syntax
begin
notation AOT-exists (binder ‹∃ › 8)
end
context AOT-no-meta-syntax
begin
no-notation AOT-exists (binder ‹∃ › 8)
end


syntax (input)
   -AOT-exists-ellipse :: ‹id-position ⇒ id-position ⇒ φ ⇒ φ› (‹∃ -...∃ - -› [1,40])
syntax (output)
   -AOT-exists-ellipse :: ‹id-position ⇒ id-position ⇒ φ ⇒ φ› (‹∃ -...∃ - '(-')› [1,40])
parse-ast-translation‹[(syntax-const ‹-AOT-exists-ellipse›, fn ctx => fn [a,b,c] =>
  Ast.mk-appl (Ast.Constant AOT-exists)
    [Ast.mk-appl (Ast.Constant -abs) [parseEllipseList -AOT-vars ctx [a,b],c]])]›
print-translation‹AOT-syntax-print-translations [
  AOT-preserve-binder-abs-tr'
    const-syntax ‹AOT-exists›
    syntax-const ‹-AOT-exists›
    (syntax-const ‹-AOT-exists-ellipse›,true) const-name ‹AOT-conj›,
  AOT-binder-trans
    @{theory}
    @{binding AOT-exists-binder}
    syntax-const ‹-AOT-exists›
]›



syntax -AOT-DDDOT :: φ (‹...›)
syntax -AOT-DDDOT :: φ (‹. . .›)
parse-translation‹[(syntax-const ‹-AOT-DDDOT›, parseDDOT)]›

print-translation‹AOT-syntax-print-translations
[(const-syntax ‹Pure.all›, fn ctxt => fn [Abs (-, -,
  Const (const-syntax ‹HOL.Trueprop›, -) $
  (Const (const-syntax ‹AOT-model-valid-in›, -) $ Bound 0 $ y))] => let
    val y = (Const (syntax-const ‹-AOT-process-frees›, dummyT) $ y)
    in (Const (syntax-const ‹-AOT-nec-theorem›, dummyT) $ y) end
```

```
| [p as Abs (name, -,
  Const (const-syntax ‹HOL.Trueprop›, -) $
  (Const (const-syntax ‹AOT-model-valid-in›, -) $ w $ y))]
=> (Const (syntax-const ‹-AOT-for-arbitrary›, dummyT) $
    (Const (-bound, dummyT) $ Free (name, dummyT)) $
    (Term.betapply (p, (Const (-bound, dummyT) $ Free (name, dummyT)))))
),

 (const-syntax ‹AOT-model-valid-in›, fn ctxt =>
  fn [w as (Const (-free, -) $ Free (v, -)), y] => let
    val is-world = (case (AOT-ProofData.get ctxt)
        of SOME (Free (w, -)) => Name.clean w = Name.clean v | - => false)
    val y = (Const (syntax-const ‹-AOT-process-frees›, dummyT) $ y)
    in if is-world then y else Const (syntax-const ‹-AOT-valid›, dummyT) $ w $ y end
  | [Const (const-syntax ‹w0›, -), y] => let
    val y = (Const (syntax-const ‹-AOT-process-frees›, dummyT) $ y)
    in case (AOT-ProofData.get ctxt) of SOME (Const (const-name ‹w0›, -)) => y |
          - => Const (syntax-const ‹-AOT-theorem›, dummyT) $ y end
  | [Const (-var, -) $ -, y] => let
    val y = (Const (syntax-const ‹-AOT-process-frees›, dummyT) $ y)
    in Const (syntax-const ‹-AOT-nec-theorem›, dummyT) $ y end
  ),
 (const-syntax ‹AOT-model-axiom›, fn ctxt => fn [trm] =>
    Const (syntax-const ‹-AOT-axiom›, dummyT) $
    (Const (syntax-const ‹-AOT-process-frees›, dummyT) $ trm)),
 (const-syntax ‹AOT-model-act-axiom›, fn ctxt => fn [trm] =>
    Const (syntax-const ‹-AOT-axiom›, dummyT) $
    (Const (syntax-const ‹-AOT-process-frees›, dummyT) $ trm)),
(syntax-const ‹-AOT-process-frees›, fn - =>  fn [t] => let
  fun mapAppls (x as Const (-free, -) $
                    Free (-, Type (-ignore-type, [Type (fun, -)])))
       = (Const (-AOT-raw-appl, dummyT) $ x)
    | mapAppls (x as Const (-free, -) $ Free (-, Type (fun, -)))
       = (Const (-AOT-raw-appl, dummyT) $ x)
    | mapAppls (x as Const (-var, -) $
                    Var (-, Type (-ignore-type, [Type (fun, -)])))
       = (Const (-AOT-raw-appl, dummyT) $ x)
    | mapAppls (x as Const (-var, -) $ Var (-, Type (fun, -)))
       = (Const (-AOT-raw-appl, dummyT) $ x)
    | mapAppls (x $ y) = mapAppls x $ mapAppls y
    | mapAppls (Abs (x,y,z)) = Abs (x,y, mapAppls z)
    | mapAppls x = x
  in mapAppls t end
)
]
›

print-ast-translation‹AOT-syntax-print-ast-translations
let
fun handleTermOfVar x kind name = (
let
val - = case kind of -free => () | -var => () | -bound => () | - => raise Match
in
  case printVarKind name
    of (SingleVariable name) => Ast.Appl [Ast.Constant kind, Ast.Variable name]
    | (Ellipses (s, e)) =>  Ast.Appl [Ast.Constant -AOT-free-var-ellipse,
    Ast.Appl [Ast.Constant kind, Ast.Variable s],
    Ast.Appl [Ast.Constant kind, Ast.Variable e]
      ]
  | Verbatim name => Ast.mk-appl (Ast.Constant -AOT-quoted)
                      [Ast.mk-appl (Ast.Constant -AOT-term-of-var) [x]]
end
)
```

```
fun termOfVar ctxt (Ast.Appl [Ast.Constant -constrain,
      x as Ast.Appl [Ast.Constant kind, Ast.Variable name], -]) = termOfVar ctxt x
  | termOfVar ctxt (x as Ast.Appl [Ast.Constant kind, Ast.Variable name])
      = handleTermOfVar x kind name
  | termOfVar ctxt (x as Ast.Appl [Ast.Constant rep, y]) = (
let
val (restr,-) = Local-Theory.raw-theory-result (fn thy => (
let
val restrs = Symtab.dest (AOT-Restriction.get thy)
val restr = List.find (fn (n,(-,Const (c,t))) => (
  c = rep orelse c = Lexicon.unmark-const rep) | - => false) restrs
in
(restr,thy)
end
)) ctxt
in
  case restr of SOME r => Ast.Appl [Ast.Constant (const-syntax ‹AOT-term-of-var›), y]
  | - => raise Match
end)

in
[(const-syntax ‹AOT-term-of-var›, fn ctxt => fn [x] => termOfVar ctxt x),
(-AOT-raw-appl, fn ctxt => fn t::a::args => let
fun applyTermOfVar (t as Ast.Appl (Ast.Constant const-syntax ‹AOT-term-of-var›::[x]))
    = (case try (termOfVar ctxt) x of SOME y => y | - => t)
  | applyTermOfVar y = (case try (termOfVar ctxt) y of SOME x => x | - => y)
val ts = fold (fn a => fn b => Ast.mk-appl (Ast.Constant syntax-const ‹-AOT-args›)
              [b,applyTermOfVar a]) args (applyTermOfVar a)
in Ast.mk-appl (Ast.Constant syntax-const ‹-AOT-appl›) [t,ts] end)]
end
›
```

context *AOT-meta-syntax*
begin
notation *AOT-denotes* (‹-↓›)
notation *AOT-imp* (**infixl** ‹→› *25*)
notation *AOT-not* (‹¬-› [*50*] *50*)
notation *AOT-box* (‹□-› [*49*] *54*)
notation *AOT-act* (‹𝓐-› [*49*] *54*)
notation *AOT-forall* (**binder** ‹∀ › *8*)
notation *AOT-eq* (**infixl** ‹=› *50*)
notation *AOT-desc* (**binder** ‹ι› *100*)
notation *AOT-lambda* (**binder** ‹λ› *100*)
notation *AOT-lambda0* (‹[λ -]›)
notation *AOT-exe* (‹⦇-,-⦈›)
notation *AOT-model-equiv-def* (**infixl** ‹$\equiv_{df}$› *10*)
notation *AOT-model-id-def* (**infixl** ‹$=_{df}$› *10*)
notation *AOT-term-of-var* (‹⟨-⟩›)
notation *AOT-concrete* (‹**E!**›)
end
context *AOT-no-meta-syntax*
begin
no-notation *AOT-denotes* (‹-↓›)
no-notation *AOT-imp* (**infixl** ‹→› *25*)
no-notation *AOT-not* (‹¬-› [*50*] *50*)
no-notation *AOT-box* (‹□-› [*49*] *54*)
no-notation *AOT-act* (‹𝓐-› [*49*] *54*)
no-notation *AOT-forall* (**binder** ‹∀ › *8*)
no-notation *AOT-eq* (**infixl** ‹=› *50*)
no-notation *AOT-desc* (**binder** ‹ι› *100*)
no-notation *AOT-lambda* (**binder** ‹λ› *100*)
no-notation *AOT-lambda0* (‹[λ -]›)
no-notation *AOT-exe* (‹⦇-,-⦈›)

```
no-notation AOT-model-equiv-def (infixl ‹≡df› 10)
no-notation AOT-model-id-def (infixl ‹=df› 10)
no-notation AOT-term-of-var (‹⟨-⟩›)
no-notation AOT-concrete (‹E!›)
end

bundle AOT-syntax
begin
declare[[show-AOT-syntax=true, show-question-marks=false, eta-contract=false]]
end

bundle AOT-no-syntax
begin
declare[[show-AOT-syntax=false, show-question-marks=true]]
end

parse-translation‹
[(-AOT-restriction, fn ctxt => fn [Const (name,-)] =>
let
val (restr, ctxt) = ctxt |> Local-Theory.raw-theory-result
  (fn thy => (Option.map fst (Symtab.lookup (AOT-Restriction.get thy) name), thy))
val restr = case restr of SOME x => x
  | - => raise Fail (Unknown restricted type: ^ name)
in restr end
)]
›

print-translation‹
AOT-syntax-print-translations
[
  (const-syntax‹AOT-model-equiv-def›, fn ctxt => fn [x,y] =>
    Const (syntax-const‹-AOT-equiv-def›, dummyT) $
    (Const (syntax-const‹-AOT-process-frees›, dummyT) $ x) $
    (Const (syntax-const‹-AOT-process-frees›, dummyT) $ y))
]
›

print-translation‹
AOT-syntax-print-translations [
(const-syntax‹AOT-model-id-def›, fn ctxt =>
  fn [lhs as Abs (lhsName, lhsTy, lhsTrm), rhs as Abs (rhsName, rhsTy, rhsTrm)] =>
    let
      val (name,-) = Name.variant lhsName
        (Syntax-Trans.declare-term-names ctxt rhsTrm
          (Name.build-context (Syntax-Trans.declare-term-names ctxt lhsTrm)));
      val lhs = Term.betapply (lhs, Const (-bound, dummyT) $ Free (name, lhsTy))
      val rhs = Term.betapply (rhs, Const (-bound, dummyT) $ Free (name, rhsTy))
    in
      Const (const-syntax‹AOT-model-id-def›, dummyT) $ lhs $ rhs
    end
  | [Const (const-syntax‹case-prod›, -) $ lhs,
     Const (const-syntax‹case-prod›, -) $ rhs] =>
    Const (const-syntax‹AOT-model-id-def›, dummyT) $ lhs $ rhs
  | [Const (const-syntax‹case-unit›, -) $ lhs,
      Const (const-syntax‹case-unit›, -) $ rhs] =>
    Const (const-syntax‹AOT-model-id-def›, dummyT) $ lhs $ rhs
  | [x, y] =>
       Const (syntax-const‹-AOT-id-def›, dummyT) $
         (Const (syntax-const‹-AOT-process-frees›, dummyT) $ x) $
         (Const (syntax-const‹-AOT-process-frees›, dummyT) $ y)
)]›
```

Special marker for printing propositions as theorems and for pretty-printing AOT terms.

definition *print-as-theorem* :: ‹o ⇒ *bool*› **where**
‹*print-as-theorem* ≡ λ φ . ∀ v . [v ⊨ φ]›
lemma *print-as-theoremI*:
assumes ‹⋀ v . [v ⊨ φ]›
shows ‹*print-as-theorem* φ›
using *assms* **by** (*simp add*: *print-as-theorem-def*)
attribute-setup *print-as-theorem* =
‹*Scan.succeed* (*Thm.rule-attribute* []
(*K* (*fn thm* => *thm RS* @{*thm print-as-theoremI*})))›
Print as theorem.
print-translation‹*AOT-syntax-print-translations* [
(***const-syntax***‹*print-as-theorem*›, *fn ctxt* => *fn* [*x*] =>
(*Const* (***syntax-const***‹*-AOT-process-frees*›, *dummyT*) $ *x*))
]›

definition *print-term* :: ‹'a ⇒ 'a› **where** ‹*print-term* ≡ λ x . x›
syntax *-AOT-print-term* :: ‹τ ⇒ 'a› (‹*AOT'-TERM*[-]›)
translations
-AOT-print-term φ => *CONST print-term* (*-AOT-process-frees* φ)
print-translation‹*AOT-syntax-print-translations* [
(***const-syntax***‹*print-term*›, *fn ctxt* => *fn* [*x*] =>
(*Const* (***syntax-const***‹*-AOT-process-frees*›, *dummyT*) $ *x*))
]›

interpretation *AOT-no-meta-syntax*.

unbundle *AOT-syntax*

5 Abstract Semantics for AOT

specification(*AOT-denotes*)
— Relate object level denoting to meta-denoting. AOT's definitions of denoting will become derivable at each type.
AOT-sem-denotes: ‹[w ⊨ τ↓] = *AOT-model-denotes* τ›
by (*rule exI*[**where** x=‹λ τ . ε_o w . *AOT-model-denotes* τ›])
(*simp add*: *AOT-model-proposition-choice-simp*)

lemma *AOT-sem-var-induct*[*induct type*: *AOT-var*]:
assumes *AOT-denoting-term-case*: ‹⋀ τ . [v ⊨ τ↓] ⟹ [v ⊨ φ{τ}]›
shows ‹[v ⊨ φ{α}]›
by (*simp add*: *AOT-denoting-term-case AOT-sem-denotes AOT-term-of-var*)

specification(*AOT-imp*)
AOT-sem-imp: ‹[w ⊨ φ → ψ] = ([w ⊨ φ] ⟶ [w ⊨ ψ])›
by (*rule exI*[**where** x=‹λ φ ψ . ε_o w . ([w ⊨ φ] ⟶ [w ⊨ ψ])›])
(*simp add*: *AOT-model-proposition-choice-simp*)

specification(*AOT-not*)
AOT-sem-not: ‹[w ⊨ ¬φ] = (¬[w ⊨ φ])›
by (*rule exI*[**where** x=‹λ φ . ε_o w . ¬[w ⊨ φ]›])
(*simp add*: *AOT-model-proposition-choice-simp*)

specification(*AOT-box*)
AOT-sem-box: ‹[w ⊨ □φ] = (∀ w . [w ⊨ φ])›
by (*rule exI*[**where** x=‹λ φ . ε_o w . ∀ w . [w ⊨ φ]›])

(*simp add*: *AOT-model-proposition-choice-simp*)

specification(*AOT-act*)
AOT-sem-act: ‹$[w \models \mathcal{A}\varphi] = [w_0 \models \varphi]$›
by (*rule exI*[**where** x=‹$\lambda\ \varphi\ .\ \varepsilon_o\ w\ .\ [w_0 \models \varphi]$›])
(*simp add*: *AOT-model-proposition-choice-simp*)

Derived semantics for basic defined connectives.

lemma *AOT-sem-conj*: ‹$[w \models \varphi\ \&\ \psi] = ([w \models \varphi] \wedge [w \models \psi])$›
using *AOT-conj AOT-model-equiv-def AOT-sem-imp AOT-sem-not* **by** *auto*
lemma *AOT-sem-equiv*: ‹$[w \models \varphi \equiv \psi] = ([w \models \varphi] = [w \models \psi])$›
using *AOT-equiv AOT-sem-conj AOT-model-equiv-def AOT-sem-imp* **by** *auto*
lemma *AOT-sem-disj*: ‹$[w \models \varphi \vee \psi] = ([w \models \varphi] \vee [w \models \psi])$›
using *AOT-disj AOT-model-equiv-def AOT-sem-imp AOT-sem-not* **by** *auto*
lemma *AOT-sem-dia*: ‹$[w \models \Diamond\varphi] = (\exists\ w\ .\ [w \models \varphi])$›
using *AOT-dia AOT-sem-box AOT-model-equiv-def AOT-sem-not* **by** *auto*

specification(*AOT-forall*)
AOT-sem-forall: ‹$[w \models \forall\alpha\ \varphi\{\alpha\}] = (\forall\ \tau\ .\ [w \models \tau\downarrow] \longrightarrow [w \models \varphi\{\tau\}])$›
by (*rule exI*[**where** x=‹$\lambda\ op\ .\ \varepsilon_o\ w\ .\ \forall\ \tau\ .\ [w \models \tau\downarrow] \longrightarrow [w \models «op\ \tau»]$›])
(*simp add*: *AOT-model-proposition-choice-simp*)

lemma *AOT-sem-exists*: ‹$[w \models \exists\alpha\ \varphi\{\alpha\}] = (\exists\ \tau\ .\ [w \models \tau\downarrow] \wedge [w \models \varphi\{\tau\}])$›
unfolding *AOT-exists*[*unfolded AOT-model-equiv-def*, *THEN spec*]
by (*simp add*: *AOT-sem-forall AOT-sem-not*)

specification(*AOT-eq*)
— Relate identity to denoting identity in the meta-logic. AOT's definitions of identity will become derivable at each type.
AOT-sem-eq: ‹$[w \models \tau = \tau'] = ([w \models \tau\downarrow] \wedge [w \models \tau'\downarrow] \wedge \tau = \tau')$›
by (*rule exI*[**where** x=‹$\lambda\ \tau\ \tau'\ .\ \varepsilon_o\ w\ .\ [w \models \tau\downarrow] \wedge [w \models \tau'\downarrow] \wedge \tau = \tau'$›])
(*simp add*: *AOT-model-proposition-choice-simp*)

specification(*AOT-desc*)
— Descriptions denote, if there is a unique denoting object satisfying the matrix in the actual world.
AOT-sem-desc-denotes: ‹$[w \models \iota x(\varphi\{x\})\downarrow] = (\exists!\ \kappa\ .\ [w_0 \models \kappa\downarrow] \wedge [w_0 \models \varphi\{\kappa\}])$›
— Denoting descriptions satisfy their matrix in the actual world.
AOT-sem-desc-prop: ‹$[w \models \iota x(\varphi\{x\})\downarrow] \Longrightarrow [w_0 \models \varphi\{\iota x(\varphi\{x\})\}]$›
— Uniqueness of denoting descriptions.
AOT-sem-desc-unique: ‹$[w \models \iota x(\varphi\{x\})\downarrow] \Longrightarrow [w \models \kappa\downarrow] \Longrightarrow [w_0 \models \varphi\{\kappa\}] \Longrightarrow$
$[w \models \iota x(\varphi\{x\}) = \kappa]$›
proof –
have ‹$\exists\ x$::$'a$. $\neg$*AOT-model-denotes* x›
using *AOT-model-nondenoting-ex*
by *blast*

Note that we may choose a distinct non-denoting object for each matrix. We do this explicitly merely to convince ourselves that our specification can still be satisfied.

then obtain *nondenoting* :: ‹$('a \Rightarrow o) \Rightarrow 'a$› **where**
nondenoting: ‹$\forall\ \varphi$. $\neg$*AOT-model-denotes* (*nondenoting* φ)›
by *fast*
define *desc* **where**
‹*desc* = ($\lambda\ \varphi$. *if* $(\exists!\ \kappa\ .\ [w_0 \models \kappa\downarrow] \wedge [w_0 \models \varphi\{\kappa\}])$
then (*THE* $\kappa\ .\ [w_0 \models \kappa\downarrow] \wedge [w_0 \models \varphi\{\kappa\}]$)
else nondenoting φ)›
{
fix φ :: ‹$'a \Rightarrow o$›
assume *ex1*: ‹$\exists!\ \kappa\ .\ [w_0 \models \kappa\downarrow] \wedge [w_0 \models \varphi\{\kappa\}]$›
then obtain κ **where** *x-prop*: $[w_0 \models \kappa\downarrow] \wedge [w_0 \models \varphi\{\kappa\}]$
unfolding *AOT-sem-denotes* **by** *blast*
moreover have (*desc* φ) = κ
unfolding *desc-def* **using** *x-prop ex1* **by** *fastforce*
ultimately have $[w_0 \models «desc\ \varphi»\downarrow] \wedge [w_0 \models «\varphi\ (desc\ \varphi)»]$

    by *blast*
  } **note** *1* = *this*
  moreover {
    fix φ :: ‹$'a \Rightarrow$ o›
    assume *nex1*: ‹$\nexists!\ \kappa\ .\ [w_0 \models \kappa\downarrow] \wedge [w_0 \models \varphi\{\kappa\}]$›
    hence (*desc* φ) = *nondenoting* φ **by** (*simp add*: *desc-def AOT-sem-denotes*)
    hence $[w \models \neg$«*desc* φ»$\downarrow]$ **for** w
      by (*simp add*: *AOT-sem-denotes nondenoting AOT-sem-not*)
  }
  ultimately have *desc-denotes-simp*:
    ‹$[w \models$ «*desc* φ»$\downarrow] = (\exists!\ \kappa\ .\ [w_0 \models \kappa\downarrow] \wedge [w_0 \models \varphi\{\kappa\}])$› **for** φ w
    by (*simp add*: *AOT-sem-denotes desc-def nondenoting*)
  have ‹$(\forall \varphi\ w.\ [w \models$ «*desc* φ»$\downarrow] = (\exists!\kappa.\ [w_0 \models \kappa\downarrow] \wedge [w_0 \models \varphi\{\kappa\}])) \wedge$
    $(\forall \varphi\ w.\ [w \models$ «*desc* φ»$\downarrow] \longrightarrow [w_0 \models$ «φ (*desc* φ)»$]) \wedge$
    $(\forall \varphi\ w\ \kappa.\ [w \models$ «*desc* φ»$\downarrow] \longrightarrow [w \models \kappa\downarrow] \longrightarrow [w_0 \models \varphi\{\kappa\}] \longrightarrow$
          $[w \models$ «*desc* φ» $= \kappa])$›
    by (*insert 1*; *auto simp*: *desc-denotes-simp AOT-sem-eq AOT-sem-denotes*
                    desc-def nondenoting)
  thus *?thesis*
    by (*safe intro*!: *exI*[**where** *x*=*desc*]; *presburger*)
qed

specification(*AOT-exe AOT-lambda*)
  — Truth conditions of exemplification formulas.
  AOT-sem-exe: ‹$[w \models [\Pi]\kappa_1...\kappa_n] = ([w \models \Pi\downarrow] \wedge [w \models \kappa_1...\kappa_n\downarrow] \wedge$
                      $[w \models$ «*Rep-rel* $\Pi\ \kappa_1\kappa_n$»$])$›
  — η-conversion for denoting terms; equivalent to AOT's axiom
  AOT-sem-lambda-eta: ‹$[w \models \Pi\downarrow] \Longrightarrow [w \models [\lambda\nu_1...\nu_n\ [\Pi]\nu_1...\nu_n] = \Pi]$›
  — β-conversion; equivalent to AOT's axiom
  AOT-sem-lambda-beta: ‹$[w \models [\lambda\nu_1...\nu_n\ \varphi\{\nu_1...\nu_n\}]\downarrow] \Longrightarrow [w \models \kappa_1...\kappa_n\downarrow] \Longrightarrow$
                  $[w \models [\lambda\nu_1...\nu_n\ \varphi\{\nu_1...\nu_n\}]\kappa_1...\kappa_n] = [w \models \varphi\{\kappa_1...\kappa_n\}]$›
  — Necessary and sufficient conditions for relations to denote. Equivalent to a theorem of AOT and used to derive the base cases of denoting relations (cqt.2).
  AOT-sem-lambda-denotes: ‹$[w \models [\lambda\nu_1...\nu_n\ \varphi\{\nu_1...\nu_n\}]\downarrow] =$
    $(\forall\ \ v\ \kappa_1\kappa_n\ \kappa_1{}'\kappa_n{}'\ .\ [v \models \kappa_1...\kappa_n\downarrow] \wedge [v \models \kappa_1{}'...\kappa_n{}'\downarrow] \wedge$
      $(\forall\ \ \Pi\ v\ .\ [v \models \Pi\downarrow] \longrightarrow [v \models [\Pi]\kappa_1...\kappa_n] = [v \models [\Pi]\kappa_1{}'...\kappa_n{}']) \longrightarrow$
          $[v \models \varphi\{\kappa_1...\kappa_n\}] = [v \models \varphi\{\kappa_1{}'...\kappa_n{}'\}])$›
  — Equivalent to AOT's coexistence axiom.
  AOT-sem-lambda-coex: ‹$[w \models [\lambda\nu_1...\nu_n\ \varphi\{\nu_1...\nu_n\}]\downarrow] \Longrightarrow$
    $(\forall\ \ w\ \kappa_1\kappa_n\ .\ [w \models \kappa_1...\kappa_n\downarrow] \longrightarrow [w \models \varphi\{\kappa_1...\kappa_n\}] = [w \models \psi\{\kappa_1...\kappa_n\}]) \Longrightarrow$
    $[w \models [\lambda\nu_1...\nu_n\ \psi\{\nu_1...\nu_n\}]\downarrow]$›
  — Only the unary case of the following should hold, but our specification has to range over all types. We might move *AOT-exe* and *AOT-lambda* to type classes in the future to solve this.
  AOT-sem-lambda-eq-prop-eq: ‹«$[\lambda\nu_1...\nu_n\ \varphi]$» = «$[\lambda\nu_1...\nu_n\ \psi]$» $\Longrightarrow \varphi = \psi$›
  — The following is solely required for validating n-ary relation identity and has the danger of implying artifactual theorems. Possibly avoidable by moving *AOT-exe* and *AOT-lambda* to type classes.
  AOT-sem-exe-denoting: ‹$[w \models \Pi\downarrow] \Longrightarrow$ *AOT-exe* $\Pi\ \kappa s$ = *Rep-rel* $\Pi\ \kappa s$›
  — The following is required for validating the base cases of denoting relations (cqt.2). A version of this meta-logical identity will become derivable in future versions of AOT, so this will ultimately not result in artifactual theorems.
  AOT-sem-exe-equiv: ‹*AOT-model-term-equiv* x y $\Longrightarrow$ *AOT-exe* Π x = *AOT-exe* Π y›
proof −
  have ‹$\exists\ \ x$ :: <$'a$> . ¬*AOT-model-denotes* x›
    by (*rule exI*[**where** *x*=‹*Abs-rel* ($\lambda\ x\ .\ \varepsilon_o\ w.\ True$)›])
      (*meson AOT-model-denotes-rel.abs-eq AOT-model-nondenoting-ex*
          AOT-model-proposition-choice-simp)
  define *exe* :: ‹<$'a$> $\Rightarrow$ $'a \Rightarrow$ o› **where**
    ‹*exe* $\equiv \lambda\ \Pi\ \kappa s$. *if AOT-model-denotes* Π
                    then Rep-rel $\Pi\ \kappa s$
                    else (ε_o w . *False*)›
  define *lambda* :: ‹($'a$⇒o) ⇒ <$'a$>› **where**
    ‹*lambda* $\equiv \lambda\ \varphi$. *if AOT-model-denotes* (*Abs-rel* φ)
      then (*Abs-rel* φ)
      else

```
    if (∀ κ κ' w . (AOT-model-denotes κ ∧ AOT-model-term-equiv κ κ') ⟶
              [w ⊨ «φ κ»] = [w ⊨ «φ κ'»])
    then
      Abs-rel (fix-irregular (λ x . if AOT-model-denotes x
                                    then φ (SOME y . AOT-model-term-equiv x y)
                                    else (εo w . False)))
    else
      Abs-rel φ›
have fix-irregular-denoting-simp[simp]:
  ‹fix-irregular (λx. if AOT-model-denotes x then φ x else ψ x) κ = φ κ›
  if ‹AOT-model-denotes κ›
  for κ :: 'a and φ ψ
  by (simp add: that fix-irregular-denoting)
have denoting-eps-cong[cong]:
  ‹[w ⊨ «φ (Eps (AOT-model-term-equiv κ))»] = [w ⊨ «φ κ»]›
  if ‹AOT-model-denotes κ›
  and ‹∀ κ κ'. AOT-model-denotes κ ∧ AOT-model-term-equiv κ κ' ⟶
          (∀ w. [w ⊨ «φ κ»] = [w ⊨ «φ κ'»])›
  for w :: w and κ :: 'a and φ :: ‹'a⇒o›
  using that AOT-model-term-equiv-eps(2) by blast
have exe-rep-rel: ‹[w ⊨ «exe Π κ1κn»] = ([w ⊨ Π↓] ∧ [w ⊨ κ1...κn↓] ∧
                                [w ⊨ «Rep-rel Π κ1κn»])› for w Π κ1κn
  by (metis AOT-model-denotes-rel.rep-eq exe-def AOT-sem-denotes
        AOT-model-proposition-choice-simp)
moreover have ‹[w ⊨ «Π»↓] ⟹ [w ⊨ «lambda (exe Π)» = «Π»]› for Π w
  by (auto simp: Rep-rel-inverse lambda-def AOT-sem-denotes AOT-sem-eq
           AOT-model-denotes-rel-def Abs-rel-inverse exe-def)
moreover have lambda-denotes-beta:
  ‹[w ⊨ «exe (lambda φ) κ »] = [w ⊨ «φ κ»]›
  if ‹[w ⊨ «lambda φ»↓]› and ‹[w ⊨ «κ»↓]›
  for φ κ w
  using that unfolding exe-def AOT-sem-denotes
  by (auto simp: lambda-def Abs-rel-inverse split: if-split-asm)
moreover have lambda-denotes-simp:
  ‹[w ⊨ «lambda φ»↓] = (∀ v κ1κn κ1'κn' . [v ⊨ κ1...κn↓] ∧ [v ⊨ κ1'...κn'↓] ∧
     (∀ Π v . [v ⊨ Π↓] ⟶ [v ⊨ «exe Π κ1κn»] = [v ⊨ «exe Π κ1'κn'»]) ⟶
     [v ⊨ φ{κ1...κn}] = [v ⊨ φ{κ1'...κn'}])› for φ w
proof
  assume ‹[w ⊨ «lambda φ»↓]›
  hence ‹AOT-model-denotes (lambda φ)›
    unfolding AOT-sem-denotes by simp
  moreover have ‹[w ⊨ «φ κ»] ⟹ [w ⊨ «φ κ'»]›
    and ‹[w ⊨ «φ κ'»] ⟹ [w ⊨ «φ κ»]›
    if ‹AOT-model-denotes κ› and ‹AOT-model-term-equiv κ κ'›
    for w κ κ'
    by (metis (no-types, lifting) AOT-model-denotes-rel.abs-eq lambda-def
                   that calculation)+
  ultimately show ‹∀ v κ1κn κ1'κn' . [v ⊨ κ1...κn↓] ∧ [v ⊨ κ1'...κn'↓] ∧
     (∀ Π v . [v ⊨ Π↓] ⟶ [v ⊨ «exe Π κ1κn»] = [v ⊨ «exe Π κ1'κn'»]) ⟶
     [v ⊨ φ{κ1...κn}] = [v ⊨ φ{κ1'...κn'}]›
    unfolding AOT-sem-denotes
    by (metis (no-types) AOT-sem-denotes lambda-denotes-beta)
next
  assume ‹∀ v κ1κn κ1'κn' . [v ⊨ κ1...κn↓] ∧ [v ⊨ κ1'...κn'↓] ∧
    (∀ Π v . [v ⊨ Π↓] ⟶ [v ⊨ «exe Π κ1κn»] = [v ⊨ «exe Π κ1'κn'»]) ⟶
    [v ⊨ φ{κ1...κn}] = [v ⊨ φ{κ1'...κn'}]›
  hence ‹[w ⊨ «φ κ»] = [w ⊨ «φ κ'»]›
    if ‹AOT-model-denotes κ ∧ AOT-model-denotes κ' ∧ AOT-model-term-equiv κ κ'›
    for w κ κ'
    using that
    by (auto simp: AOT-sem-denotes)
       (meson AOT-model-term-equiv-rel-equiv AOT-sem-denotes exe-rep-rel)+
  hence ‹[w ⊨ «φ κ»] = [w ⊨ «φ κ'»]›
```

```
      if ‹AOT-model-denotes κ ∧ AOT-model-term-equiv κ κ'›
      for w κ κ'
      using that AOT-model-term-equiv-denotes by blast
    hence ‹AOT-model-denotes (lambda φ)›
      by (auto simp: lambda-def Abs-rel-inverse AOT-model-denotes-rel.abs-eq
                       AOT-model-irregular-equiv AOT-model-term-equiv-eps(3)
                       AOT-model-term-equiv-regular fix-irregular-def AOT-sem-denotes
                       AOT-model-term-equiv-denotes AOT-model-proposition-choice-simp
                       AOT-model-irregular-false
               split: if-split-asm
               intro: AOT-model-irregular-eqI)
    thus ‹[w ⊨ «lambda φ»↓]›
      by (simp add: AOT-sem-denotes)
  qed
  moreover have ‹[w ⊨ «lambda ψ»↓]›
    if ‹[w ⊨ «lambda φ»↓]›
    and ‹∀ w κ1κn . [w ⊨ κ1...κn↓] ⟶ [w ⊨ φ{κ1...κn}] = [w ⊨ ψ{κ1...κn}]›
    for φ ψ w using that unfolding lambda-denotes-simp by auto
  moreover have ‹[w ⊨ Π↓] ⟹ exe Π κs = Rep-rel Π κs› for Π κs w
    by (simp add: exe-def AOT-sem-denotes)
  moreover have ‹lambda (λx. p) = lambda (λx. q) ⟹ p = q› for p q
    unfolding lambda-def
    by (auto split: if-split-asm simp: Abs-rel-inject fix-irregular-def)
      (meson AOT-model-irregular-nondenoting AOT-model-denoting-ex)+
  moreover have ‹AOT-model-term-equiv x y ⟹ exe Π x = exe Π y› for x y Π
    unfolding exe-def
    by (meson AOT-model-denotes-rel.rep-eq)
  note calculation = calculation this
  show ?thesis
    apply (safe intro!: exI[where x=exe] exI[where x=lambda])
    using calculation apply simp-all
    using lambda-denotes-simp by blast+
qed

lemma AOT-model-lambda-denotes:
  ‹AOT-model-denotes (AOT-lambda φ) = (∀ v κ κ' .
    AOT-model-denotes κ ∧ AOT-model-denotes κ' ∧ AOT-model-term-equiv κ κ' ⟶
    [v ⊨ «φ κ»] = [v ⊨ «φ κ'»])›
proof(safe)
  fix v and κ κ' :: 'a
  assume ‹AOT-model-denotes (AOT-lambda φ)›
  hence 1: ‹AOT-model-denotes κ1κn ∧
        AOT-model-denotes κ1'κn' ∧
        (∀Π v. AOT-model-denotes Π ⟶ [v ⊨ [Π]κ1...κn] = [v ⊨ [Π]κ1'...κn']) ⟶
        [v ⊨ φ{κ1...κn}] = [v ⊨ φ{κ1'...κn'}]› for κ1κn κ1'κn' v
    using AOT-sem-lambda-denotes[simplified AOT-sem-denotes] by blast
  {
    fix v and κ1κn κ1'κn' :: 'a
    assume d: ‹AOT-model-denotes κ1κn ∧ AOT-model-denotes κ1'κn' ∧
             AOT-model-term-equiv κ1κn κ1'κn'›
    hence ‹∀Π w. AOT-model-denotes Π ⟶ [w ⊨ [Π]κ1...κn] = [w ⊨ [Π]κ1'...κn']›
      by (metis AOT-sem-exe-equiv)
    hence ‹[v ⊨ φ{κ1...κn}] = [v ⊨ φ{κ1'...κn'}]› using d 1 by auto
  }
  moreover assume ‹AOT-model-denotes κ›
  moreover assume ‹AOT-model-denotes κ'›
  moreover assume ‹AOT-model-term-equiv κ κ'›
  ultimately show ‹[v ⊨ «φ κ»] ⟹ [v ⊨ «φ κ'»]›
            and ‹[v ⊨ «φ κ'»] ⟹ [v ⊨ «φ κ»]›
    by auto
next
  assume 0: ‹∀ v κ κ' . AOT-model-denotes κ ∧ AOT-model-denotes κ' ∧
                     AOT-model-term-equiv κ κ' ⟶ [v ⊨ «φ κ»] = [v ⊨ «φ κ'»]›
```

{
fix $\kappa_1\kappa_n$ $\kappa_1'\kappa_n'$ :: *'a*
assume *den*: ‹*AOT-model-denotes* $\kappa_1\kappa_n$›
moreover assume *den'*: ‹*AOT-model-denotes* $\kappa_1'\kappa_n'$›
assume ‹$\forall\,\Pi\ v$. *AOT-model-denotes* $\Pi \longrightarrow$
$[v \models [\Pi]\kappa_1...\kappa_n] = [v \models [\Pi]\kappa_1'...\kappa_n']$›
hence ‹$\forall\,\Pi\ v$. *AOT-model-denotes* $\Pi \longrightarrow$
$[v \models$ «*Rep-rel* Π $\kappa_1\kappa_n$»$] = [v \models$ «*Rep-rel* Π $\kappa_1'\kappa_n'$»$]$›
by (*simp add*: *AOT-sem-denotes AOT-sem-exe den den'*)
hence *AOT-model-term-equiv* $\kappa_1\kappa_n$ $\kappa_1'\kappa_n'$
unfolding *AOT-model-term-equiv-rel-equiv*[*OF den*, *OF den'*]
by *argo*
hence ‹$[v \models \varphi\{\kappa_1...\kappa_n\}] = [v \models \varphi\{\kappa_1'...\kappa_n'\}]$› **for** *v*
using *den den' 0* **by** *blast*
}
thus ‹*AOT-model-denotes* (*AOT-lambda* φ)›
unfolding *AOT-sem-lambda-denotes*[*simplified AOT-sem-denotes*]
by *blast*
qed

specification (*AOT-lambda0*)
AOT-sem-lambda0: *AOT-lambda0* $\varphi = \varphi$
by (*rule exI*[**where** *x*=‹$\lambda x.\ x$›]) *simp*

specification(*AOT-concrete*)
AOT-sem-concrete: ‹$[w \models [E!]\kappa]$ =
AOT-model-concrete $w\ \kappa$›
by (*rule exI*[**where** *x*=‹*AOT-var-of-term* (*Abs-rel*
($\lambda\ x$. $\varepsilon_\circ$ w . *AOT-model-concrete* $w\ x$))›];
subst AOT-var-of-term-inverse)
(*auto simp*: *AOT-model-unary-regular AOT-model-concrete-denotes*
AOT-model-concrete-equiv AOT-model-regular-κ-def
AOT-model-proposition-choice-simp AOT-sem-exe Abs-rel-inverse
AOT-model-denotes-rel-def AOT-sem-denotes)

lemma *AOT-sem-equiv-defI*:
assumes ‹$\bigwedge\ v$. $[v \models \varphi] \Longrightarrow [v \models \psi]$›
and ‹$\bigwedge\ v$. $[v \models \psi] \Longrightarrow [v \models \varphi]$›
shows ‹*AOT-model-equiv-def* $\varphi\ \psi$›
using *AOT-model-equiv-def assms* **by** *blast*

lemma *AOT-sem-id-defI*:
assumes ‹$\bigwedge\ \alpha\ v$. $[v \models$ «$\sigma\ \alpha$»$\downarrow] \Longrightarrow [v \models$ «$\tau\ \alpha$» = «$\sigma\ \alpha$»$]$›
assumes ‹$\bigwedge\ \alpha\ v$. $\neg[v \models$ «$\sigma\ \alpha$»$\downarrow] \Longrightarrow [v \models \neg$«$\tau\ \alpha$»$\downarrow]$›
shows ‹*AOT-model-id-def* $\tau\ \sigma$›
using *assms*
unfolding *AOT-sem-denotes AOT-sem-eq AOT-sem-not*
using *AOT-model-id-def*[*THEN iffD2*]
by *metis*

lemma *AOT-sem-id-def2I*:
assumes ‹$\bigwedge\ \alpha\ \beta\ v$. $[v \models$ «$\sigma\ \alpha\ \beta$»$\downarrow] \Longrightarrow [v \models$ «$\tau\ \alpha\ \beta$» = «$\sigma\ \alpha\ \beta$»$]$›
assumes ‹$\bigwedge\ \alpha\ \beta\ v$. $\neg[v \models$ «$\sigma\ \alpha\ \beta$»$\downarrow] \Longrightarrow [v \models \neg$«$\tau\ \alpha\ \beta$»$\downarrow]$›
shows ‹*AOT-model-id-def* (*case-prod* τ) (*case-prod* σ)›
apply (*rule AOT-sem-id-defI*)
using *assms* **by** (*auto simp*: *AOT-sem-conj AOT-sem-not AOT-sem-eq AOT-sem-denotes*
AOT-model-denotes-prod-def)

lemma *AOT-sem-id-defE1*:
assumes ‹*AOT-model-id-def* $\tau\ \sigma$›
and ‹$[v \models$ «$\sigma\ \alpha$»$\downarrow]$›
shows ‹$[v \models$ «$\tau\ \alpha$» = «$\sigma\ \alpha$»$]$›
using *assms* **by** (*simp add*: *AOT-model-id-def AOT-sem-denotes AOT-sem-eq*)

lemma *AOT-sem-id-defE2*:
assumes ‹*AOT-model-id-def* τ σ›
and ‹¬[v ⊨ «σ α»↓]›
shows ‹¬[v ⊨ «τ α»↓]›
using *assms* **by** (*simp add*: *AOT-model-id-def AOT-sem-denotes AOT-sem-eq*)

lemma *AOT-sem-id-def0I*:
assumes ‹⋀ v . [v ⊨ σ↓] ⟹ [v ⊨ τ = σ]›
and ‹⋀ v . ¬[v ⊨ σ↓] ⟹ [v ⊨ ¬τ↓]›
shows ‹*AOT-model-id-def* (*case-unit* τ) (*case-unit* σ)›
apply (*rule AOT-sem-id-defI*)
using *assms*
by (*simp-all add*: *AOT-sem-conj AOT-sem-eq AOT-sem-not AOT-sem-denotes*
AOT-model-denotes-unit-def case-unit-Unity)

lemma *AOT-sem-id-def0E1*:
assumes ‹*AOT-model-id-def* (*case-unit* τ) (*case-unit* σ)›
and ‹[v ⊨ σ↓]›
shows ‹[v ⊨ τ = σ]›
by (*metis* (*full-types*) *AOT-sem-id-defE1 assms*(*1*) *assms*(*2*) *case-unit-Unity*)

lemma *AOT-sem-id-def0E2*:
assumes ‹*AOT-model-id-def* (*case-unit* τ) (*case-unit* σ)›
and ‹¬[v ⊨ σ↓]›
shows ‹¬[v ⊨ τ↓]›
by (*metis AOT-sem-id-defE2 assms*(*1*) *assms*(*2*) *case-unit-Unity*)

lemma *AOT-sem-id-def0E3*:
assumes ‹*AOT-model-id-def* (*case-unit* τ) (*case-unit* σ)›
and ‹[v ⊨ σ↓]›
shows ‹[v ⊨ τ↓]›
using *AOT-sem-id-def0E1*[*OF assms*]
by (*simp add*: *AOT-sem-eq AOT-sem-denotes*)

lemma *AOT-sem-ordinary-def-denotes*: ‹[w ⊨ [λx ◇[E!]x]↓]›
unfolding *AOT-sem-denotes AOT-model-lambda-denotes*
by (*auto simp*: *AOT-sem-dia AOT-model-concrete-equiv*
AOT-sem-concrete AOT-sem-denotes)
lemma *AOT-sem-abstract-def-denotes*: ‹[w ⊨ [λx ¬◇[E!]x]↓]›
unfolding *AOT-sem-denotes AOT-model-lambda-denotes*
by (*auto simp*: *AOT-sem-dia AOT-model-concrete-equiv*
AOT-sem-concrete AOT-sem-denotes AOT-sem-not)

Relation identity is constructed using an auxiliary abstract projection mechanism with suitable instantiations for κ and products.

class *AOT-RelationProjection* =
fixes *AOT-sem-proj-id* :: ‹'a::*AOT-IndividualTerm* ⇒ ('a ⇒ o) ⇒ ('a ⇒ o) ⇒ o›
assumes *AOT-sem-proj-id-prop*:
‹[v ⊨ Π = Π'] =
[v ⊨ Π↓ & Π'↓ & ∀α («*AOT-sem-proj-id* α (λ τ . «[Π]τ») (λ τ . «[Π']τ»)»)]›
and *AOT-sem-proj-id-refl*:
‹[v ⊨ τ↓] ⟹ [v ⊨ [λν_1...ν_n φ{ν_1...ν_n}] = [λν_1...ν_n φ{ν_1...ν_n}]] ⟹
[v ⊨ «*AOT-sem-proj-id* τ φ φ»]›

class *AOT-UnaryRelationProjection* = *AOT-RelationProjection* +
assumes *AOT-sem-unary-proj-id*:
‹*AOT-sem-proj-id* κ φ ψ = «[λν_1...ν_n φ{ν_1...ν_n}] = [λν_1...ν_n ψ{ν_1...ν_n}]»›

instantiation κ :: *AOT-UnaryRelationProjection*
begin
definition *AOT-sem-proj-id-κ* :: ‹κ ⇒ (κ ⇒ o) ⇒ (κ ⇒ o) ⇒ o› **where**
‹*AOT-sem-proj-id-κ* κ φ ψ = «[λz φ{z}] = [λz ψ{z}]»›

```
instance proof
  show ‹[v ⊨ Π = Π'] =
        [v ⊨ Π↓ & Π'↓ & ∀α («AOT-sem-proj-id α (λ τ . «[Π]τ») (λ τ . «[Π']τ»)»)]›
    for v and Π Π' :: ‹<κ>›
    unfolding AOT-sem-proj-id-κ-def
    by (simp add: AOT-sem-eq AOT-sem-conj AOT-sem-denotes AOT-sem-forall)
       (metis AOT-sem-denotes AOT-model-denoting-ex AOT-sem-eq AOT-sem-lambda-eta)
next
  show ‹AOT-sem-proj-id κ φ ψ = «[λν1...νn φ{ν1...νn}] = [λν1...νn ψ{ν1...νn}]»›
    for κ :: κ and φ ψ
    unfolding AOT-sem-proj-id-κ-def ..
next
  show ‹[v ⊨ «AOT-sem-proj-id τ φ φ»]›
    if ‹[v ⊨ τ↓]› and ‹[v ⊨ [λν1...νn φ{ν1...νn}] = [λν1...νn φ{ν1...νn}]]›
    for τ :: κ and v φ
    using that by (simp add: AOT-sem-proj-id-κ-def AOT-sem-eq)
qed
end

instantiation prod ::
  ({AOT-UnaryRelationProjection, AOT-UnaryIndividualTerm}, AOT-RelationProjection)
  AOT-RelationProjection
begin
definition AOT-sem-proj-id-prod :: ‹'a×'b ⇒ ('a×'b ⇒ o) ⇒ ('a×'b ⇒ o) ⇒ o› where
  ‹AOT-sem-proj-id-prod ≡ λ (x,y) φ ψ . «[λz «φ (z,y)»] = [λz «ψ (z,y)»] &
    «AOT-sem-proj-id y (λ a . φ (x,a)) (λ a . ψ (x,a))»»›
instance proof
```

This is the main proof that allows to derive the definition of n-ary relation identity. We need to show that our defined projection identity implies relation identity for relations on pairs of individual terms.

```
  fix v and Π Π' :: ‹<'a×'b>›
  have AOT-meta-proj-denotes1: ‹AOT-model-denotes (Abs-rel (λz. AOT-exe Π (z, β)))›
    if ‹AOT-model-denotes Π› for Π :: ‹<'a×'b>› and β
    using that unfolding AOT-model-denotes-rel.rep-eq
    apply (simp add: Abs-rel-inverse AOT-meta-prod-equivI(2) AOT-sem-denotes
                  that)
    by (metis (no-types, lifting) AOT-meta-prod-equivI(2) AOT-model-denotes-prod-def
             AOT-model-unary-regular AOT-sem-exe AOT-sem-exe-equiv case-prodD)
  {
    fix κ :: 'a and Π :: ‹<'a×'b>›
    assume Π-denotes: ‹AOT-model-denotes Π›
    assume α-denotes: ‹AOT-model-denotes κ›
    hence ‹AOT-exe Π (κ, x) = AOT-exe Π (κ, y)›
      if ‹AOT-model-term-equiv x y› for x y :: 'b
      by (simp add: AOT-meta-prod-equivI(1) AOT-sem-exe-equiv that)
    moreover have ‹AOT-model-denotes κ1'κn'›
            if ‹[w ⊨ [Π]κ κ1'...κn']› for w κ1'κn'
      by (metis that AOT-model-denotes-prod-def AOT-sem-exe
              AOT-sem-denotes case-prodD)
    moreover {
      fix x :: 'b
      assume x-irregular: ‹¬AOT-model-regular x›
      hence prod-irregular: ‹¬AOT-model-regular (κ, x)›
        by (metis (no-types, lifting) AOT-model-irregular-nondenoting
                                   AOT-model-regular-prod-def case-prodD)
      hence ‹(¬AOT-model-denotes κ ∨ ¬AOT-model-regular x) ∧
            (AOT-model-denotes κ ∨ ¬AOT-model-denotes x)›
        unfolding AOT-model-regular-prod-def by blast
      hence x-nonden: ‹¬AOT-model-regular x›
        by (simp add: α-denotes)
      have ‹Rep-rel Π (κ, x) = AOT-model-irregular (Rep-rel Π) (κ, x)›
        using AOT-model-denotes-rel.rep-eq Π-denotes prod-irregular by blast
      moreover have ‹AOT-model-irregular (Rep-rel Π) (κ, x) =
```

AOT-model-irregular (λz. *Rep-rel* Π (κ, z)) x›
using Π-*denotes* *x-irregular prod-irregular x-nonden*
using *AOT-model-irregular-prod-generic*
apply (*induct arbitrary*: Π x *rule*: *AOT-model-irregular-prod.induct*)
by (*auto simp*: α-*denotes AOT-model-irregular-nondenoting*
AOT-model-regular-prod-def AOT-meta-prod-equivI(*2*)
AOT-model-denotes-rel.rep-eq AOT-model-term-equiv-eps(*1*)
intro!: *AOT-model-irregular-eqI*)
ultimately have
‹*AOT-exe* Π (κ, x) = *AOT-model-irregular* (λz. *AOT-exe* Π (κ, z)) x›
unfolding *AOT-sem-exe-denoting*[*simplified AOT-sem-denotes*, *OF* Π-*denotes*]
by *auto*
}
ultimately have ‹*AOT-model-denotes* (*Abs-rel* (λz. *AOT-exe* Π (κ, z)))›
by (*simp add*: *Abs-rel-inverse AOT-model-denotes-rel.rep-eq*)
} **note** *AOT-meta-proj-denotes2* = *this*
{
fix $\kappa_1{'}\kappa_n{'}$ :: *'b* **and** Π :: ‹<'a×'b>›
assume Π-*denotes*: ‹*AOT-model-denotes* Π›
assume β-*denotes*: ‹*AOT-model-denotes* $\kappa_1{'}\kappa_n{'}$›
hence ‹*AOT-exe* Π (x, $\kappa_1{'}\kappa_n{'}$) = *AOT-exe* Π (y, $\kappa_1{'}\kappa_n{'}$)›
if ‹*AOT-model-term-equiv* x y› **for** x y :: *'a*
by (*simp add*: *AOT-meta-prod-equivI*(*2*) *AOT-sem-exe-equiv that*)
moreover have ‹*AOT-model-denotes* κ›
if ‹[w ⊨ [Π]κ $\kappa_1{'}...\kappa_n{'}$]› **for** w κ
by (*metis that AOT-model-denotes-prod-def AOT-sem-exe*
AOT-sem-denotes case-prodD)
moreover {
fix x :: *'a*
assume ‹¬*AOT-model-regular* x›
hence ‹*False*›
using *AOT-model-unary-regular* **by** *blast*
hence
‹*AOT-exe* Π (x,$\kappa_1{'}\kappa_n{'}$) = *AOT-model-irregular* (λz. *AOT-exe* Π (z,$\kappa_1{'}\kappa_n{'}$)) x›
unfolding *AOT-sem-exe-denoting*[*simplified AOT-sem-denotes*, *OF* Π-*denotes*]
by *auto*
}
ultimately have ‹*AOT-model-denotes* (*Abs-rel* (λz. *AOT-exe* Π (z,$\kappa_1{'}\kappa_n{'}$)))›
by (*simp add*: *Abs-rel-inverse AOT-model-denotes-rel.rep-eq*)
} **note** *AOT-meta-proj-denotes1* = *this*
{
assume Π-*denotes*: ‹*AOT-model-denotes* Π›
assume Π'-*denotes*: ‹*AOT-model-denotes* Π'›
have Π-*proj2-den*: ‹*AOT-model-denotes* (*Abs-rel* (λz. *Rep-rel* Π (α, z)))›
if ‹*AOT-model-denotes* α› **for** α
using *that AOT-meta-proj-denotes2*[*OF* Π-*denotes*]
AOT-sem-exe-denoting[*simplified AOT-sem-denotes*,*OF* Π-*denotes*] **by** *simp*
have Π'-*proj2-den*: ‹*AOT-model-denotes* (*Abs-rel* (λz. *Rep-rel* Π' (α, z)))›
if ‹*AOT-model-denotes* α› **for** α
using *that AOT-meta-proj-denotes2*[*OF* Π'-*denotes*]
AOT-sem-exe-denoting[*simplified AOT-sem-denotes*,*OF* Π'-*denotes*] **by** *simp*
have Π-*proj1-den*: ‹*AOT-model-denotes* (*Abs-rel* (λz. *Rep-rel* Π (z, α)))›
if ‹*AOT-model-denotes* α› **for** α
using *that AOT-meta-proj-denotes1*[*OF* Π-*denotes*]
AOT-sem-exe-denoting[*simplified AOT-sem-denotes*,*OF* Π-*denotes*] **by** *simp*
have Π'-*proj1-den*: ‹*AOT-model-denotes* (*Abs-rel* (λz. *Rep-rel* Π' (z, α)))›
if ‹*AOT-model-denotes* α› **for** α
using *that AOT-meta-proj-denotes1*[*OF* Π'-*denotes*]
AOT-sem-exe-denoting[*simplified AOT-sem-denotes*,*OF* Π'-*denotes*] **by** *simp*
{
fix κ :: *'a* **and** $\kappa_1{'}\kappa_n{'}$ :: *'b*
assume ‹Π = Π'›
assume ‹*AOT-model-denotes* (κ,$\kappa_1{'}\kappa_n{'}$)›

```
  hence ‹AOT-model-denotes κ› and beta-denotes: ‹AOT-model-denotes κ1'κn'›
    by (auto simp: AOT-model-denotes-prod-def)
  have ‹AOT-model-denotes «[λz [Π]z κ1'...κn']»›
    by (rule AOT-model-lambda-denotes[THEN iffD2])
      (metis AOT-sem-exe-denoting AOT-meta-prod-equivI(2)
             AOT-model-denotes-rel.rep-eq AOT-sem-denotes
             AOT-sem-exe-denoting Π-denotes)
  moreover have ‹«[λz [Π]z κ1'...κn']» = «[λz [Π']z κ1'...κn']»›
    by (simp add: ‹Π = Π'›)
  moreover have ‹[v ⊨ «AOT-sem-proj-id κ1'κn' (λκ1'κn'. «[Π]κ κ1'...κn'»)
                                            (λκ1'κn'. «[Π']κ κ1'...κn'»)»]›
    unfolding ‹Π = Π'› using beta-denotes
    by (rule AOT-sem-proj-id-refl[unfolded AOT-sem-denotes];
        simp add: AOT-sem-denotes AOT-sem-eq AOT-model-lambda-denotes)
      (metis AOT-meta-prod-equivI(1) AOT-model-denotes-rel.rep-eq
             AOT-sem-exe AOT-sem-exe-denoting Π'-denotes)
  ultimately have ‹[v ⊨ «AOT-sem-proj-id (κ,κ1'κn') (λ κ1κn . «[Π]κ1...κn»)
                                                (λ κ1κn . «[Π']κ1...κn»)»]›
    unfolding AOT-sem-proj-id-prod-def
    by (simp add: AOT-sem-denotes AOT-sem-conj AOT-sem-eq)
}
moreover {
  assume ‹⋀ α . AOT-model-denotes α ⟹
    [v ⊨ «AOT-sem-proj-id α (λ κ1κn . «[Π]κ1...κn») (λ κ1κn . «[Π']κ1...κn»)»]›
  hence 0: ‹AOT-model-denotes κ ⟹ AOT-model-denotes κ1'κn' ⟹
         AOT-model-denotes «[λz [Π]z κ1'...κn']» ∧
         AOT-model-denotes «[λz [Π']z κ1'...κn']» ∧
         «[λz [Π]z κ1'...κn']» = «[λz [Π']z κ1'...κn']» ∧
         [v ⊨ «AOT-sem-proj-id κ1'κn' (λκ1κn. «[Π]κ κ1...κn»)
                                     (λκ1κn. «[Π']κ κ1...κn»)»]› for κ κ1'κn'
    unfolding AOT-sem-proj-id-prod-def
    by (auto simp: AOT-sem-denotes AOT-sem-conj AOT-sem-eq
                   AOT-model-denotes-prod-def)
  obtain αden :: 'a and βden :: 'b where
    αden: ‹AOT-model-denotes αden› and βden: ‹AOT-model-denotes βden›
    using AOT-model-denoting-ex by metis
  {
    fix κ :: 'a
    assume αdenotes: ‹AOT-model-denotes κ›
    have 1: ‹[v ⊨ «AOT-sem-proj-id κ1'κn' (λκ1'κn'. «[Π]κ κ1'...κn'»)
                                         (λκ1'κn'. «[Π']κ κ1'...κn'»)»]›
      if ‹AOT-model-denotes κ1'κn'› for κ1'κn'
      using that 0 using αdenotes by blast
    hence ‹[v ⊨ «AOT-sem-proj-id β (λz. Rep-rel Π (κ, z))
                                  (λz. Rep-rel Π' (κ, z))»]›
      if ‹AOT-model-denotes β› for β
      using that
      unfolding AOT-sem-exe-denoting[simplified AOT-sem-denotes, OF Π-denotes]
                AOT-sem-exe-denoting[simplified AOT-sem-denotes, OF Π'-denotes]
      by blast
    hence ‹Abs-rel (λz. Rep-rel Π (κ, z)) = Abs-rel (λz. Rep-rel Π' (κ, z))›
      using AOT-sem-proj-id-prop[of v ‹Abs-rel (λz. Rep-rel Π (κ, z))›
                                      ‹Abs-rel (λz. Rep-rel Π' (κ, z))›,
            simplified AOT-sem-eq AOT-sem-conj AOT-sem-forall
                       AOT-sem-denotes, THEN iffD2]
          Π-proj2-den[OF αdenotes] Π'-proj2-den[OF αdenotes]
      unfolding AOT-sem-exe-denoting[simplified AOT-sem-denotes, OF Π-denotes]
                AOT-sem-exe-denoting[simplified AOT-sem-denotes,
                                     OF Π-proj2-den[OF αdenotes]]
                AOT-sem-exe-denoting[simplified AOT-sem-denotes,
                                     OF Π'-proj2-den[OF αdenotes]]
      by (metis Abs-rel-inverse UNIV-I)
    hence Rep-rel Π (κ,β) = Rep-rel Π' (κ,β) for β
```

```
        by (simp add: Abs-rel-inject[simplified]) meson
    } note αdenotes = this
    {
      fix κ1'κn' :: 'b
      assume βden: ‹AOT-model-denotes κ1'κn'›
      have 1: ‹«[λz [Π]z κ1'...κn']» = «[λz [Π']z κ1'...κn']»›
        using 0 βden AOT-model-denoting-ex by blast
      hence ‹Abs-rel (λz. Rep-rel Π (z, κ1'κn')) =
             Abs-rel (λz. Rep-rel Π' (z, κ1'κn'))› (is ‹?a = ?b›)
        apply (safe intro!: AOT-sem-proj-id-prop[of v ‹?a› ‹?b›,
                simplified AOT-sem-eq AOT-sem-conj AOT-sem-forall
                AOT-sem-denotes, THEN iffD2, THEN conjunct2, THEN conjunct2]
                Π-proj1-den[OF βden] Π'-proj1-den[OF βden])
        unfolding AOT-sem-exe-denoting[simplified AOT-sem-denotes, OF Π-denotes]
                  AOT-sem-exe-denoting[simplified AOT-sem-denotes, OF Π'-denotes]
                  AOT-sem-exe-denoting[simplified AOT-sem-denotes,
                                       OF Π-proj1-den[OF βden]]
                  AOT-sem-exe-denoting[simplified AOT-sem-denotes,
                                       OF Π'-proj1-den[OF βden]]
        by (subst (0 1) Abs-rel-inverse; simp?)
           (metis (no-types, lifting) AOT-model-denotes-rel.abs-eq
                    AOT-model-lambda-denotes AOT-sem-denotes AOT-sem-eq
                    AOT-sem-unary-proj-id Π-proj1-den[OF βden])
      hence ‹Rep-rel Π (x,κ1'κn') = Rep-rel Π' (x,κ1'κn')› for x
        by (simp add: Abs-rel-inject)
           metis
    } note βdenotes = this
    {
      fix α :: 'a and β :: 'b
      assume ‹AOT-model-regular (α, β)›
      moreover {
        assume ‹AOT-model-denotes α ∧ AOT-model-regular β›
        hence ‹Rep-rel Π (α,β) = Rep-rel Π' (α,β)›
          using αdenotes by presburger
      }
      moreover {
        assume ‹¬AOT-model-denotes α ∧ AOT-model-denotes β›
        hence ‹Rep-rel Π (α,β) = Rep-rel Π' (α,β)›
          by (simp add: βdenotes)
      }
      ultimately have ‹Rep-rel Π (α,β) = Rep-rel Π' (α,β)›
        by (metis (no-types, lifting) AOT-model-regular-prod-def case-prodD)
    }
    hence ‹Rep-rel Π = Rep-rel Π'›
      using Π-denotes[unfolded AOT-model-denotes-rel.rep-eq,
                  THEN conjunct2, THEN conjunct2, THEN spec, THEN mp]
      using Π'-denotes[unfolded AOT-model-denotes-rel.rep-eq,
                   THEN conjunct2, THEN conjunct2, THEN spec, THEN mp]
      using AOT-model-irregular-eqI[of ‹Rep-rel Π› ‹Rep-rel Π'› ‹(-,-)›]
      using AOT-model-irregular-nondenoting by fastforce
    hence ‹Π = Π'›
      by (rule Rep-rel-inject[THEN iffD1])
  }
  ultimately have ‹Π = Π' = (∀ κ . AOT-model-denotes κ ⟶
      [v ⊨ «AOT-sem-proj-id κ (λ κ1κn . «[Π]κ1...κn»)
                             (λ κ1κn . «[Π']κ1...κn»)»])›
    by auto
}
thus ‹[v ⊨ Π = Π'] = [v ⊨ Π↓ & Π'↓ &
    ∀α («AOT-sem-proj-id α (λ κ1κn . «[Π]κ1...κn») (λ κ1κn . «[Π']κ1...κn»)»)]›
  by (auto simp: AOT-sem-eq AOT-sem-denotes AOT-sem-forall AOT-sem-conj)
next
  fix v and φ :: ‹'a×'b⇒o› and τ :: ‹'a×'b›
```

```
  assume ‹[v ⊨ τ↓]›
  moreover assume ‹[v ⊨ [λz1...zn «φ z1zn»] = [λz1...zn «φ z1zn»]]›
  ultimately show ‹[v ⊨ «AOT-sem-proj-id τ φ φ»]›
    unfolding AOT-sem-proj-id-prod-def
    using AOT-sem-proj-id-refl[of v snd τ λb. φ (fst τ, b)]
    by (auto simp: AOT-sem-eq AOT-sem-conj AOT-sem-denotes
                   AOT-model-denotes-prod-def AOT-model-lambda-denotes
                   AOT-meta-prod-equivI)
qed
end
```

Sanity-check to verify that n-ary relation identity follows.

```
lemma ‹[v ⊨ Π = Π'] = [v ⊨ Π↓ & Π'↓ & ∀x∀y([λz [Π]z y] = [λz [Π']z y] &
                                          [λz [Π]x z] = [λz [Π']x z])]›
  for Π :: ‹<κ×κ>›
  by (auto simp: AOT-sem-proj-id-prop[of v Π Π'] AOT-sem-proj-id-prod-def
                 AOT-sem-conj AOT-sem-denotes AOT-sem-forall AOT-sem-unary-proj-id
                 AOT-model-denotes-prod-def)
lemma ‹[v ⊨ Π = Π'] = [v ⊨ Π↓ & Π'↓ & ∀x1∀x2∀x3 (
  [λz [Π]z x2 x3] = [λz [Π']z x2 x3] &
  [λz [Π]x1 z x3] = [λz [Π']x1 z x3] &
  [λz [Π]x1 x2 z] = [λz [Π']x1 x2 z])]›
  for Π :: ‹<κ×κ×κ>›
  by (auto simp: AOT-sem-proj-id-prop[of v Π Π'] AOT-sem-proj-id-prod-def
                 AOT-sem-conj AOT-sem-denotes AOT-sem-forall AOT-sem-unary-proj-id
                 AOT-model-denotes-prod-def)
lemma ‹[v ⊨ Π = Π'] = [v ⊨ Π↓ & Π'↓ & ∀x1∀x2∀x3∀x4 (
    [λz [Π]z x2 x3 x4] = [λz [Π']z x2 x3 x4] &
    [λz [Π]x1 z x3 x4] = [λz [Π']x1 z x3 x4] &
    [λz [Π]x1 x2 z x4] = [λz [Π']x1 x2 z x4] &
    [λz [Π]x1 x2 x3 z] = [λz [Π']x1 x2 x3 z])]›
  for Π :: ‹<κ×κ×κ×κ>›
  by (auto simp: AOT-sem-proj-id-prop[of v Π Π'] AOT-sem-proj-id-prod-def
                 AOT-sem-conj AOT-sem-denotes AOT-sem-forall AOT-sem-unary-proj-id
                 AOT-model-denotes-prod-def)
```

n-ary Encoding is constructed using a similar mechanism as n-ary relation identity using an auxiliary notion of projection-encoding.

```
class AOT-Enc =
  fixes AOT-enc :: ‹'a ⇒ <'a::AOT-IndividualTerm> ⇒ o›
    and AOT-proj-enc :: ‹'a ⇒ ('a ⇒ o) ⇒ o›
  assumes AOT-sem-enc-denotes:
    ‹[v ⊨ «AOT-enc κ1κn Π»] ⟹ [v ⊨ κ1...κn↓] ∧ [v ⊨ Π↓]›
  assumes AOT-sem-enc-proj-enc:
    ‹[v ⊨ «AOT-enc κ1κn Π»] =
     [v ⊨ Π↓ & «AOT-proj-enc κ1κn (λ κ1κn.  «[Π]κ1...κn»)»]›
  assumes AOT-sem-proj-enc-denotes:
    ‹[v ⊨ «AOT-proj-enc κ1κn φ»] ⟹ [v ⊨ κ1...κn↓]›
  assumes AOT-sem-enc-nec:
    ‹[v ⊨ «AOT-enc κ1κn Π»] ⟹ [w ⊨ «AOT-enc κ1κn Π»]›
  assumes AOT-sem-proj-enc-nec:
    ‹[v ⊨ «AOT-proj-enc κ1κn φ»] ⟹ [w ⊨ «AOT-proj-enc κ1κn φ»]›
  assumes AOT-sem-universal-encoder:
    ‹∃ κ1κn. [v ⊨ κ1...κn↓] ∧ (∀ Π . [v ⊨ Π↓] ⟶ [v ⊨ «AOT-enc κ1κn Π»]) ∧
            (∀ φ . [v ⊨ [λz1...zn φ{z1...zn}]↓] ⟶ [v ⊨ «AOT-proj-enc κ1κn φ»])›

AOT-syntax-print-translations
  -AOT-enc (-AOT-individual-term κ) (-AOT-relation-term Π) <= CONST AOT-enc κ Π

context AOT-meta-syntax
begin
notation AOT-enc (‹⦃-,-⦄›)
end
```

context *AOT-no-meta-syntax*
begin
no-notation *AOT-enc* (‹⦃-,-⦄›)
end

Unary encoding additionally has to satisfy the axioms of unary encoding and the definition of property identity.

class *AOT-UnaryEnc* = *AOT-UnaryIndividualTerm* +
assumes *AOT-sem-enc-eq*: ‹$[v \models \Pi\!\downarrow \;\&\; \Pi'\!\downarrow \;\&\; \Box\forall\nu\,(\nu[\Pi] \equiv \nu[\Pi']) \rightarrow \Pi = \Pi']$›
and *AOT-sem-A-objects*: ‹$[v \models \exists x\,(\neg\Diamond[E!]x \;\&\; \forall F\,(x[F] \equiv \varphi\{F\}))]$›
and *AOT-sem-unary-proj-enc*: ‹*AOT-proj-enc* $x\ \psi$ = *AOT-enc* x «$[\lambda z\ \psi\{z\}]$»›
and *AOT-sem-nocoder*: ‹$[v \models [E!]\kappa] \Longrightarrow \neg[w \models$ «*AOT-enc* $\kappa\ \Pi$»$]$›
and *AOT-sem-ind-eq*: ‹$(([v \models \kappa\!\downarrow] \wedge [v \models \kappa'\!\downarrow] \wedge \kappa = (\kappa')) =$
$(([v \models [\lambda x\ \Diamond[E!]x]\kappa] \wedge$
$[v \models [\lambda x\ \Diamond[E!]x]\kappa'] \wedge$
$(\forall\ v\ \Pi\ .\ [v \models \Pi\!\downarrow] \longrightarrow [v \models [\Pi]\kappa] = [v \models [\Pi]\kappa']))$
$\vee\ ([v \models [\lambda x\ \neg\Diamond[E!]x]\kappa] \wedge$
$[v \models [\lambda x\ \neg\Diamond[E!]x]\kappa'] \wedge$
$(\forall\ v\ \Pi\ .\ [v \models \Pi\!\downarrow] \longrightarrow [v \models \kappa[\Pi]] = [v \models \kappa'[\Pi]]))))$›

and *AOT-sem-enc-indistinguishable-all*:
‹*AOT-ExtendedModel* $\Longrightarrow$
$[v \models [\lambda x\ \neg\Diamond[E!]x]\kappa] \Longrightarrow$
$[v \models [\lambda x\ \neg\Diamond[E!]x]\kappa'] \Longrightarrow$
$(\bigwedge \Pi'\ .\ [v \models \Pi'\!\downarrow] \Longrightarrow (\bigwedge w\ .\ [w \models [\Pi']\kappa] = [w \models [\Pi']\kappa'])) \Longrightarrow$
$[v \models \Pi\!\downarrow] \Longrightarrow$
$(\bigwedge \Pi'\ .\ [v \models \Pi'\!\downarrow] \Longrightarrow (\bigwedge \kappa_0\ .\ [v \models [\lambda x\ \Diamond[E!]x]\kappa_0] \Longrightarrow$
$(\bigwedge w\ .\ [w \models [\Pi']\kappa_0] = [w \models [\Pi]\kappa_0])) \Longrightarrow [v \models \kappa[\Pi']]) \Longrightarrow$
$(\bigwedge \Pi'\ .\ [v \models \Pi'\!\downarrow] \Longrightarrow (\bigwedge \kappa_0\ .\ [v \models [\lambda x\ \Diamond[E!]x]\kappa_0] \Longrightarrow$
$(\bigwedge w\ .\ [w \models [\Pi']\kappa_0] = [w \models [\Pi]\kappa_0])) \Longrightarrow [v \models \kappa'[\Pi']])$›
and *AOT-sem-enc-indistinguishable-ex*:
‹*AOT-ExtendedModel* $\Longrightarrow$
$[v \models [\lambda x\ \neg\Diamond[E!]x]\kappa] \Longrightarrow$
$[v \models [\lambda x\ \neg\Diamond[E!]x]\kappa'] \Longrightarrow$
$(\bigwedge \Pi'\ .\ [v \models \Pi'\!\downarrow] \Longrightarrow (\bigwedge w\ .\ [w \models [\Pi']\kappa] = [w \models [\Pi']\kappa'])) \Longrightarrow$
$[v \models \Pi\!\downarrow] \Longrightarrow$
$\exists\ \Pi'\ .\ [v \models \Pi'\!\downarrow] \wedge [v \models \kappa[\Pi']] \wedge$
$(\forall\ \kappa_0\ .\ [v \models [\lambda x\ \Diamond[E!]x]\kappa_0] \longrightarrow$
$(\forall\ w\ .\ [w \models [\Pi']\kappa_0] = [w \models [\Pi]\kappa_0])) \Longrightarrow$
$\exists\ \Pi'\ .\ [v \models \Pi'\!\downarrow] \wedge [v \models \kappa'[\Pi']] \wedge$
$(\forall\ \kappa_0\ .\ [v \models [\lambda x\ \Diamond[E!]x]\kappa_0] \longrightarrow$
$(\forall\ w\ .\ [w \models [\Pi']\kappa_0] = [w \models [\Pi]\kappa_0]))$›

We specify encoding to align with the model-construction of encoding.

consts *AOT-sem-enc-κ* :: ‹$\kappa \Rightarrow$ <κ> $\Rightarrow$ o›
specification(*AOT-sem-enc-κ*)
AOT-sem-enc-κ:
‹$[v \models$ «*AOT-sem-enc-κ* $\kappa\ \Pi$»$]$ =
(*AOT-model-denotes* κ ∧ *AOT-model-denotes* Π ∧ *AOT-model-enc* κ Π)›
by (*rule exI*[**where** x=‹$\lambda\ \kappa\ \Pi\ .\ \varepsilon_o\ w$. *AOT-model-denotes* κ ∧ *AOT-model-denotes* Π ∧
AOT-model-enc κ Π›])
(*simp add*: *AOT-model-proposition-choice-simp AOT-model-enc-κ-def κ.case-eq-if*)

We show that κ satisfies the generic properties of n-ary encoding.

instantiation κ :: *AOT-Enc*
begin
definition *AOT-enc-κ* :: ‹$\kappa \Rightarrow$ <κ> $\Rightarrow$ o› **where**
‹*AOT-enc-κ* ≡ *AOT-sem-enc-κ*›
definition *AOT-proj-enc-κ* :: ‹$\kappa \Rightarrow (\kappa \Rightarrow o) \Rightarrow o$› **where**
‹*AOT-proj-enc-κ* ≡ $\lambda\ \kappa\ \varphi$. *AOT-enc* κ «$[\lambda z$ «$\varphi\ z$»$]$»›
lemma *AOT-enc-κ-meta*:
‹$[v \models \kappa[\Pi]]$ = (*AOT-model-denotes* κ ∧ *AOT-model-denotes* Π ∧ *AOT-model-enc* κ Π)›

```
  for κ::κ
  using AOT-sem-enc-κ unfolding AOT-enc-κ-def by auto
instance proof
  fix v and κ :: κ and Π
  show ‹[v ⊨ «AOT-enc κ Π»] ⟹ [v ⊨ κ↓] ∧ [v ⊨ Π↓]›
    unfolding AOT-sem-denotes
    using AOT-enc-κ-meta by blast
next
  fix v and κ :: κ and Π
  show ‹[v ⊨ κ[Π]] = [v ⊨ Π↓ & «AOT-proj-enc κ (λ κ'. «[Π]κ'»)»]›
  proof
    assume enc: ‹[v ⊨ κ[Π]]›
    hence Π-denotes: ‹AOT-model-denotes Π›
      by (simp add: AOT-enc-κ-meta)
    hence Π-eta-denotes: ‹AOT-model-denotes «[λz [Π]z]»›
      using AOT-sem-denotes AOT-sem-eq AOT-sem-lambda-eta by metis
    show ‹[v ⊨ Π↓ & «AOT-proj-enc κ (λ κ. «[Π]κ»)»]›
      using AOT-sem-lambda-eta[simplified AOT-sem-denotes AOT-sem-eq, OF Π-denotes]
      using Π-eta-denotes Π-denotes
      by (simp add: AOT-sem-conj AOT-sem-denotes AOT-proj-enc-κ-def enc)
  next
    assume ‹[v ⊨ Π↓ & «AOT-proj-enc κ (λ κ. «[Π]κ»)»]›
    hence Π-denotes: AOT-model-denotes Π and eta-enc: [v ⊨ κ[λz [Π]z]]
      by (auto simp: AOT-sem-conj AOT-sem-denotes AOT-proj-enc-κ-def)
    thus ‹[v ⊨ κ[Π]]›
      using AOT-sem-lambda-eta[simplified AOT-sem-denotes AOT-sem-eq, OF Π-denotes]
      by auto
  qed
next
  show ‹[v ⊨ «AOT-proj-enc κ φ»] ⟹ [v ⊨ κ↓]› for v and κ :: κ and φ
    by (simp add: AOT-enc-κ-meta AOT-sem-denotes AOT-proj-enc-κ-def)
next
  fix v w and κ :: κ and Π
  assume ‹[v ⊨ κ[Π]]›
  thus ‹[w ⊨ κ[Π]]›
    by (simp add: AOT-enc-κ-meta)
next
  fix v w and κ :: κ and φ
  assume ‹[v ⊨ «AOT-proj-enc κ φ»]›
  thus ‹[w ⊨ «AOT-proj-enc κ φ»]›
    by (simp add: AOT-enc-κ-meta AOT-proj-enc-κ-def)
next
  show ‹∃ κ::κ. [v ⊨ κ↓] ∧ (∀ Π . [v ⊨ Π↓] ⟶ [v ⊨ κ[Π]]) ∧
              (∀ φ . [v ⊨ [λz φ{z}]↓] ⟶ [v ⊨ «AOT-proj-enc κ φ»])› for v
    by (rule exI[where x=‹ακ UNIV›])
      (simp add: AOT-sem-denotes AOT-enc-κ-meta AOT-model-enc-κ-def
                 AOT-model-denotes-κ-def AOT-proj-enc-κ-def)
qed
end
```

We show that κ satisfies the properties of unary encoding.

```
instantiation κ :: AOT-UnaryEnc
begin
instance proof
  fix v and Π Π' :: ‹<κ>›
  show ‹[v ⊨ Π↓ & Π'↓ & □∀ν (ν[Π] ≡ ν[Π']) → Π = Π']›
    apply (simp add: AOT-sem-forall AOT-sem-eq AOT-sem-imp AOT-sem-equiv
                     AOT-enc-κ-meta AOT-sem-conj AOT-sem-denotes AOT-sem-box)
    using AOT-meta-A-objects-κ by fastforce
next
  fix v and φ:: ‹<κ> ⇒ o›
  show ‹[v ⊨ ∃ x (¬◇[E!]x & ∀ F (x[F] ≡ φ{F}))]›
    using AOT-model-A-objects[of λ Π . [v ⊨ φ{Π}]]
```

```
    by (auto simp: AOT-sem-denotes AOT-sem-exists AOT-sem-conj AOT-sem-not
                   AOT-sem-dia AOT-sem-concrete AOT-enc-κ-meta AOT-sem-equiv
                   AOT-sem-forall)
next
  show ‹AOT-proj-enc x ψ = AOT-enc x (AOT-lambda ψ)› for x :: κ and ψ
    by (simp add: AOT-proj-enc-κ-def)
next
  show ‹[v ⊨ [E!]κ] ⟹ ¬ [w ⊨ κ[Π]]› for v w and κ :: κ and Π
    by (simp add: AOT-enc-κ-meta AOT-sem-concrete AOT-model-nocoder)
next
  fix v and κ κ' :: κ
  show ‹([v ⊨ κ↓] ∧ [v ⊨ κ'↓] ∧ κ = κ') =
        (([v ⊨ [λx ◇[E!]x]κ] ∧
          [v ⊨ [λx ◇[E!]x]κ'] ∧
          (∀ v Π . [v ⊨ Π↓] ⟶ [v ⊨ [Π]κ] = [v ⊨ [Π]κ']))
         ∨ ([v ⊨ [λx ¬◇[E!]x]κ] ∧
            [v ⊨ [λx ¬◇[E!]x]κ'] ∧
            (∀ v Π . [v ⊨ Π↓] ⟶ [v ⊨ κ[Π]] = [v ⊨ κ'[Π]])))›
    (is ‹?lhs = (?ordeq ∨ ?abseq)›)
  proof -
  {
    assume 0: ‹[v ⊨ κ↓] ∧ [v ⊨ κ'↓] ∧ κ = κ'›
    {
      assume ‹is-ωκ κ'›
      hence ‹[v ⊨ [λx ◇[E!]x]κ']›
        apply (subst AOT-sem-lambda-beta[OF AOT-sem-ordinary-def-denotes, of v κ'])
         apply (simp add: 0)
        apply (simp add: AOT-sem-dia)
        using AOT-sem-concrete AOT-model-ω-concrete-in-some-world is-ωκ-def by force
      hence ‹?ordeq› unfolding 0[THEN conjunct2, THEN conjunct2] by auto
    }
    moreover {
      assume ‹is-ακ κ'›
      hence ‹[v ⊨ [λx ¬◇[E!]x]κ']›
        apply (subst AOT-sem-lambda-beta[OF AOT-sem-abstract-def-denotes, of v κ'])
         apply (simp add: 0)
        apply (simp add: AOT-sem-not AOT-sem-dia)
        using AOT-sem-concrete is-ακ-def by force
      hence ‹?abseq› unfolding 0[THEN conjunct2, THEN conjunct2] by auto
    }
    ultimately have ‹?ordeq ∨ ?abseq›
      by (meson 0 AOT-sem-denotes AOT-model-denotes-κ-def κ.exhaust-disc)
  }
  moreover {
    assume ordeq: ‹?ordeq›
    hence κ-denotes: ‹[v ⊨ κ↓]› and κ'-denotes: ‹[v ⊨ κ'↓]›
      by (simp add: AOT-sem-denotes AOT-sem-exe)+
    hence ‹is-ωκ κ› and ‹is-ωκ κ'›
      by (metis AOT-model-concrete-κ.simps(2) AOT-model-denotes-κ-def
                AOT-sem-concrete AOT-sem-denotes AOT-sem-dia AOT-sem-lambda-beta
                AOT-sem-ordinary-def-denotes κ.collapse(2) κ.exhaust-disc ordeq)+
    have denotes: ‹[v ⊨ [λz «εₒ w . κυ z = κυ κ»]↓]›
      unfolding AOT-sem-denotes AOT-model-lambda-denotes
      by (simp add: AOT-model-term-equiv-κ-def)
    hence [v ⊨ [λz «εₒ w . κυ z = κυ κ»]κ] = [v ⊨ [λz «εₒ w . κυ z = κυ κ»]κ']
      using ordeq by (simp add: AOT-sem-denotes)
    hence ‹[v ⊨ «κ»↓] ∧ [v ⊨ «κ'»↓] ∧ κ = κ'›
      unfolding AOT-sem-lambda-beta[OF denotes, OF κ-denotes]
                AOT-sem-lambda-beta[OF denotes, OF κ'-denotes]
      using κ'-denotes ‹is-ωκ κ'› ‹is-ωκ κ› is-ωκ-def
            AOT-model-proposition-choice-simp by force
  }
  moreover {
```

```
    assume 0: ‹?abseq›
    hence κ-denotes: ‹[v ⊨ κ↓]› and κ'-denotes: ‹[v ⊨ κ'↓]›
      by (simp add: AOT-sem-denotes AOT-sem-exe)+
    hence ‹¬is-ωκ κ› and ‹¬is-ωκ κ'›
      by (metis AOT-model-ω-concrete-in-some-world AOT-model-concrete-κ.simps(1)
                AOT-sem-concrete AOT-sem-dia AOT-sem-exe AOT-sem-lambda-beta
                AOT-sem-not κ.collapse(1) 0)+
    hence ‹is-ακ κ› and ‹is-ακ κ'›
      by (meson AOT-sem-denotes AOT-model-denotes-κ-def κ.exhaust-disc
                κ-denotes κ'-denotes)+
    then obtain x y where κ-def: ‹κ = ακ x› and κ'-def: ‹κ' = ακ y›
      using is-ακ-def by auto
    {
      fix r
      assume ‹r ∈ x›
      hence ‹[v ⊨ κ[«urrel-to-rel r»]]›
        unfolding κ-def
        unfolding AOT-enc-κ-meta
        unfolding AOT-model-enc-κ-def
        apply (simp add: AOT-model-denotes-κ-def)
        by (metis (mono-tags) AOT-rel-equiv-def Quotient-def urrel-quotient)
      hence ‹[v ⊨ κ'[«urrel-to-rel r»]]›
        using AOT-enc-κ-meta 0 by (metis AOT-sem-enc-denotes)
      hence ‹r ∈ y›
        unfolding κ'-def
        unfolding AOT-enc-κ-meta
        unfolding AOT-model-enc-κ-def
        apply (simp add: AOT-model-denotes-κ-def)
        using Quotient-abs-rep urrel-quotient by fastforce
    }
    moreover {
      fix r
      assume ‹r ∈ y›
      hence ‹[v ⊨ κ'[«urrel-to-rel r»]]›
        unfolding κ'-def
        unfolding AOT-enc-κ-meta
        unfolding AOT-model-enc-κ-def
        apply (simp add: AOT-model-denotes-κ-def)
        by (metis (mono-tags) AOT-rel-equiv-def Quotient-def urrel-quotient)
      hence ‹[v ⊨ κ[«urrel-to-rel r»]]›
        using AOT-enc-κ-meta 0 by (metis AOT-sem-enc-denotes)
      hence ‹r ∈ x›
        unfolding κ-def
        unfolding AOT-enc-κ-meta
        unfolding AOT-model-enc-κ-def
        apply (simp add: AOT-model-denotes-κ-def)
        using Quotient-abs-rep urrel-quotient by fastforce
    }
    ultimately have x = y by blast
    hence ‹[v ⊨ κ↓] ∧ [v ⊨ κ'↓] ∧ κ = κ'›
      using κ'-def κ'-denotes κ-def by blast
  }
  ultimately show ?thesis
    unfolding AOT-sem-denotes
    by auto
  qed

next
  fix v and κ κ' :: κ and Π Π' :: ‹<κ>›
  assume ext: ‹AOT-ExtendedModel›
  assume ‹[v ⊨ [λx ¬◇[E!]x]κ]›
  hence ‹is-ακ κ›
    by (metis AOT-model-ω-concrete-in-some-world AOT-model-concrete-κ.simps(1)
```

```
          AOT-model-denotes-κ-def AOT-sem-concrete AOT-sem-denotes AOT-sem-dia
          AOT-sem-exe AOT-sem-lambda-beta AOT-sem-not κ.collapse(1) κ.exhaust-disc)
hence κ-abs: ‹¬(∃ w . AOT-model-concrete w κ)›
  using is-ακ-def by fastforce
have κ-den: ‹AOT-model-denotes κ›
  by (simp add: AOT-model-denotes-κ-def κ.distinct-disc(5) ‹is-ακ κ›)
assume ‹[v ⊨ [λx ¬◇[E!]x]κ']›
hence ‹is-ακ κ'›
  by (metis AOT-model-ω-concrete-in-some-world AOT-model-concrete-κ.simps(1)
          AOT-model-denotes-κ-def AOT-sem-concrete AOT-sem-denotes AOT-sem-dia
          AOT-sem-exe AOT-sem-lambda-beta AOT-sem-not κ.collapse(1)
          κ.exhaust-disc)
hence κ'-abs: ‹¬(∃ w . AOT-model-concrete w κ')›
  using is-ακ-def by fastforce
have κ'-den: ‹AOT-model-denotes κ'›
  by (meson AOT-model-denotes-κ-def κ.distinct-disc(6) ‹is-ακ κ'›)
assume ‹[v ⊨ Π'↓] ⟹ [w ⊨ [Π']κ] = [w ⊨ [Π']κ']› for Π' w
hence indist: ‹[v ⊨ «Rep-rel Π' κ»] = [v ⊨ «Rep-rel Π' κ'»]›
  if ‹AOT-model-denotes Π'› for Π' v
  by (metis AOT-sem-denotes AOT-sem-exe κ'-den κ-den that)
assume κ-enc-cond: ‹[v ⊨ Π'↓] ⟹
  (⋀ κ₀ w. [v ⊨ [λx ◇[E!]x]κ₀] ⟹
          [w ⊨ [Π']κ₀] = [w ⊨ [Π]κ₀]) ⟹
  [v ⊨ κ[Π']]› for Π'
assume Π-den': ‹[v ⊨ Π↓]›
hence Π-den: ‹AOT-model-denotes Π›
  using AOT-sem-denotes by blast
{
  fix Π' :: ‹<κ>›
  assume Π'-den: ‹AOT-model-denotes Π'›
  hence Π'-den': ‹[v ⊨ Π'↓]›
    by (simp add: AOT-sem-denotes)
  assume 1: ‹∃ w. AOT-model-concrete w x ⟹
            [v ⊨ «Rep-rel Π' x»] = [v ⊨ «Rep-rel Π x»]› for v x
  {
    fix κ₀ :: κ and w
    assume ‹[v ⊨ [λx ◇[E!]x]κ₀]›
    hence ‹is-ωκ κ₀›
      by (smt (z3) AOT-model-concrete-κ.simps(2) AOT-model-denotes-κ-def
                AOT-sem-concrete AOT-sem-denotes AOT-sem-dia AOT-sem-exe
                AOT-sem-lambda-beta κ.exhaust-disc is-ακ-def)
    then obtain x where x-prop: ‹κ₀ = ωκ x›
      using is-ωκ-def by blast
    have ‹∃ w . AOT-model-concrete w (ωκ x)›
      by (simp add: AOT-model-ω-concrete-in-some-world)
    hence ‹[v ⊨ «Rep-rel Π' (ωκ x)»] = [v ⊨ «Rep-rel Π (ωκ x)»]› for v
      using 1 by blast
    hence ‹[w ⊨ [Π']κ₀] = [w ⊨ [Π]κ₀]› unfolding x-prop
      by (simp add: AOT-sem-exe AOT-sem-denotes AOT-model-denotes-κ-def
                Π'-den Π-den)
  } note 2 = this
  have ‹[v ⊨ κ[Π']]›
    using κ-enc-cond[OF Π'-den', OF 2]
    by metis
  hence ‹AOT-model-enc κ Π'›
    using AOT-enc-κ-meta by blast
} note κ-enc-cond = this
hence ‹AOT-model-denotes Π' ⟹
    (⋀v x. ∃ w. AOT-model-concrete w x ⟹
          [v ⊨ «Rep-rel Π' x»] = [v ⊨ «Rep-rel Π x»]) ⟹
    AOT-model-enc κ Π'› for Π'
  by blast
assume Π'-den': ‹[v ⊨ Π'↓]›
```

```
  hence Π'-den: ‹AOT-model-denotes Π'›
    using AOT-sem-denotes by blast
  assume ord-indist: ‹[v ⊨ [λx ◇[E!]x]κ₀] ⟹
                    [w ⊨ [Π']κ₀] = [w ⊨ [Π]κ₀]› for κ₀ w
  {
    fix w and κ₀ :: κ
    assume 0: ‹∃ w. AOT-model-concrete w κ₀›
    hence ‹[v ⊨ [λx ◇[E!]x]κ₀]›
      using AOT-model-concrete-denotes AOT-sem-concrete AOT-sem-denotes AOT-sem-dia
            AOT-sem-lambda-beta AOT-sem-ordinary-def-denotes by blast
    hence ‹[w ⊨ [Π']κ₀] = [w ⊨ [Π]κ₀]›
      using ord-indist by metis
    hence ‹[w ⊨ «Rep-rel Π' κ₀»] = [w ⊨ «Rep-rel Π κ₀»]›
      by (metis AOT-model-concrete-denotes AOT-sem-denotes AOT-sem-exe Π'-den Π-den 0)
  } note ord-indist = this
  have ‹AOT-model-enc κ' Π'›
    using AOT-model-enc-indistinguishable-all
          [OF ext, OF κ-den, OF κ-abs, OF κ'-den, OF κ'-abs, OF Π-den]
        indist κ-enc-cond Π'-den ord-indist by blast
  thus ‹[v ⊨ κ'[Π']]›
    using AOT-enc-κ-meta Π'-den κ'-den by blast
next
  fix v and κ κ' :: κ and Π Π' :: ‹<κ>›
  assume ext: ‹AOT-ExtendedModel›
  assume ‹[v ⊨ [λx ¬◇[E!]x]κ]›
  hence ‹is-ακ κ›
    by (metis AOT-model-ω-concrete-in-some-world AOT-model-concrete-κ.simps(1)
              AOT-model-denotes-κ-def AOT-sem-concrete AOT-sem-denotes AOT-sem-dia
              AOT-sem-exe AOT-sem-lambda-beta AOT-sem-not κ.collapse(1)
              κ.exhaust-disc)
  hence κ-abs: ‹¬(∃ w . AOT-model-concrete w κ)›
    using is-ακ-def by fastforce
  have κ-den: ‹AOT-model-denotes κ›
    by (simp add: AOT-model-denotes-κ-def κ.distinct-disc(5) ‹is-ακ κ›)
  assume ‹[v ⊨ [λx ¬◇[E!]x]κ']›
  hence ‹is-ακ κ'›
    by (metis AOT-model-ω-concrete-in-some-world AOT-model-concrete-κ.simps(1)
              AOT-model-denotes-κ-def AOT-sem-concrete AOT-sem-denotes AOT-sem-dia
              AOT-sem-exe AOT-sem-lambda-beta AOT-sem-not κ.collapse(1)
              κ.exhaust-disc)
  hence κ'-abs: ‹¬(∃ w . AOT-model-concrete w κ')›
    using is-ακ-def by fastforce
  have κ'-den: ‹AOT-model-denotes κ'›
    by (meson AOT-model-denotes-κ-def κ.distinct-disc(6) ‹is-ακ κ'›)
  assume ‹[v ⊨ Π'↓] ⟹ [w ⊨ [Π']κ] = [w ⊨ [Π']κ']› for Π' w
  hence indist: ‹[v ⊨ «Rep-rel Π' κ»] = [v ⊨ «Rep-rel Π' κ'»]›
    if ‹AOT-model-denotes Π'› for Π' v
    by (metis AOT-sem-denotes AOT-sem-exe κ'-den κ-den that)
  assume Π-den': ‹[v ⊨ Π↓]›
  hence Π-den: ‹AOT-model-denotes Π›
    using AOT-sem-denotes by blast
  assume ‹∃Π'. [v ⊨ Π'↓] ∧ [v ⊨ κ[Π']] ∧
            (∀κ₀. [v ⊨ [λx ◇[E!]x]κ₀] ⟶
                  (∀w. [w ⊨ [Π']κ₀] = [w ⊨ [Π]κ₀]))›
  then obtain Π' where
      Π'-den: ‹[v ⊨ Π'↓]› and
      Π'-enc: ‹[v ⊨ κ[Π']]› and
      Π'-prop: ‹∀κ₀. [v ⊨ [λx ◇[E!]x]κ₀] ⟶
                  (∀w. [w ⊨ [Π']κ₀] = [w ⊨ [Π]κ₀])›
    by blast
  have ‹AOT-model-denotes Π'›
    using AOT-enc-κ-meta Π'-enc by force
  moreover have ‹AOT-model-enc κ Π'›
```

using *AOT-enc-κ-meta* Π′*-enc* **by** *blast*
moreover have ‹(∃ *w*. *AOT-model-concrete w* κ_0) ⟹
[*v* ⊨ «*Rep-rel* Π′ κ_0»] = [*v* ⊨ «*Rep-rel* Π κ_0»]› **for** κ_0 *v*
proof −
assume *0*: ‹∃ *w*. *AOT-model-concrete w* κ_0›
hence ‹[*v* ⊨ [λ*x* ◊[*E*!]*x*]κ_0]› **for** *v*
using *AOT-model-concrete-denotes AOT-sem-concrete AOT-sem-denotes AOT-sem-dia AOT-sem-lambda-beta AOT-sem-ordinary-def-denotes* **by** *blast*
hence ‹∀ *w*. [*w* ⊨ [Π′]κ_0] = [*w* ⊨ [Π]κ_0]› **using** Π′*-prop* **by** *blast*
thus ‹[*v* ⊨ «*Rep-rel* Π′ κ_0»] = [*v* ⊨ «*Rep-rel* Π κ_0»]›
by (*meson 0 AOT-model-concrete-denotes AOT-sem-denotes AOT-sem-exe* Π*-den calculation*(*1*))
qed
ultimately have ‹∃ Π′. *AOT-model-denotes* Π′ ∧ *AOT-model-enc* κ Π′ ∧
(∀ *v x*. (∃ *w*. *AOT-model-concrete w x*) ⟶
[*v* ⊨ «*Rep-rel* Π′ *x*»] = [*v* ⊨ «*Rep-rel* Π *x*»])›
by *blast*
hence ‹∃ Π′. *AOT-model-denotes* Π′ ∧ *AOT-model-enc* κ′ Π′ ∧
(∀ *v x*. (∃ *w*. *AOT-model-concrete w x*) ⟶
[*v* ⊨ «*Rep-rel* Π′ *x*»] = [*v* ⊨ «*Rep-rel* Π *x*»])›
using *AOT-model-enc-indistinguishable-ex*
[*OF ext*, *OF* κ*-den*, *OF* κ*-abs*, *OF* κ′*-den*, *OF* κ′*-abs*, *OF* Π*-den*]
indist **by** *blast*
then obtain Π′′ **where**
Π′′*-den*: ‹*AOT-model-denotes* Π′′›
and Π′′*-enc*: ‹*AOT-model-enc* κ′ Π′′›
and Π′′*-prop*: ‹(∃ *w*. *AOT-model-concrete w x*) ⟹
[*v* ⊨ «*Rep-rel* Π′′ *x*»] = [*v* ⊨ «*Rep-rel* Π *x*»]› **for** *v x*
by *blast*
have ‹[*v* ⊨ Π′′↓]›
by (*simp add*: *AOT-sem-denotes* Π′′*-den*)
moreover have ‹[*v* ⊨ κ′[Π′′]]›
by (*simp add*: *AOT-enc-κ-meta* Π′′*-den* Π′′*-enc* κ′*-den*)
moreover have ‹[*v* ⊨ [λ*x* ◊[*E*!]*x*]κ_0] ⟹
(∀ *w*. [*w* ⊨ [Π′′]κ_0] = [*w* ⊨ [Π]κ_0])› **for** κ_0
proof −
assume ‹[*v* ⊨ [λ*x* ◊[*E*!]*x*]κ_0]›
hence ‹∃ *w*. *AOT-model-concrete w* κ_0›
by (*metis AOT-sem-concrete AOT-sem-dia AOT-sem-exe AOT-sem-lambda-beta*)
thus ‹∀ *w*. [*w* ⊨ [Π′′]κ_0] = [*w* ⊨ [Π]κ_0]›
using Π′′*-prop*
by (*metis AOT-sem-denotes AOT-sem-exe* Π′′*-den* Π*-den*)
qed
ultimately show ‹∃ Π′. [*v* ⊨ Π′↓] ∧ [*v* ⊨ κ′[Π′]] ∧
(∀ κ_0. [*v* ⊨ [λ*x* ◊[*E*!]*x*]κ_0] ⟶
(∀ *w*. [*w* ⊨ [Π′]κ_0] = [*w* ⊨ [Π]κ_0]))›
by (*safe intro*!: *exI*[**where** *x*=Π′′]) *blast*+
qed
end

Define encoding for products using projection-encoding.

instantiation *prod* :: (*AOT-UnaryEnc*, *AOT-Enc*) *AOT-Enc*
begin
definition *AOT-proj-enc-prod* :: ‹′*a*×′*b* ⇒ (′*a*×′*b* ⇒ o) ⇒ o› **where**
‹*AOT-proj-enc-prod* ≡ λ (κ,κ′) φ . «κ[λν «φ (ν,κ′)»] &
«*AOT-proj-enc* κ′ (λν. φ (κ,ν))»»›
definition *AOT-enc-prod* :: ‹′*a*×′*b* ⇒ <′*a*×′*b*> ⇒ o› **where**
‹*AOT-enc-prod* ≡ λ κ Π . «Π↓ & «*AOT-proj-enc* κ (λ $\kappa_1{}'\kappa_n{}'$. «[Π]$\kappa_1{}'...\kappa_n{}'$»)»»›
instance proof
show ‹[*v* ⊨ $\kappa_1...\kappa_n$[Π]] ⟹ [*v* ⊨ $\kappa_1...\kappa_n$↓] ∧ [*v* ⊨ Π↓]›
for *v* **and** $\kappa_1\kappa_n$ :: ‹′*a*×′*b*› **and** Π
unfolding *AOT-enc-prod-def*
apply (*induct* $\kappa_1\kappa_n$; *simp add*: *AOT-sem-conj AOT-sem-denotes AOT-proj-enc-prod-def*)

by (*metis AOT-sem-denotes AOT-model-denotes-prod-def AOT-sem-enc-denotes AOT-sem-proj-enc-denotes case-prodI*)
next
show ‹[v ⊨ $\kappa_1...\kappa_n$[Π]] =
[v ⊨ «Π»↓ & «*AOT-proj-enc* $\kappa_1\kappa_n$ (λ $\kappa_1\kappa_n$. «[Π]$\kappa_1...\kappa_n$»)»]›
for v **and** $\kappa_1\kappa_n$ :: ‹'a×'b› **and** Π
unfolding *AOT-enc-prod-def* **..**
next
show ‹[v ⊨ «*AOT-proj-enc* κs φ»] ⟹ [v ⊨ «κs»↓]›
for v **and** κs :: ‹'a×'b› **and** φ
by (*metis* (*mono-tags*, *lifting*)
AOT-sem-conj AOT-sem-denotes AOT-model-denotes-prod-def AOT-sem-enc-denotes AOT-sem-proj-enc-denotes AOT-proj-enc-prod-def case-prod-unfold)
next
fix v w Π **and** $\kappa_1\kappa_n$ :: ‹'a×'b›
show ‹[w ⊨ $\kappa_1...\kappa_n$[Π]]› **if** ‹[v ⊨ $\kappa_1...\kappa_n$[Π]]› **for** v w Π **and** $\kappa_1\kappa_n$ :: ‹'a×'b›
by (*metis* (*mono-tags*, *lifting*)
AOT-enc-prod-def AOT-sem-enc-proj-enc AOT-sem-conj AOT-sem-denotes AOT-sem-proj-enc-nec AOT-proj-enc-prod-def case-prod-unfold that)
next
show ‹[w ⊨ «*AOT-proj-enc* $\kappa_1\kappa_n$ φ»]› **if** ‹[v ⊨ «*AOT-proj-enc* $\kappa_1\kappa_n$ φ»]›
for v w φ **and** $\kappa_1\kappa_n$ :: ‹'a×'b›
by (*metis* (*mono-tags*, *lifting*)
that AOT-sem-enc-proj-enc AOT-sem-conj AOT-sem-denotes AOT-sem-proj-enc-nec AOT-proj-enc-prod-def case-prod-unfold)
next
fix v
obtain κ :: 'a **where** *a-prop*: ‹[v ⊨ κ↓] ∧ (∀ Π . [v ⊨ Π↓] ⟶ [v ⊨ κ[Π]])›
using *AOT-sem-universal-encoder* **by** *blast*
obtain $\kappa_1'\kappa_n'$ :: 'b **where** *b-prop*:
‹[v ⊨ $\kappa_1'...\kappa_n'$↓] ∧ (∀ φ . [v ⊨ [λ$\nu_1...\nu_n$ «φ $\nu_1\nu_n$»]↓] ⟶
[v ⊨ «*AOT-proj-enc* $\kappa_1'\kappa_n'$ φ»])›
using *AOT-sem-universal-encoder* **by** *blast*
have ‹*AOT-model-denotes* «[λ$\nu_1...\nu_n$ [«Π»]$\nu_1...\nu_n$ $\kappa_1'...\kappa_n'$]»›
if ‹*AOT-model-denotes* Π› **for** Π :: ‹<'a×'b>›
unfolding *AOT-model-lambda-denotes*
by (*metis AOT-meta-prod-equivI*(*2*) *AOT-sem-exe-equiv*)
moreover have ‹*AOT-model-denotes* «[λ$\nu_1...\nu_n$ [«Π»]κ $\nu_1...\nu_n$]»›
if ‹*AOT-model-denotes* Π› **for** Π :: ‹<'a×'b>›
unfolding *AOT-model-lambda-denotes*
by (*metis AOT-meta-prod-equivI*(*1*) *AOT-sem-exe-equiv*)
ultimately have *1*: ‹[v ⊨ «(κ,$\kappa_1'\kappa_n'$)»↓]›
and *2*: ‹(∀ Π . [v ⊨ Π↓] ⟶ [v ⊨ κ $\kappa_1'...\kappa_n'$[Π]])›
using *a-prop b-prop*
by (*auto simp*: *AOT-sem-denotes AOT-enc-κ-meta AOT-model-enc-κ-def AOT-model-denotes-κ-def AOT-model-denotes-prod-def AOT-enc-prod-def AOT-proj-enc-prod-def AOT-sem-conj*)
have ‹*AOT-model-denotes* «[λ$z_1...z_n$ «φ (z_1z_n, $\kappa_1'\kappa_n'$)»]»›
if ‹*AOT-model-denotes* «[λ$z_1...z_m$ φ{$z_1...z_m$}]»› **for** φ :: ‹'a×'b ⇒ o›
using *that*
unfolding *AOT-model-lambda-denotes*
by (*metis* (*no-types*, *lifting*) *AOT-sem-denotes AOT-model-denotes-prod-def AOT-meta-prod-equivI*(*2*) *b-prop case-prodI*)
moreover have ‹*AOT-model-denotes* «[λ$z_1...z_n$ «φ (κ, z_1z_n)»]»›
if ‹*AOT-model-denotes* «[λ$z_1...z_m$ φ{$z_1...z_m$}]»› **for** φ :: ‹'a×'b ⇒ o›
using *that*
unfolding *AOT-model-lambda-denotes*
by (*metis* (*no-types*, *lifting*) *AOT-sem-denotes AOT-model-denotes-prod-def AOT-meta-prod-equivI*(*1*) *a-prop case-prodI*)
ultimately have *3*:
‹[v ⊨ «(κ,$\kappa_1'\kappa_n'$)»↓] ∧ (∀ φ . [v ⊨ [λ$z_1...z_n$ φ{$z_1...z_n$}]↓] ⟶
[v ⊨ «*AOT-proj-enc* (κ,$\kappa_1'\kappa_n'$) φ»])›

using *a-prop b-prop*
by (*auto simp*: *AOT-sem-denotes AOT-enc-κ-meta AOT-model-enc-κ-def AOT-model-denotes-κ-def AOT-enc-prod-def AOT-proj-enc-prod-def AOT-sem-conj AOT-model-denotes-prod-def*)
show ‹$\exists\, \kappa_1\kappa_n$::'a×'b. $[v \models \kappa_1...\kappa_n\downarrow] \wedge (\forall\ \Pi\ .\ [v \models \Pi\downarrow] \longrightarrow [v \models \kappa_1...\kappa_n[\Pi]]) \wedge$
$(\forall\ \varphi\ .\ [v \models [\lambda z_1...z_n$ «$\varphi\ z_1z_n$»$]\downarrow] \longrightarrow$
$[v \models$ «*AOT-proj-enc* $\kappa_1\kappa_n\ \varphi$»$])$›
apply (*rule exI*[**where** x=‹$(\kappa,\kappa_1{}'\kappa_n{}')$›]) **using** *1 2 3* **by** *blast*
qed
end

Sanity-check to verify that n-ary encoding follows.

lemma ‹$[v \models \kappa_1\kappa_2[\Pi]] = [v \models \Pi\downarrow\ \&\ \kappa_1[\lambda\nu\ [\Pi]\nu\kappa_2]\ \&\ \kappa_2[\lambda\nu\ [\Pi]\kappa_1\nu]]$›
for κ_1 :: 'a::*AOT-UnaryEnc* **and** κ_2 :: 'b::*AOT-UnaryEnc*
by (*simp add*: *AOT-sem-conj AOT-enc-prod-def AOT-proj-enc-prod-def AOT-sem-unary-proj-enc*)
lemma ‹$[v \models \kappa_1\kappa_2\kappa_3[\Pi]] =$
$[v \models \Pi\downarrow\ \&\ \kappa_1[\lambda\nu\ [\Pi]\nu\kappa_2\kappa_3]\ \&\ \kappa_2[\lambda\nu\ [\Pi]\kappa_1\nu\kappa_3]\ \&\ \kappa_3[\lambda\nu\ [\Pi]\kappa_1\kappa_2\nu]]$›
for $\kappa_1\ \kappa_2\ \kappa_3$ :: 'a::*AOT-UnaryEnc*
by (*simp add*: *AOT-sem-conj AOT-enc-prod-def AOT-proj-enc-prod-def AOT-sem-unary-proj-enc*)

lemma *AOT-sem-vars-denote*: ‹$[v \models \alpha_1...\alpha_n\downarrow]$›
by *induct simp*

Combine the introduced type classes and register them as type constraints for individual terms.

class *AOT-κs* = *AOT-IndividualTerm* + *AOT-RelationProjection* + *AOT-Enc*
class *AOT-κ* = *AOT-κs* + *AOT-UnaryIndividualTerm* +
AOT-UnaryRelationProjection + *AOT-UnaryEnc*

instance κ :: *AOT-κ* **by** *standard*
instance *prod* :: (*AOT-κ*, *AOT-κs*) *AOT-κs* **by** *standard*

AOT-register-type-constraints
Individual: ‹-::*AOT-κ*› ‹-::*AOT-κs*› **and**
Relation: ‹<-::*AOT-κs*>›

We define semantic predicates to capture the conditions of cqt.2 (i.e. the base cases of denoting terms) on matrices of λ-expressions.

definition *AOT-instance-of-cqt-2* :: ‹('a::*AOT-κs* ⇒ o) ⇒ *bool*› **where**
‹*AOT-instance-of-cqt-2* ≡ λ φ . ∀ x y . *AOT-model-denotes* x ∧ *AOT-model-denotes* y ∧
AOT-model-term-equiv x y ⟶ φ x = φ y›
definition *AOT-instance-of-cqt-2-exe-arg* :: ‹('a::*AOT-κs* ⇒ 'b::*AOT-κs*) ⇒ *bool*› **where**
‹*AOT-instance-of-cqt-2-exe-arg* ≡ λ φ . ∀ x y .
AOT-model-denotes x ∧ *AOT-model-denotes* y ∧ *AOT-model-term-equiv* x y ⟶
AOT-model-term-equiv (φ x) (φ y)›

λ-expressions with a matrix that satisfies our predicate denote.

lemma *AOT-sem-cqt-2*:
assumes ‹*AOT-instance-of-cqt-2* φ›
shows ‹$[v \models [\lambda\nu_1...\nu_n\ \varphi\{\nu_1...\nu_n\}]\downarrow]$›
using *assms*
by (*metis AOT-instance-of-cqt-2-def AOT-model-lambda-denotes AOT-sem-denotes*)

syntax *AOT-instance-of-cqt-2* :: ‹*id-position* ⇒ *AOT-prop*›
(‹*INSTANCE'-OF'-CQT'-2'*(-')›)

Prove introduction rules for the predicates that match the natural language restrictions of the axiom.

named-theorems *AOT-instance-of-cqt-2-intro*
lemma *AOT-instance-of-cqt-2-intros-const*[*AOT-instance-of-cqt-2-intro*]:
‹*AOT-instance-of-cqt-2* (λα. φ)›
by (*simp add*: *AOT-instance-of-cqt-2-def AOT-sem-denotes AOT-model-lambda-denotes*)

lemma *AOT-instance-of-cqt-2-intros-not*[*AOT-instance-of-cqt-2-intro*]:
assumes ‹*AOT-instance-of-cqt-2* φ›
shows ‹*AOT-instance-of-cqt-2* ($\lambda\tau$. «$\neg\varphi\{\tau\}$»)›
using *assms*
by (*metis* (*no-types*, *lifting*) *AOT-instance-of-cqt-2-def*)
lemma *AOT-instance-of-cqt-2-intros-imp*[*AOT-instance-of-cqt-2-intro*]:
assumes ‹*AOT-instance-of-cqt-2* φ› **and** ‹*AOT-instance-of-cqt-2* ψ›
shows ‹*AOT-instance-of-cqt-2* ($\lambda\tau$. «$\varphi\{\tau\} \rightarrow \psi\{\tau\}$»)›
using *assms*
by (*auto simp*: *AOT-instance-of-cqt-2-def AOT-sem-denotes AOT-model-lambda-denotes AOT-sem-imp*)
lemma *AOT-instance-of-cqt-2-intros-box*[*AOT-instance-of-cqt-2-intro*]:
assumes ‹*AOT-instance-of-cqt-2* φ›
shows ‹*AOT-instance-of-cqt-2* ($\lambda\tau$. «$\Box\varphi\{\tau\}$»)›
using *assms*
by (*auto simp*: *AOT-instance-of-cqt-2-def AOT-sem-denotes AOT-model-lambda-denotes AOT-sem-box*)
lemma *AOT-instance-of-cqt-2-intros-act*[*AOT-instance-of-cqt-2-intro*]:
assumes ‹*AOT-instance-of-cqt-2* φ›
shows ‹*AOT-instance-of-cqt-2* ($\lambda\tau$. «$\mathcal{A}\varphi\{\tau\}$»)›
using *assms*
by (*auto simp*: *AOT-instance-of-cqt-2-def AOT-sem-denotes AOT-model-lambda-denotes AOT-sem-act*)
lemma *AOT-instance-of-cqt-2-intros-diamond*[*AOT-instance-of-cqt-2-intro*]:
assumes ‹*AOT-instance-of-cqt-2* φ›
shows ‹*AOT-instance-of-cqt-2* ($\lambda\tau$. «$\Diamond\varphi\{\tau\}$»)›
using *assms*
by (*auto simp*: *AOT-instance-of-cqt-2-def AOT-sem-denotes AOT-model-lambda-denotes AOT-sem-dia*)
lemma *AOT-instance-of-cqt-2-intros-conj*[*AOT-instance-of-cqt-2-intro*]:
assumes ‹*AOT-instance-of-cqt-2* φ› **and** ‹*AOT-instance-of-cqt-2* ψ›
shows ‹*AOT-instance-of-cqt-2* ($\lambda\tau$. «$\varphi\{\tau\}$ & $\psi\{\tau\}$»)›
using *assms*
by (*auto simp*: *AOT-instance-of-cqt-2-def AOT-sem-denotes AOT-model-lambda-denotes AOT-sem-conj*)
lemma *AOT-instance-of-cqt-2-intros-disj*[*AOT-instance-of-cqt-2-intro*]:
assumes ‹*AOT-instance-of-cqt-2* φ› **and** ‹*AOT-instance-of-cqt-2* ψ›
shows ‹*AOT-instance-of-cqt-2* ($\lambda\tau$. «$\varphi\{\tau\} \vee \psi\{\tau\}$»)›
using *assms*
by (*auto simp*: *AOT-instance-of-cqt-2-def AOT-sem-denotes AOT-model-lambda-denotes AOT-sem-disj*)
lemma *AOT-instance-of-cqt-2-intros-equib*[*AOT-instance-of-cqt-2-intro*]:
assumes ‹*AOT-instance-of-cqt-2* φ› **and** ‹*AOT-instance-of-cqt-2* ψ›
shows ‹*AOT-instance-of-cqt-2* ($\lambda\tau$. «$\varphi\{\tau\} \equiv \psi\{\tau\}$»)›
using *assms*
by (*auto simp*: *AOT-instance-of-cqt-2-def AOT-sem-denotes AOT-model-lambda-denotes AOT-sem-equiv*)
lemma *AOT-instance-of-cqt-2-intros-forall*[*AOT-instance-of-cqt-2-intro*]:
assumes ‹$\bigwedge$ α . *AOT-instance-of-cqt-2* (Φ α)›
shows ‹*AOT-instance-of-cqt-2* ($\lambda\tau$. «$\forall\,\alpha$ $\Phi\{\alpha,\tau\}$»)›
using *assms*
by (*auto simp*: *AOT-instance-of-cqt-2-def AOT-sem-denotes AOT-model-lambda-denotes AOT-sem-forall*)
lemma *AOT-instance-of-cqt-2-intros-exists*[*AOT-instance-of-cqt-2-intro*]:
assumes ‹$\bigwedge$ α . *AOT-instance-of-cqt-2* (Φ α)›
shows ‹*AOT-instance-of-cqt-2* ($\lambda\tau$. «$\exists\,\alpha$ $\Phi\{\alpha,\tau\}$»)›
using *assms*
by (*auto simp*: *AOT-instance-of-cqt-2-def AOT-sem-denotes AOT-model-lambda-denotes AOT-sem-exists*)
lemma *AOT-instance-of-cqt-2-intros-exe-arg-self*[*AOT-instance-of-cqt-2-intro*]:
‹*AOT-instance-of-cqt-2-exe-arg* (λx. x)›
unfolding *AOT-instance-of-cqt-2-exe-arg-def AOT-instance-of-cqt-2-def AOT-sem-lambda-denotes*

by (*auto simp*: *AOT-model-term-equiv-part-equivp equivp-reflp AOT-sem-denotes*)
lemma *AOT-instance-of-cqt-2-intros-exe-arg-const*[*AOT-instance-of-cqt-2-intro*]:
‹*AOT-instance-of-cqt-2-exe-arg* (λ*x*. κ)›
unfolding *AOT-instance-of-cqt-2-exe-arg-def AOT-instance-of-cqt-2-def*
by (*auto simp*: *AOT-model-term-equiv-part-equivp equivp-reflp*
AOT-sem-denotes AOT-sem-lambda-denotes)
lemma *AOT-instance-of-cqt-2-intros-exe-arg-fst*[*AOT-instance-of-cqt-2-intro*]:
‹*AOT-instance-of-cqt-2-exe-arg fst*›
unfolding *AOT-instance-of-cqt-2-exe-arg-def AOT-instance-of-cqt-2-def*
by (*simp add*: *AOT-model-term-equiv-prod-def case-prod-beta*)
lemma *AOT-instance-of-cqt-2-intros-exe-arg-snd*[*AOT-instance-of-cqt-2-intro*]:
‹*AOT-instance-of-cqt-2-exe-arg snd*›
unfolding *AOT-instance-of-cqt-2-exe-arg-def AOT-instance-of-cqt-2-def*
by (*simp add*: *AOT-model-term-equiv-prod-def AOT-sem-denotes AOT-sem-lambda-denotes*)
lemma *AOT-instance-of-cqt-2-intros-exe-arg-Pair*[*AOT-instance-of-cqt-2-intro*]:
assumes ‹*AOT-instance-of-cqt-2-exe-arg* φ› **and** ‹*AOT-instance-of-cqt-2-exe-arg* ψ›
shows ‹*AOT-instance-of-cqt-2-exe-arg* (λτ. *Pair* (φ τ) (ψ τ))›
using *assms*
unfolding *AOT-instance-of-cqt-2-exe-arg-def AOT-instance-of-cqt-2-def*
AOT-sem-denotes AOT-sem-lambda-denotes AOT-model-term-equiv-prod-def
AOT-model-denotes-prod-def
by *auto*
lemma *AOT-instance-of-cqt-2-intros-desc*[*AOT-instance-of-cqt-2-intro*]:
assumes ‹⋀*z* :: *'a*::*AOT-κ*. *AOT-instance-of-cqt-2* (Φ *z*)›
shows ‹*AOT-instance-of-cqt-2-exe-arg* (λ κ :: *'b*::*AOT-κ* . «ι*z*(Φ{*z*,κ})»)›
proof −
have *0*: ‹⋀ κ κ'. *AOT-model-denotes* κ ∧ *AOT-model-denotes* κ' ∧
AOT-model-term-equiv κ κ' ⟹
Φ *z* κ = Φ *z* κ'› **for** *z*
using *assms*
unfolding *AOT-instance-of-cqt-2-def*
AOT-sem-denotes AOT-model-lambda-denotes **by** *force*
{
fix κ κ'
have ‹«ι*z*(Φ{*z*,κ})» = «ι*z*(Φ{*z*,κ'})»›
if ‹*AOT-model-denotes* κ ∧ *AOT-model-denotes* κ' ∧ *AOT-model-term-equiv* κ κ'›
using *0*[*OF that*]
by *auto*
moreover have ‹*AOT-model-term-equiv x x*› **for** *x* :: ‹*'a*::*AOT-κ*›
by (*metis AOT-instance-of-cqt-2-exe-arg-def*
AOT-instance-of-cqt-2-intros-exe-arg-const
AOT-model-A-objects AOT-model-term-equiv-denotes
AOT-model-term-equiv-eps(*1*))
ultimately have ‹*AOT-model-term-equiv* «ι*z*(Φ{*z*,κ})» «ι*z*(Φ{*z*,κ'})»›
if ‹*AOT-model-denotes* κ ∧ *AOT-model-denotes* κ' ∧ *AOT-model-term-equiv* κ κ'›
using *that* **by** *simp*
}
thus *?thesis* **using** *0*
unfolding *AOT-instance-of-cqt-2-exe-arg-def*
by *simp*
qed

lemma *AOT-instance-of-cqt-2-intros-exe-const*[*AOT-instance-of-cqt-2-intro*]:
assumes ‹*AOT-instance-of-cqt-2-exe-arg* κ*s*›
shows ‹*AOT-instance-of-cqt-2* (λ*x* :: *'b*::*AOT-*κ*s*. *AOT-exe* Π (κ*s x*))›
using *assms*
unfolding *AOT-instance-of-cqt-2-def AOT-sem-denotes AOT-model-lambda-denotes*
AOT-sem-disj AOT-sem-conj
AOT-sem-not AOT-sem-box AOT-sem-act AOT-instance-of-cqt-2-exe-arg-def
AOT-sem-equiv AOT-sem-imp AOT-sem-forall AOT-sem-exists AOT-sem-dia
by (*auto intro*!: *AOT-sem-exe-equiv*)
lemma *AOT-instance-of-cqt-2-intros-exe-lam*[*AOT-instance-of-cqt-2-intro*]:
assumes ‹⋀ *y* . *AOT-instance-of-cqt-2* (λ*x*. φ *x y*)›

```
    and ‹AOT-instance-of-cqt-2-exe-arg κs›
  shows ‹AOT-instance-of-cqt-2 (λκ₁κₙ :: 'b::AOT-κs.
          «[λν₁...νₙ φ{κ₁...κₙ,ν₁...νₙ}]«κs κ₁κₙ»»)›
proof −
  {
    fix x y :: 'b
    assume ‹AOT-model-denotes x›
    moreover assume ‹AOT-model-denotes y›
    moreover assume ‹AOT-model-term-equiv x y›
    moreover have 1: ‹φ x = φ y›
      using assms calculation unfolding AOT-instance-of-cqt-2-def by blast
    ultimately have ‹AOT-exe (AOT-lambda (φ x)) (κs x) =
                     AOT-exe (AOT-lambda (φ y)) (κs y)›
      unfolding 1
      apply (safe intro!: AOT-sem-exe-equiv)
      by (metis AOT-instance-of-cqt-2-exe-arg-def assms(2))
  }
  thus ?thesis
  unfolding AOT-instance-of-cqt-2-def
          AOT-instance-of-cqt-2-exe-arg-def
  by blast
qed
lemma AOT-instance-of-cqt-2-intro-prod[AOT-instance-of-cqt-2-intro]:
  assumes ‹⋀ x . AOT-instance-of-cqt-2 (φ x)›
    and ‹⋀ x . AOT-instance-of-cqt-2 (λ z . φ z x)›
  shows ‹AOT-instance-of-cqt-2 (λ(x,y) . φ x y)›
  using assms unfolding AOT-instance-of-cqt-2-def
  by (auto simp add: AOT-model-lambda-denotes AOT-sem-denotes
              AOT-model-denotes-prod-def
              AOT-model-term-equiv-prod-def)
```

The following are already derivable semantically, but not yet added to *AOT-instance-of-cqt-2-intro*. They will be added with the next planned extension of axiom cqt:2.

```
named-theorems AOT-instance-of-cqt-2-intro-next
definition AOT-instance-of-cqt-2-enc-arg :: ‹('a::AOT-κs ⇒ 'b::AOT-κs) ⇒ bool› where
  ‹AOT-instance-of-cqt-2-enc-arg ≡ λ φ . ∀ x y z .
     AOT-model-denotes x ∧ AOT-model-denotes y ∧ AOT-model-term-equiv x y ⟶
     AOT-enc (φ x) z = AOT-enc (φ y) z›
definition AOT-instance-of-cqt-2-enc-rel :: ‹('a::AOT-κs ⇒ <'b::AOT-κs>) ⇒ bool› where
  ‹AOT-instance-of-cqt-2-enc-rel ≡ λ φ . ∀ x y z .
     AOT-model-denotes x ∧ AOT-model-denotes y ∧ AOT-model-term-equiv x y ⟶
     AOT-enc z (φ x) = AOT-enc z (φ y)›
lemma AOT-instance-of-cqt-2-intros-enc[AOT-instance-of-cqt-2-intro-next]:
  assumes ‹AOT-instance-of-cqt-2-enc-rel Π› and ‹AOT-instance-of-cqt-2-enc-arg κs›
  shows ‹AOT-instance-of-cqt-2 (λx . AOT-enc (κs x) «[«Π x»]»)›
  using assms
  unfolding AOT-instance-of-cqt-2-def AOT-sem-denotes AOT-model-lambda-denotes
          AOT-instance-of-cqt-2-enc-rel-def AOT-sem-not AOT-sem-box AOT-sem-act
          AOT-sem-dia AOT-sem-conj AOT-sem-disj AOT-sem-equiv AOT-sem-imp
          AOT-sem-forall AOT-sem-exists AOT-instance-of-cqt-2-enc-arg-def
  by fastforce+
lemma AOT-instance-of-cqt-2-enc-arg-intro-const[AOT-instance-of-cqt-2-intro-next]:
  ‹AOT-instance-of-cqt-2-enc-arg (λx. c)›
  unfolding AOT-instance-of-cqt-2-enc-arg-def by simp
lemma AOT-instance-of-cqt-2-enc-arg-intro-desc[AOT-instance-of-cqt-2-intro-next]:
  assumes ‹⋀z :: 'a::AOT-κ. AOT-instance-of-cqt-2 (Φ z)›
  shows ‹AOT-instance-of-cqt-2-enc-arg (λ κ :: 'b::AOT-κ . «ιz(Φ{z,κ})»)›
proof −
  have 0: ‹⋀ κ κ'. AOT-model-denotes κ ∧ AOT-model-denotes κ' ∧
                AOT-model-term-equiv κ κ' ⟹
                Φ z κ = Φ z κ'› for z
    using assms
    unfolding AOT-instance-of-cqt-2-def
```

AOT-sem-denotes AOT-model-lambda-denotes **by** *force*
{
fix κ κ'
have ‹«$\iota z(\Phi\{z,\kappa\})$» = «$\iota z(\Phi\{z,\kappa'\})$»›
if ‹*AOT-model-denotes* κ ∧ *AOT-model-denotes* κ' ∧ *AOT-model-term-equiv* κ κ'›
using *0*[*OF that*]
by *auto*
}
thus *?thesis* **using** *0*
unfolding *AOT-instance-of-cqt-2-enc-arg-def* **by** *meson*
qed
lemma *AOT-instance-of-cqt-2-enc-rel-intro*[*AOT-instance-of-cqt-2-intro-next*]:
assumes ‹$\bigwedge$ κ . *AOT-instance-of-cqt-2* ($\lambda\kappa'$ :: *'b*::*AOT-κs* . φ κ κ')›
assumes ‹$\bigwedge$ κ' . *AOT-instance-of-cqt-2* ($\lambda\kappa$ :: *'a*::*AOT-κs* . φ κ κ')›
shows ‹*AOT-instance-of-cqt-2-enc-rel* ($\lambda\kappa$ :: *'a*::*AOT-κs*. *AOT-lambda* ($\lambda\kappa'$. φ κ κ'))›
proof –
{
fix x y :: *'a* **and** z ::*'b*
assume ‹*AOT-model-term-equiv x y*›
moreover assume ‹*AOT-model-denotes x*›
moreover assume ‹*AOT-model-denotes y*›
ultimately have ‹φ x = φ y›
using *assms* **unfolding** *AOT-instance-of-cqt-2-def* **by** *blast*
hence ‹*AOT-enc z* (*AOT-lambda* (φ x)) = *AOT-enc z* (*AOT-lambda* (φ y))›
by *simp*
}
thus *?thesis*
unfolding *AOT-instance-of-cqt-2-enc-rel-def* **by** *auto*
qed

Further restrict unary individual variables to type κ (rather than class *AOT-κ* only) and define being ordinary and being abstract.

AOT-register-type-constraints
Individual: ‹κ› ‹-::*AOT-κs*›

AOT-define *AOT-ordinary* :: ‹Π› (‹*O*!›) ‹*O*! $=_{df}$ [λx ◇*E*!*x*]›
declare *AOT-ordinary*[*AOT del*, *AOT-defs del*]
AOT-define *AOT-abstract* :: ‹Π› (‹*A*!›) ‹*A*! $=_{df}$ [λx ¬◇*E*!*x*]›
declare *AOT-abstract*[*AOT del*, *AOT-defs del*]

context *AOT-meta-syntax*
begin
notation *AOT-ordinary* (‹**O!**›)
notation *AOT-abstract* (‹**A!**›)
end
context *AOT-no-meta-syntax*
begin
no-notation *AOT-ordinary* (‹**O!**›)
no-notation *AOT-abstract* (‹**A!**›)
end

no-translations
-AOT-concrete => *CONST AOT-term-of-var* (*CONST AOT-concrete*)
parse-translation‹
[(***syntax-const***‹*-AOT-concrete*›, *fn* - => *fn* [] =>
Const (***const-name***‹*AOT-term-of-var*›, *dummyT*)
$ *Const* (***const-name***‹*AOT-concrete*›, ***typ***‹<κ> *AOT-var*›))]
›

Auxiliary lemmata.

lemma *AOT-sem-ordinary*: «*O*!» = «[λx ◇*E*!*x*]»
using *AOT-ordinary*[*THEN AOT-sem-id-def0E1*] *AOT-sem-ordinary-def-denotes*
by (*auto simp*: *AOT-sem-eq*)

```
lemma AOT-sem-abstract: «A!» = «[λx ¬◇E!x]»
  using AOT-abstract[THEN AOT-sem-id-def0E1]  AOT-sem-abstract-def-denotes
  by (auto simp: AOT-sem-eq)
lemma AOT-sem-ordinary-denotes: ‹[w ⊨ O!↓]›
  by (simp add: AOT-sem-ordinary AOT-sem-ordinary-def-denotes)
lemma AOT-meta-abstract-denotes: ‹[w ⊨ A!↓]›
  by (simp add: AOT-sem-abstract AOT-sem-abstract-def-denotes)
lemma AOT-model-abstract-ακ: ‹∃ a . κ = ακ a› if ‹[v ⊨ A!κ]›
  using that[unfolded AOT-sem-abstract, simplified
      AOT-meta-abstract-denotes[unfolded AOT-sem-abstract, THEN AOT-sem-lambda-beta,
          OF that[simplified AOT-sem-exe, THEN conjunct2, THEN conjunct1]]]
  apply (simp add: AOT-sem-not AOT-sem-dia AOT-sem-concrete)
  by (metis AOT-model-ω-concrete-in-some-world AOT-model-concrete-κ.simps(1)
            AOT-model-denotes-κ-def AOT-sem-denotes AOT-sem-exe κ.exhaust-disc
            is-ακ-def is-ωκ-def that)
lemma AOT-model-ordinary-ωκ: ‹∃ a . κ = ωκ a› if ‹[v ⊨ O!κ]›
  using that[unfolded AOT-sem-ordinary, simplified
      AOT-sem-ordinary-denotes[unfolded AOT-sem-ordinary, THEN AOT-sem-lambda-beta,
        OF that[simplified AOT-sem-exe, THEN conjunct2, THEN conjunct1]]]
  apply (simp add: AOT-sem-dia AOT-sem-concrete)
  by (metis AOT-model-concrete-κ.simps(2) AOT-model-concrete-κ.simps(3)
            κ.exhaust-disc is-ακ-def is-ωκ-def is-nullκ-def)
lemma AOT-model-ωκ-ordinary: ‹[v ⊨ O!«ωκ x»]›
  by (metis AOT-model-abstract-ακ AOT-model-denotes-κ-def AOT-sem-abstract
            AOT-sem-denotes AOT-sem-ind-eq AOT-sem-ordinary κ.disc(7) κ.distinct(1))
lemma AOT-model-ακ-ordinary: ‹[v ⊨ A!«ακ x»]›
  by (metis AOT-model-denotes-κ-def AOT-model-ordinary-ωκ AOT-sem-abstract
            AOT-sem-denotes AOT-sem-ind-eq AOT-sem-ordinary κ.disc(8) κ.distinct(1))
AOT-theorem prod-denotesE: assumes ‹«(κ1,κ2)»↓› shows ‹κ1↓ & κ2↓›
  using assms by (simp add: AOT-sem-denotes AOT-sem-conj AOT-model-denotes-prod-def)
declare prod-denotesE[AOT del]
AOT-theorem prod-denotesI: assumes ‹κ1↓ & κ2↓› shows ‹«(κ1,κ2)»↓›
  using assms by (simp add: AOT-sem-denotes AOT-sem-conj AOT-model-denotes-prod-def)
declare prod-denotesI[AOT del]
```

Prepare the derivation of the additional axioms that are validated by our extended models.

```
locale AOT-ExtendedModel =
  assumes AOT-ExtendedModel: ‹AOT-ExtendedModel›
begin
lemma AOT-sem-indistinguishable-ord-enc-all:
  assumes Π-den: ‹[v ⊨ Π↓]›
  assumes Ax: ‹[v ⊨ A!x]›
  assumes Ay: ‹[v ⊨ A!y]›
  assumes indist: ‹[v ⊨ ∀ F □([F]x ≡ [F]y)]›
  shows
  ‹[v ⊨ ∀ G(∀ z(O!z → □([G]z ≡ [Π]z)) → x[G])] =
   [v ⊨ ∀ G(∀ z(O!z → □([G]z ≡ [Π]z)) → y[G])]›
proof −
    have 0: ‹[v ⊨ [λx ¬◇[E!]x]x]›
      using Ax by (simp add: AOT-sem-abstract)
    have 1: ‹[v ⊨ [λx ¬◇[E!]x]y]›
      using Ay by (simp add: AOT-sem-abstract)
    {
      assume ‹[v ⊨ ∀ G(∀ z (O!z → □([G]z ≡ [Π]z)) → x[G])]›
      hence 3: ‹[v ⊨ ∀ G(∀ z([λx ◇[E!]x]z → □([G]z ≡ [Π]z)) → x[G])]›
        by (simp add: AOT-sem-ordinary)
      {
        fix Π' :: ‹<κ>›
        assume 1: ‹[v ⊨ Π'↓]›
        assume 2: ‹[v ⊨ [λx ◇[E!]x]z → □([Π']z ≡ [Π]z)]› for z
        have ‹[v ⊨ x[Π']]›
          using 3
          by (auto simp: AOT-sem-forall AOT-sem-imp AOT-sem-box AOT-sem-denotes)
```

```
      (metis (no-types, lifting) 1 2 AOT-term-of-var-cases AOT-sem-box
                               AOT-sem-denotes AOT-sem-imp)
  } note 3 = this
  fix Π' :: ‹<κ>›
  assume Π-den: ‹[v ⊨ Π'↓]›
  assume 4: ‹[v ⊨ ∀ z (O!z → □([Π']z ≡ [Π]z))]›
  {
    fix κ₀
    assume ‹[v ⊨ [λx ◇[E!]x]κ₀]›
    hence ‹[v ⊨ O!κ₀]›
      using AOT-sem-ordinary by metis
    moreover have ‹[v ⊨ κ₀↓]›
      using calculation by (simp add: AOT-sem-exe)
    ultimately have ‹[v ⊨ □([Π']κ₀ ≡ [Π]κ₀)]›
      using 4 by (auto simp: AOT-sem-forall AOT-sem-imp)
  } note 4 = this
  {
    fix Π'' :: ‹<κ>›
    assume ‹[v ⊨ Π''↓]›
    moreover assume ‹[w ⊨ [Π'']κ₀] = [w ⊨ [Π']κ₀]› if ‹[v ⊨ [λx ◇E!x]κ₀]› for κ₀ w
    ultimately have 5: ‹[v ⊨ x[Π'']]›
      using 4 3
      by (auto simp: AOT-sem-imp AOT-sem-equiv AOT-sem-box)
  } note 5 = this
  have ‹[v ⊨ y[Π']]›
    apply (rule AOT-sem-enc-indistinguishable-all[OF AOT-ExtendedModel])
    apply (fact 0)
    by (auto simp: 5 0 1 Π-den indist[simplified AOT-sem-forall
                AOT-sem-box AOT-sem-equiv])
}
moreover {
{
  assume ‹[v ⊨ ∀ G(∀ z (O!z → □([G]z ≡ [Π]z)) → y[G])]›
  hence 3: ‹[v ⊨ ∀ G(∀ z ([λx ◇[E!]x]z → □([G]z ≡ [Π]z)) → y[G])]›
    by (simp add: AOT-sem-ordinary)
  {
    fix Π' :: ‹<κ>›
    assume 1: ‹[v ⊨ Π'↓]›
    assume 2: ‹[v ⊨ [λx ◇[E!]x]z → □([Π']z ≡ [Π]z)]› for z
    have ‹[v ⊨ y[Π']]›
      using 3
      apply (simp add: AOT-sem-forall AOT-sem-imp AOT-sem-box AOT-sem-denotes)
      by (metis (no-types, lifting) 1 2 AOT-model.AOT-term-of-var-cases
                               AOT-sem-box AOT-sem-denotes AOT-sem-imp)
  } note 3 = this
  fix Π' :: ‹<κ>›
  assume Π-den: ‹[v ⊨ Π'↓]›
  assume 4: ‹[v ⊨ ∀ z (O!z → □([Π']z ≡ [Π]z))]›
  {
    fix κ₀
    assume ‹[v ⊨ [λx ◇[E!]x]κ₀]›
    hence ‹[v ⊨ O!κ₀]›
      using AOT-sem-ordinary by metis
    moreover have ‹[v ⊨ κ₀↓]›
      using calculation by (simp add: AOT-sem-exe)
    ultimately have ‹[v ⊨ □([Π']κ₀ ≡ [Π]κ₀)]›
      using 4 by (auto simp: AOT-sem-forall AOT-sem-imp)
  } note 4 = this
  {
    fix Π'' :: ‹<κ>›
    assume ‹[v ⊨ Π''↓]›
    moreover assume ‹[w ⊨ [Π'']κ₀] = [w ⊨ [Π']κ₀]› if ‹[v ⊨ [λx ◇E!x]κ₀]› for w κ₀
    ultimately have ‹[v ⊨ y[Π'']]›
```

```
        using 3 4 by (auto simp: AOT-sem-imp AOT-sem-equiv AOT-sem-box)
    } note 5 = this
    have ‹[v ⊨ x[Π']]›
      apply (rule AOT-sem-enc-indistinguishable-all[OF AOT-ExtendedModel])
           apply (fact 1)
      by (auto simp: 5 0 1 Π-den indist[simplified AOT-sem-forall
                    AOT-sem-box AOT-sem-equiv])
    }
  }
  ultimately show ‹[v ⊨ ∀ G (∀ z (O!z → □([G]z ≡ [Π]z)) → x[G])] =
        [v ⊨ ∀ G (∀ z (O!z → □([G]z ≡ [Π]z)) → y[G])]›
    by (auto simp: AOT-sem-forall AOT-sem-imp)
qed

lemma AOT-sem-indistinguishable-ord-enc-ex:
  assumes Π-den: ‹[v ⊨ Π↓]›
  assumes Ax: ‹[v ⊨ A!x]›
  assumes Ay: ‹[v ⊨ A!y]›
  assumes indist: ‹[v ⊨ ∀ F □([F]x ≡ [F]y)]›
  shows ‹[v ⊨ ∃ G(∀ z (O!z → □([G]z ≡ [Π]z)) & x[G])] =
         [v ⊨ ∃ G(∀ z(O!z → □([G]z ≡ [Π]z)) & y[G])]›
proof –
  have Aux: ‹[v ⊨ [λx ◇[E!]x]κ] = ([v ⊨ [λx ◇[E!]x]κ] ∧ [v ⊨ κ↓])› for v κ
    using AOT-sem-exe by blast
  AOT-modally-strict {
    fix x y
    AOT-assume Π-den: ‹[Π]↓›
    AOT-assume 2: ‹∀ F □([F]x ≡ [F]y)›
    AOT-assume ‹A!x›
    AOT-hence 0: ‹[λx ¬◇[E!]x]x›
      by (simp add: AOT-sem-abstract)
    AOT-assume ‹A!y›
    AOT-hence 1: ‹[λx ¬◇[E!]x]y›
      by (simp add: AOT-sem-abstract)
    {
      AOT-assume ‹∃ G(∀ z (O!z → □([G]z ≡ [Π]z)) & x[G])›
      then AOT-obtain Π'
        where Π'-den: ‹Π'↓›
          and Π'-indist: ‹∀ z (O!z → □([Π']z ≡ [Π]z))›
          and x-enc-Π': ‹x[Π']›
        by (meson AOT-sem-conj AOT-sem-exists)
      {
        fix κ₀
        AOT-assume ‹[λx ◇[E!]x]κ₀›
        AOT-hence ‹□([Π']κ₀ ≡ [Π]κ₀)›
          using Π'-indist
          by (auto simp: AOT-sem-exe AOT-sem-imp AOT-sem-exists AOT-sem-conj
                    AOT-sem-ordinary AOT-sem-forall)
      } note 3 = this
      AOT-have ‹∀ z ([λx ◇[E!]x]z → □([Π']z ≡ [Π]z))›
        using Π'-indist by (simp add: AOT-sem-ordinary)
      AOT-obtain Π'' where
          Π''-den: ‹Π''↓› and
          Π''-indist: ‹[λx ◇[E!]x]κ₀ → □([Π'']κ₀ ≡ [Π]κ₀)› and
          y-enc-Π'': ‹y[Π'']› for κ₀
        using AOT-sem-enc-indistinguishable-ex[OF AOT-ExtendedModel,
                OF 0, OF 1, rotated, OF Π-den,
                OF exI[where x=Π'], OF conjI, OF Π'-den, OF conjI,
                OF x-enc-Π', OF allI, OF impI,
                OF 3[simplified AOT-sem-box AOT-sem-equiv], simplified, OF
                2[simplified AOT-sem-forall AOT-sem-equiv AOT-sem-box,
                  THEN spec, THEN mp, THEN spec], simplified]
        unfolding AOT-sem-imp AOT-sem-box AOT-sem-equiv by blast
```

{
  AOT-have ‹$\Pi''\downarrow$›
    and ‹$\forall x$ ($[\lambda x\ \Diamond[E!]x]x \rightarrow \Box([\Pi'']x \equiv [\Pi]x)$)›
    and ‹$y[\Pi'']$›
   apply (*simp add*: Π''*-den*)
   apply (*simp add*: *AOT-sem-forall* Π''*-indist*)
   by (*simp add*: *y-enc-*Π'')
} **note** *2* = *this*
AOT-have ‹$\exists G(\forall z\ (O!z \rightarrow \Box([G]z \equiv [\Pi]z))\ \&\ y[G])$›
  apply (*simp add*: *AOT-sem-exists AOT-sem-ordinary*
    AOT-sem-imp AOT-sem-box AOT-sem-forall AOT-sem-equiv AOT-sem-conj)
  using *2*[*simplified AOT-sem-box AOT-sem-equiv AOT-sem-imp AOT-sem-forall*]
  by *blast*
}
} **note** *0* = *this*
AOT-modally-strict {
 {
  fix *x y*
  AOT-assume Π-*den*: ‹$[\Pi]\downarrow$›
  moreover AOT-assume ‹$\forall F\ \Box([F]x \equiv [F]y)$›
  moreover AOT-have ‹$\forall F\ \Box([F]y \equiv [F]x)$›
   using *calculation*(*2*)
   by (*auto simp*: *AOT-sem-forall AOT-sem-box AOT-sem-equiv*)
  moreover AOT-assume ‹$A!x$›
  moreover AOT-assume ‹$A!y$›
  ultimately AOT-have ‹$\exists G\ (\forall z\ (O!z \rightarrow \Box([G]z \equiv [\Pi]z))\ \&\ x[G]) \equiv$
      $\exists G\ (\forall z\ (O!z \rightarrow \Box([G]z \equiv [\Pi]z))\ \&\ y[G])$›
   using *0* **by** (*auto simp*: *AOT-sem-equiv*)
 }
 have *1*: ‹$[v \models \forall F\ \Box([F]y \equiv [F]x)]$›
  using *indist*
  by (*auto simp*: *AOT-sem-forall AOT-sem-box AOT-sem-equiv*)
 thus ‹$[v \models \exists G\ (\forall z\ (O!z \rightarrow \Box([G]z \equiv [\Pi]z))\ \&\ x[G])] =$
   $[v \models \exists G\ (\forall z\ (O!z \rightarrow \Box([G]z \equiv [\Pi]z))\ \&\ y[G])]$›
  using *assms*
  by (*auto simp*: *AOT-sem-imp AOT-sem-conj AOT-sem-equiv 0*)
}
qed
end

setup‹*setup-AOT-no-atp*›
bundle *AOT-no-atp* **begin declare** *AOT-no-atp*[*no-atp*] **end**

theory *AOT-Definitions*
 imports *AOT-semantics*
begin

6 Definitions of AOT

AOT-theorem *conventions*:*1*: ‹$\varphi\ \&\ \psi \equiv_{df} \neg(\varphi \rightarrow \neg\psi)$›
 using *AOT-conj*.
AOT-theorem *conventions*:*2*: ‹$\varphi \vee \psi \equiv_{df} \neg\varphi \rightarrow \psi$›
 using *AOT-disj*.
AOT-theorem *conventions*:*3*: ‹$\varphi \equiv \psi \equiv_{df} (\varphi \rightarrow \psi)\ \&\ (\psi \rightarrow \varphi)$›
 using *AOT-equiv*.
AOT-theorem *conventions*:*4*: ‹$\exists \alpha\ \varphi\{\alpha\} \equiv_{df} \neg\forall \alpha\ \neg\varphi\{\alpha\}$›
 using *AOT-exists*.
AOT-theorem *conventions*:*5*: ‹$\Diamond\varphi \equiv_{df} \neg\Box\neg\varphi$›
 using *AOT-dia*.

declare *conventions:1[AOT-defs]* *conventions:2[AOT-defs]*
conventions:3[AOT-defs] *conventions:4[AOT-defs]*
conventions:5[AOT-defs]

notepad
begin
fix φ ψ χ

have *conventions3[1]*: ‹«$\varphi \to \psi \equiv \neg\psi \to \neg\varphi$» = «$(\varphi \to \psi) \equiv (\neg\psi \to \neg\varphi)$»›
by *blast*
have *conventions3[2]*: ‹«φ & $\psi \to \chi$» = «$(\varphi$ & $\psi) \to \chi$»›
and ‹«$\varphi \vee \psi \to \chi$» = «$(\varphi \vee \psi) \to \chi$»›
by *blast+*
have *conventions3[3]*: ‹«$\varphi \vee \psi$ & χ» = «$(\varphi \vee \psi)$ & χ»›
and ‹«φ & $\psi \vee \chi$» = «$(\varphi$ & $\psi) \vee \chi$»›
by *blast+* — Note that PLM instead generally uses parenthesis in these cases.
end

AOT-theorem *existence:1*: ‹$\kappa\downarrow \equiv_{df} \exists F\ [F]\kappa$›
by (*simp add*: *AOT-sem-denotes AOT-sem-exists AOT-model-equiv-def*)
(*metis AOT-sem-denotes AOT-sem-exe AOT-sem-lambda-beta AOT-sem-lambda-denotes*)
AOT-theorem *existence:2*: ‹$\Pi\downarrow \equiv_{df} \exists x_1 ... \exists x_n\ x_1 ... x_n[\Pi]$›
using *AOT-sem-denotes AOT-sem-enc-denotes AOT-sem-universal-encoder*
by (*simp add*: *AOT-sem-denotes AOT-sem-exists AOT-model-equiv-def*) *blast*
AOT-theorem *existence:2[1]*: ‹$\Pi\downarrow \equiv_{df} \exists x\ x[\Pi]$›
using *existence:2[of* Π*]* **by** *simp*
AOT-theorem *existence:2[2]*: ‹$\Pi\downarrow \equiv_{df} \exists x \exists y\ xy[\Pi]$›
using *existence:2[of* Π*]*
by (*simp add*: *AOT-sem-denotes AOT-sem-exists AOT-model-equiv-def*
AOT-model-denotes-prod-def)
AOT-theorem *existence:2[3]*: ‹$\Pi\downarrow \equiv_{df} \exists x \exists y \exists z\ xyz[\Pi]$›
using *existence:2[of* Π*]*
by (*simp add*: *AOT-sem-denotes AOT-sem-exists AOT-model-equiv-def*
AOT-model-denotes-prod-def)
AOT-theorem *existence:2[4]*: ‹$\Pi\downarrow \equiv_{df} \exists x_1 \exists x_2 \exists x_3 \exists x_4\ x_1x_2x_3x_4[\Pi]$›
using *existence:2[of* Π*]*
by (*simp add*: *AOT-sem-denotes AOT-sem-exists AOT-model-equiv-def*
AOT-model-denotes-prod-def)

AOT-theorem *existence:3*: ‹$\varphi\downarrow \equiv_{df} [\lambda x\ \varphi]\downarrow$›
by (*simp add*: *AOT-sem-denotes AOT-model-denotes-*o*-def AOT-model-equiv-def*
AOT-model-lambda-denotes)

declare *existence:1[AOT-defs]* *existence:2[AOT-defs]* *existence:2[1][AOT-defs]*
existence:2[2][AOT-defs] *existence:2[3][AOT-defs]*
existence:2[4][AOT-defs] *existence:3[AOT-defs]*

AOT-theorem *oa:1*: ‹$O! =_{df} [\lambda x\ \Diamond E!x]$› **using** *AOT-ordinary* **.**
AOT-theorem *oa:2*: ‹$A! =_{df} [\lambda x\ \neg\Diamond E!x]$› **using** *AOT-abstract* **.**

declare *oa:1[AOT-defs]* *oa:2[AOT-defs]*

AOT-theorem *identity:1*:
‹$x = y \equiv_{df} ([O!]x$ & $[O!]y$ & $\Box\forall F\ ([F]x \equiv [F]y)) \vee$
$([A!]x$ & $[A!]y$ & $\Box\forall F\ (x[F] \equiv y[F]))$›
unfolding *AOT-model-equiv-def*
using *AOT-sem-ind-eq[of - x y]*
by (*simp add*: *AOT-sem-ordinary AOT-sem-abstract AOT-sem-conj*
AOT-sem-box AOT-sem-equiv AOT-sem-forall AOT-sem-disj AOT-sem-eq
AOT-sem-denotes)

AOT-theorem *identity:2*:

‹$F = G \equiv_{df} F\downarrow$ & $G\downarrow$ & $\Box\forall x(x[F] \equiv x[G])$›
using *AOT-sem-enc-eq*[*of - F G*]
by (*auto simp*: *AOT-model-equiv-def AOT-sem-imp AOT-sem-denotes AOT-sem-eq AOT-sem-conj AOT-sem-forall AOT-sem-box AOT-sem-equiv*)

AOT-theorem *identity*:*3*[*2*]:
‹$F = G \equiv_{df} F\downarrow$ & $G\downarrow$ & $\forall y([\lambda z\ [F]zy] = [\lambda z\ [G]zy]$ & $[\lambda z\ [F]yz] = [\lambda z\ [G]yz])$›
by (*auto simp*: *AOT-model-equiv-def AOT-sem-proj-id-prop*[*of - F G*] *AOT-sem-proj-id-prod-def AOT-sem-conj AOT-sem-denotes AOT-sem-forall AOT-sem-unary-proj-id AOT-model-denotes-prod-def*)
AOT-theorem *identity*:*3*[*3*]:
‹$F = G \equiv_{df} F\downarrow$ & $G\downarrow$ & $\forall y_1\forall y_2([\lambda z\ [F]zy_1y_2] = [\lambda z\ [G]zy_1y_2]$ &
$[\lambda z\ [F]y_1zy_2] = [\lambda z\ [G]y_1zy_2]$ &
$[\lambda z\ [F]y_1y_2z] = [\lambda z\ [G]y_1y_2z])$›
by (*auto simp*: *AOT-model-equiv-def AOT-sem-proj-id-prop*[*of - F G*] *AOT-sem-proj-id-prod-def AOT-sem-conj AOT-sem-denotes AOT-sem-forall AOT-sem-unary-proj-id AOT-model-denotes-prod-def*)
AOT-theorem *identity*:*3*[*4*]:
‹$F = G \equiv_{df} F\downarrow$ & $G\downarrow$ & $\forall y_1\forall y_2\forall y_3([\lambda z\ [F]zy_1y_2y_3] = [\lambda z\ [G]zy_1y_2y_3]$ &
$[\lambda z\ [F]y_1zy_2y_3] = [\lambda z\ [G]y_1zy_2y_3]$ &
$[\lambda z\ [F]y_1y_2zy_3] = [\lambda z\ [G]y_1y_2zy_3]$ &
$[\lambda z\ [F]y_1y_2y_3z] = [\lambda z\ [G]y_1y_2y_3z])$›
by (*auto simp*: *AOT-model-equiv-def AOT-sem-proj-id-prop*[*of - F G*] *AOT-sem-proj-id-prod-def AOT-sem-conj AOT-sem-denotes AOT-sem-forall AOT-sem-unary-proj-id AOT-model-denotes-prod-def*)
AOT-theorem *identity*:*3*:
‹$F = G \equiv_{df} F\downarrow$ & $G\downarrow$ & $\forall x_1 ... \forall x_n$ «*AOT-sem-proj-id* x_1x_n ($\lambda\ \tau$. *AOT-exe* $F\ \tau$)
($\lambda\ \tau$. *AOT-exe* $G\ \tau$)»›
by (*auto simp*: *AOT-model-equiv-def AOT-sem-proj-id-prop*[*of - F G*] *AOT-sem-proj-id-prod-def AOT-sem-conj AOT-sem-denotes AOT-sem-forall AOT-sem-unary-proj-id AOT-model-denotes-prod-def*)

AOT-theorem *identity*:*4*:
‹$p = q \equiv_{df} p\downarrow$ & $q\downarrow$ & $[\lambda x\ p] = [\lambda x\ q]$›
by (*auto simp*: *AOT-model-equiv-def AOT-sem-eq AOT-sem-denotes AOT-sem-conj AOT-model-lambda-denotes AOT-sem-lambda-eq-prop-eq*)

declare *identity*:*1*[*AOT-defs*] *identity*:*2*[*AOT-defs*] *identity*:*3*[*2*][*AOT-defs*] *identity*:*3*[*3*][*AOT-defs*] *identity*:*3*[*4*][*AOT-defs*] *identity*:*3*[*AOT-defs*] *identity*:*4*[*AOT-defs*]

AOT-define *AOT-nonidentical* :: ‹$\tau \Rightarrow \tau \Rightarrow \varphi$› (**infixl** ‹$\neq$› *50*)
=−infix: ‹$\tau \neq \sigma \equiv_{df} \neg(\tau = \sigma)$›

context *AOT-meta-syntax*
begin
notation *AOT-nonidentical* (**infixl** ‹$\neq$› *50*)
end
context *AOT-no-meta-syntax*
begin
no-notation *AOT-nonidentical* (**infixl** ‹$\neq$› *50*)
end

The following are purely technical pseudo-definitions required due to our internal implementation of n-ary relations and ellipses using tuples.

AOT-theorem *tuple-denotes*: ‹«(τ,τ')»$\downarrow \equiv_{df} \tau\downarrow$ & $\tau'\downarrow$›
by (*simp add*: *AOT-model-denotes-prod-def AOT-model-equiv-def AOT-sem-conj AOT-sem-denotes*)
AOT-theorem *tuple-identity-1*: ‹«(τ,τ')» = «(σ, σ')» $\equiv_{df} (\tau = \sigma)$ & $(\tau' = \sigma')$›
by (*auto simp*: *AOT-model-equiv-def AOT-sem-conj AOT-sem-eq AOT-model-denotes-prod-def AOT-sem-denotes*)
AOT-theorem *tuple-forall*: ‹$\forall\alpha_1 ... \forall\alpha_n\ \varphi\{\alpha_1 ... \alpha_n\} \equiv_{df} \forall\alpha_1(\forall\alpha_2 ... \forall\alpha_n\ \varphi\{$«$(\alpha_1,\ \alpha_2\alpha_n)$»$\})$›
by (*auto simp*: *AOT-model-equiv-def AOT-sem-forall AOT-sem-denotes*

AOT-model-denotes-prod-def)
AOT-theorem *tuple-exists*: ‹$\exists\,\alpha_1...\exists\,\alpha_n\ \varphi\{\alpha_1...\alpha_n\} \equiv_{df} \exists\,\alpha_1(\exists\,\alpha_2...\exists\,\alpha_n\ \varphi\{«(\alpha_1,\ \alpha_2\alpha_n)»\})$›
by (*auto simp*: *AOT-model-equiv-def AOT-sem-exists AOT-sem-denotes*
AOT-model-denotes-prod-def)
declare *tuple-denotes*[*AOT-defs*] *tuple-identity-1*[*AOT-defs*] *tuple-forall*[*AOT-defs*]
tuple-exists[*AOT-defs*]

end

7 Axioms of PLM

AOT-axiom *pl*:*1*: ‹$\varphi \to (\psi \to \varphi)$›
by (*auto simp*: *AOT-sem-imp AOT-model-axiomI*)
AOT-axiom *pl*:*2*: ‹$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$›
by (*auto simp*: *AOT-sem-imp AOT-model-axiomI*)
AOT-axiom *pl*:*3*: ‹$(\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$›
by (*auto simp*: *AOT-sem-imp AOT-sem-not AOT-model-axiomI*)

AOT-axiom *cqt*:*1*: ‹$\forall\,\alpha\ \varphi\{\alpha\} \to (\tau\downarrow \to \varphi\{\tau\})$›
by (*auto simp*: *AOT-sem-denotes AOT-sem-forall AOT-sem-imp AOT-model-axiomI*)

AOT-axiom *cqt*:*2*[*const-var*]: ‹$\alpha\downarrow$›
using *AOT-sem-vars-denote* **by** (*rule AOT-model-axiomI*)
AOT-axiom *cqt*:*2*[*lambda*]:
assumes ‹*INSTANCE-OF-CQT-2*(φ)›
shows ‹$[\lambda\nu_1...\nu_n\ \varphi\{\nu_1...\nu_n\}]\downarrow$›
by (*auto intro*!: *AOT-model-axiomI AOT-sem-cqt-2*[*OF assms*])
AOT-axiom *cqt*:*2*[*lambda0*]:
shows ‹$[\lambda\ \varphi]\downarrow$›
by (*auto intro*!: *AOT-model-axiomI*
simp: *AOT-sem-lambda-denotes existence*:*3*[*unfolded AOT-model-equiv-def*])

AOT-axiom *cqt*:*3*: ‹$\forall\,\alpha\ (\varphi\{\alpha\} \to \psi\{\alpha\}) \to (\forall\,\alpha\ \varphi\{\alpha\} \to \forall\,\alpha\ \psi\{\alpha\})$›
by (*simp add*: *AOT-sem-forall AOT-sem-imp AOT-model-axiomI*)
AOT-axiom *cqt*:*4*: ‹$\varphi \to \forall\,\alpha\ \varphi$›
by (*simp add*: *AOT-sem-forall AOT-sem-imp AOT-model-axiomI*)
AOT-axiom *cqt*:*5*:*a*: ‹$[\Pi]\kappa_1...\kappa_n \to (\Pi\downarrow\ \&\ \kappa_1...\kappa_n\downarrow)$›
by (*simp add*: *AOT-sem-conj AOT-sem-denotes AOT-sem-exe*
AOT-sem-imp AOT-model-axiomI)
AOT-axiom *cqt*:*5*:*a*[*1*]: ‹$[\Pi]\kappa \to (\Pi\downarrow\ \&\ \kappa\downarrow)$›
using *cqt*:*5*:*a AOT-model-axiomI* **by** *blast*
AOT-axiom *cqt*:*5*:*a*[*2*]: ‹$[\Pi]\kappa_1\kappa_2 \to (\Pi\downarrow\ \&\ \kappa_1\downarrow\ \&\ \kappa_2\downarrow)$›
by (*rule AOT-model-axiomI*)
(*metis AOT-model-denotes-prod-def AOT-sem-conj AOT-sem-denotes AOT-sem-exe*
AOT-sem-imp case-prodD)
AOT-axiom *cqt*:*5*:*a*[*3*]: ‹$[\Pi]\kappa_1\kappa_2\kappa_3 \to (\Pi\downarrow\ \&\ \kappa_1\downarrow\ \&\ \kappa_2\downarrow\ \&\ \kappa_3\downarrow)$›
by (*rule AOT-model-axiomI*)
(*metis AOT-model-denotes-prod-def AOT-sem-conj AOT-sem-denotes AOT-sem-exe*
AOT-sem-imp case-prodD)
AOT-axiom *cqt*:*5*:*a*[*4*]: ‹$[\Pi]\kappa_1\kappa_2\kappa_3\kappa_4 \to (\Pi\downarrow\ \&\ \kappa_1\downarrow\ \&\ \kappa_2\downarrow\ \&\ \kappa_3\downarrow\ \&\ \kappa_4\downarrow)$›
by (*rule AOT-model-axiomI*)
(*metis AOT-model-denotes-prod-def AOT-sem-conj AOT-sem-denotes AOT-sem-exe*
AOT-sem-imp case-prodD)
AOT-axiom *cqt*:*5*:*b*: ‹$\kappa_1...\kappa_n[\Pi] \to (\Pi\downarrow\ \&\ \kappa_1...\kappa_n\downarrow)$›
using *AOT-sem-enc-denotes*
by (*auto intro*!: *AOT-model-axiomI simp*: *AOT-sem-conj AOT-sem-denotes AOT-sem-imp*)+
AOT-axiom *cqt*:*5*:*b*[*1*]: ‹$\kappa[\Pi] \to (\Pi\downarrow\ \&\ \kappa\downarrow)$›
using *cqt*:*5*:*b AOT-model-axiomI* **by** *blast*
AOT-axiom *cqt*:*5*:*b*[*2*]: ‹$\kappa_1\kappa_2[\Pi] \to (\Pi\downarrow\ \&\ \kappa_1\downarrow\ \&\ \kappa_2\downarrow)$›
by (*rule AOT-model-axiomI*)
(*metis AOT-model-denotes-prod-def AOT-sem-conj AOT-sem-denotes*
AOT-sem-enc-denotes AOT-sem-imp case-prodD)
AOT-axiom *cqt*:*5*:*b*[*3*]: ‹$\kappa_1\kappa_2\kappa_3[\Pi] \to (\Pi\downarrow\ \&\ \kappa_1\downarrow\ \&\ \kappa_2\downarrow\ \&\ \kappa_3\downarrow)$›

by (*rule AOT-model-axiomI*)
(*metis AOT-model-denotes-prod-def AOT-sem-conj AOT-sem-denotes AOT-sem-enc-denotes AOT-sem-imp case-prodD*)
AOT-axiom *cqt:5:b[4]*: ‹$\kappa_1\kappa_2\kappa_3\kappa_4[\Pi] \rightarrow (\Pi\downarrow \;\&\; \kappa_1\downarrow \;\&\; \kappa_2\downarrow \;\&\; \kappa_3\downarrow \;\&\; \kappa_4\downarrow)$›
by (*rule AOT-model-axiomI*)
(*metis AOT-model-denotes-prod-def AOT-sem-conj AOT-sem-denotes AOT-sem-enc-denotes AOT-sem-imp case-prodD*)

AOT-axiom *l−identity*: ‹$\alpha = \beta \rightarrow (\varphi\{\alpha\} \rightarrow \varphi\{\beta\})$›
by (*rule AOT-model-axiomI*)
(*simp add*: *AOT-sem-eq AOT-sem-imp*)

AOT-act-axiom *logic−actual*: ‹$\boldsymbol{\mathcal{A}}\varphi \rightarrow \varphi$›
by (*rule AOT-model-act-axiomI*)
(*simp add*: *AOT-sem-act AOT-sem-imp*)

AOT-axiom *logic−actual−nec:1*: ‹$\boldsymbol{\mathcal{A}}\neg\varphi \equiv \neg\boldsymbol{\mathcal{A}}\varphi$›
by (*rule AOT-model-axiomI*)
(*simp add*: *AOT-sem-act AOT-sem-equiv AOT-sem-not*)
AOT-axiom *logic−actual−nec:2*: ‹$\boldsymbol{\mathcal{A}}(\varphi \rightarrow \psi) \equiv (\boldsymbol{\mathcal{A}}\varphi \rightarrow \boldsymbol{\mathcal{A}}\psi)$›
by (*rule AOT-model-axiomI*)
(*simp add*: *AOT-sem-act AOT-sem-equiv AOT-sem-imp*)

AOT-axiom *logic−actual−nec:3*: ‹$\boldsymbol{\mathcal{A}}(\forall\,\alpha\;\varphi\{\alpha\}) \equiv \forall\,\alpha\;\boldsymbol{\mathcal{A}}\varphi\{\alpha\}$›
by (*rule AOT-model-axiomI*)
(*simp add*: *AOT-sem-act AOT-sem-equiv AOT-sem-forall AOT-sem-denotes*)
AOT-axiom *logic−actual−nec:4*: ‹$\boldsymbol{\mathcal{A}}\varphi \equiv \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}\varphi$›
by (*rule AOT-model-axiomI*)
(*simp add*: *AOT-sem-act AOT-sem-equiv*)

AOT-axiom *qml:1*: ‹$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$›
by (*rule AOT-model-axiomI*)
(*simp add*: *AOT-sem-box AOT-sem-imp*)
AOT-axiom *qml:2*: ‹$\Box\varphi \rightarrow \varphi$›
by (*rule AOT-model-axiomI*)
(*simp add*: *AOT-sem-box AOT-sem-imp*)
AOT-axiom *qml:3*: ‹$\Diamond\varphi \rightarrow \Box\Diamond\varphi$›
by (*rule AOT-model-axiomI*)
(*simp add*: *AOT-sem-box AOT-sem-dia AOT-sem-imp*)

AOT-axiom *qml:4*: ‹$\Diamond\exists\,x\;(E!x \;\&\; \neg\boldsymbol{\mathcal{A}}E!x)$›
using *AOT-sem-concrete AOT-model-contingent*
by (*auto intro*!: *AOT-model-axiomI*
simp: *AOT-sem-box AOT-sem-dia AOT-sem-imp AOT-sem-exists AOT-sem-denotes AOT-sem-conj AOT-sem-not AOT-sem-act AOT-sem-exe*)+

AOT-axiom *qml−act:1*: ‹$\boldsymbol{\mathcal{A}}\varphi \rightarrow \Box\boldsymbol{\mathcal{A}}\varphi$›
by (*rule AOT-model-axiomI*)
(*simp add*: *AOT-sem-act AOT-sem-box AOT-sem-imp*)
AOT-axiom *qml−act:2*: ‹$\Box\varphi \equiv \boldsymbol{\mathcal{A}}\Box\varphi$›
by (*rule AOT-model-axiomI*)
(*simp add*: *AOT-sem-act AOT-sem-box AOT-sem-equiv*)

AOT-axiom *descriptions*: ‹$x = \iota x(\varphi\{x\}) \equiv \forall\,z(\boldsymbol{\mathcal{A}}\varphi\{z\} \equiv z = x)$›
proof (*rule AOT-model-axiomI*)
AOT-modally-strict {
AOT-show ‹$x = \iota x(\varphi\{x\}) \equiv \forall\,z(\boldsymbol{\mathcal{A}}\varphi\{z\} \equiv z = x)$›
by (*induct*; *simp add*: *AOT-sem-equiv AOT-sem-forall AOT-sem-act AOT-sem-eq*)
(*metis* (*no-types*, *opaque-lifting*) *AOT-sem-desc-denotes AOT-sem-desc-prop AOT-sem-denotes*)
}
qed

AOT-axiom *lambda−predicates:1*:
‹$[\lambda\nu_1...\nu_n\ \varphi\{\nu_1...\nu_n\}]\downarrow \rightarrow [\lambda\nu_1...\nu_n\ \varphi\{\nu_1...\nu_n\}] = [\lambda\mu_1...\mu_n\ \varphi\{\mu_1...\mu_n\}]$›
by (*rule AOT-model-axiomI*)
(*simp add*: *AOT-sem-denotes AOT-sem-eq AOT-sem-imp*)
AOT-axiom *lambda−predicates:1[zero]*: ‹$[\lambda\ p]\downarrow \rightarrow [\lambda\ p] = [\lambda\ p]$›
by (*rule AOT-model-axiomI*)
(*simp add*: *AOT-sem-denotes AOT-sem-eq AOT-sem-imp*)
AOT-axiom *lambda−predicates:2*:
‹$[\lambda x_1...x_n\ \varphi\{x_1...x_n\}]\downarrow \rightarrow ([\lambda x_1...x_n\ \varphi\{x_1...x_n\}]x_1...x_n \equiv \varphi\{x_1...x_n\})$›
by (*rule AOT-model-axiomI*)
(*simp add*: *AOT-sem-equiv AOT-sem-imp AOT-sem-lambda-beta AOT-sem-vars-denote*)
AOT-axiom *lambda−predicates:3*: ‹$[\lambda x_1...x_n\ [F]x_1...x_n] = F$›
by (*rule AOT-model-axiomI*)
(*simp add*: *AOT-sem-lambda-eta AOT-sem-vars-denote*)
AOT-axiom *lambda−predicates:3[zero]*: ‹$[\lambda\ p] = p$›
by (*rule AOT-model-axiomI*)
(*simp add*: *AOT-sem-eq AOT-sem-lambda0 AOT-sem-vars-denote*)

AOT-axiom *safe−ext*:
‹$([\lambda\nu_1...\nu_n\ \varphi\{\nu_1...\nu_n\}]\downarrow\ \&\ \Box\forall\nu_1...\forall\nu_n\ (\varphi\{\nu_1...\nu_n\} \equiv \psi\{\nu_1...\nu_n\})) \rightarrow$
$[\lambda\nu_1...\nu_n\ \psi\{\nu_1...\nu_n\}]\downarrow$›
using *AOT-sem-lambda-coex*
by (*auto intro*!: *AOT-model-axiomI simp*: *AOT-sem-imp AOT-sem-denotes AOT-sem-conj AOT-sem-equiv AOT-sem-box AOT-sem-forall*)
AOT-axiom *safe−ext[2]*:
‹$([\lambda\nu_1\nu_2\ \varphi\{\nu_1,\nu_2\}]\downarrow\ \&\ \Box\forall\nu_1\forall\nu_2\ (\varphi\{\nu_1,\ \nu_2\} \equiv \psi\{\nu_1,\ \nu_2\})) \rightarrow$
$[\lambda\nu_1\nu_2\ \psi\{\nu_1,\nu_2\}]\downarrow$›
using *safe−ext*[**where** φ=λ(*x*,*y*). φ *x y*]
by (*simp add*: *AOT-model-axiom-def AOT-sem-denotes AOT-model-denotes-prod-def AOT-sem-forall AOT-sem-imp AOT-sem-conj AOT-sem-equiv AOT-sem-box*)
AOT-axiom *safe−ext[3]*:
‹$([\lambda\nu_1\nu_2\nu_3\ \varphi\{\nu_1,\nu_2,\nu_3\}]\downarrow\ \&\ \Box\forall\nu_1\forall\nu_2\forall\nu_3\ (\varphi\{\nu_1,\ \nu_2,\ \nu_3\} \equiv \psi\{\nu_1,\ \nu_2,\ \nu_3\})) \rightarrow$
$[\lambda\nu_1\nu_2\nu_3\ \psi\{\nu_1,\nu_2,\nu_3\}]\downarrow$›
using *safe−ext*[**where** φ=λ(*x*,*y*,*z*). φ *x y z*]
by (*simp add*: *AOT-model-axiom-def AOT-model-denotes-prod-def AOT-sem-forall AOT-sem-denotes AOT-sem-imp AOT-sem-conj AOT-sem-equiv AOT-sem-box*)
AOT-axiom *safe−ext[4]*:
‹$([\lambda\nu_1\nu_2\nu_3\nu_4\ \varphi\{\nu_1,\nu_2,\nu_3,\nu_4\}]\downarrow\ \&$
$\Box\forall\nu_1\forall\nu_2\forall\nu_3\forall\nu_4\ (\varphi\{\nu_1,\ \nu_2,\ \nu_3,\ \nu_4\} \equiv \psi\{\nu_1,\ \nu_2,\ \nu_3,\ \nu_4\})) \rightarrow$
$[\lambda\nu_1\nu_2\nu_3\nu_4\ \psi\{\nu_1,\nu_2,\nu_3,\nu_4\}]\downarrow$›
using *safe−ext*[**where** φ=λ(*x*,*y*,*z*,*w*). φ *x y z w*]
by (*simp add*: *AOT-model-axiom-def AOT-model-denotes-prod-def AOT-sem-forall AOT-sem-denotes AOT-sem-imp AOT-sem-conj AOT-sem-equiv AOT-sem-box*)

AOT-axiom *nary−encoding[2]*:
‹$x_1x_2[F] \equiv x_1[\lambda y\ [F]yx_2]\ \&\ x_2[\lambda y\ [F]x_1y]$›
by (*rule AOT-model-axiomI*)
(*simp add*: *AOT-sem-conj AOT-sem-equiv AOT-enc-prod-def AOT-proj-enc-prod-def AOT-sem-unary-proj-enc AOT-sem-vars-denote*)
AOT-axiom *nary−encoding[3]*:
‹$x_1x_2x_3[F] \equiv x_1[\lambda y\ [F]yx_2x_3]\ \&\ x_2[\lambda y\ [F]x_1yx_3]\ \&\ x_3[\lambda y\ [F]x_1x_2y]$›
by (*rule AOT-model-axiomI*)
(*simp add*: *AOT-sem-conj AOT-sem-equiv AOT-enc-prod-def AOT-proj-enc-prod-def AOT-sem-unary-proj-enc AOT-sem-vars-denote*)
AOT-axiom *nary−encoding[4]*:
‹$x_1x_2x_3x_4[F] \equiv x_1[\lambda y\ [F]yx_2x_3x_4]\ \&$
$x_2[\lambda y\ [F]x_1yx_3x_4]\ \&$
$x_3[\lambda y\ [F]x_1x_2yx_4]\ \&$
$x_4[\lambda y\ [F]x_1x_2x_3y]$›
by (*rule AOT-model-axiomI*)
(*simp add*: *AOT-sem-conj AOT-sem-equiv AOT-enc-prod-def AOT-proj-enc-prod-def AOT-sem-unary-proj-enc AOT-sem-vars-denote*)

AOT-axiom *encoding*: ‹$x[F] \rightarrow \Box x[F]$›
using *AOT-sem-enc-nec*
by (*auto intro*!: *AOT-model-axiomI simp*: *AOT-sem-imp AOT-sem-box*)

AOT-axiom *nocoder*: ‹$O!x \rightarrow \neg\exists F\ x[F]$›
by (*auto intro*!: *AOT-model-axiomI*
simp: *AOT-sem-imp AOT-sem-not AOT-sem-exists AOT-sem-ordinary*
AOT-sem-dia
AOT-sem-lambda-beta[*OF AOT-sem-ordinary-def-denotes*,
OF AOT-sem-vars-denote])
(*metis AOT-sem-nocoder*)

AOT-axiom *A−objects*: ‹$\exists x\ (A!x\ \&\ \forall F(x[F] \equiv \varphi\{F\}))$›
proof(*rule AOT-model-axiomI*)
AOT-modally-strict {
AOT-obtain κ **where** ‹$\kappa\downarrow\ \&\ \Box\neg E!\kappa\ \&\ \forall F\ (\kappa[F] \equiv \varphi\{F\})$›
using *AOT-sem-A-objects*[*of - φ*]
by (*auto simp*: *AOT-sem-imp AOT-sem-box AOT-sem-forall AOT-sem-exists*
AOT-sem-conj AOT-sem-not AOT-sem-dia AOT-sem-denotes
AOT-sem-equiv) *blast*
AOT-thus ‹$\exists x\ (A!x\ \&\ \forall F(x[F] \equiv \varphi\{F\}))$›
unfolding *AOT-sem-exists*
by (*auto intro*!: *exI*[**where** *x*=κ]
simp: *AOT-sem-lambda-beta*[*OF AOT-sem-abstract-def-denotes*]
AOT-sem-box AOT-sem-dia AOT-sem-not AOT-sem-denotes
AOT-var-of-term-inverse AOT-sem-conj
AOT-sem-equiv AOT-sem-forall AOT-sem-abstract)
}
qed

AOT-theorem *universal-closure*:
assumes ‹*for arbitrary* α: $\varphi\{\alpha\} \in \Lambda_\Box$›
shows ‹$\forall \alpha\ \varphi\{\alpha\} \in \Lambda_\Box$›
using *assms*
by (*metis AOT-term-of-var-cases AOT-model-axiom-def AOT-sem-denotes AOT-sem-forall*)

AOT-theorem *act-closure*:
assumes ‹$\varphi \in \Lambda_\Box$›
shows ‹$\boldsymbol{\mathcal{A}}\varphi \in \Lambda_\Box$›
using *assms* **by** (*simp add*: *AOT-model-axiom-def AOT-sem-act*)

AOT-theorem *nec-closure*:
assumes ‹$\varphi \in \Lambda_\Box$›
shows ‹$\Box\varphi \in \Lambda_\Box$›
using *assms* **by** (*simp add*: *AOT-model-axiom-def AOT-sem-box*)

AOT-theorem *universal-closure-act*:
assumes ‹*for arbitrary* α: $\varphi\{\alpha\} \in \Lambda$›
shows ‹$\forall \alpha\ \varphi\{\alpha\} \in \Lambda$›
using *assms*
by (*metis AOT-term-of-var-cases AOT-model-act-axiom-def AOT-sem-denotes*
AOT-sem-forall)

The following are not part of PLM and only hold in the extended models. They are a generalization of the predecessor axiom.

context *AOT-ExtendedModel*
begin
AOT-axiom *indistinguishable-ord-enc-all*:
‹$\Pi\downarrow\ \&\ A!x\ \&\ A!y\ \&\ \forall F\ \Box([F]x \equiv [F]y) \rightarrow$
$((\forall G(\forall z(O!z \rightarrow \Box([G]z \equiv [\Pi]z)) \rightarrow x[G])) \equiv$
$\forall G(\forall z(O!z \rightarrow \Box([G]z \equiv [\Pi]z)) \rightarrow y[G]))$›
by (*rule AOT-model-axiomI*)

(*auto simp*: *AOT-sem-equiv AOT-sem-imp AOT-sem-conj*
AOT-sem-indistinguishable-ord-enc-all)
AOT-axiom *indistinguishable-ord-enc-ex*:
‹$\Pi\downarrow$ & $A!x$ & $A!y$ & $\forall F\ \Box([F]x \equiv [F]y) \rightarrow$
$((\exists\ G(\forall\ z(O!z \rightarrow \Box([G]z \equiv [\Pi]z))\ \&\ x[G])) \equiv$
$\exists\ G(\forall\ z(O!z \rightarrow \Box([G]z \equiv [\Pi]z))\ \&\ y[G]))$›
by (*rule AOT-model-axiomI*)
(*auto simp*: *AOT-sem-equiv AOT-sem-imp AOT-sem-conj*
AOT-sem-indistinguishable-ord-enc-ex)
end

8 The Deductive System PLM

unbundle *AOT-no-atp*

8.1 Primitive Rule of PLM: Modus Ponens

AOT-theorem *modus−ponens*:
assumes ‹φ› **and** ‹$\varphi \rightarrow \psi$›
shows ‹ψ›

using *assms* **by** (*simp add*: *AOT-sem-imp*)
lemmas *MP* = *modus−ponens*

8.2 (Modally Strict) Proofs and Derivations

AOT-theorem *non−con−thm−thm*:
assumes ‹$\vdash_\Box\ \varphi$›
shows ‹$\vdash\ \varphi$›
using *assms* **by** *simp*

AOT-theorem *vdash−properties*:*1*[*1*]:
assumes ‹$\varphi \in \Lambda$›
shows ‹$\vdash\ \varphi$›

using *assms* **unfolding** *AOT-model-act-axiom-def* **by** *blast*

Convenience attribute for instantiating modally-fragile axioms.

attribute-setup *act-axiom-inst* =
‹*Scan.succeed* (*Thm.rule-attribute* []
(*K* (*fn thm* => *thm RS* @{*thm vdash−properties*:*1*[*1*]})))›
Instantiate modally fragile axiom as modally fragile theorem.

AOT-theorem *vdash−properties*:*1*[*2*]:
assumes ‹$\varphi \in \Lambda_\Box$›
shows ‹$\vdash_\Box\ \varphi$›

using *assms* **unfolding** *AOT-model-axiom-def* **by** *blast*

Convenience attribute for instantiating modally-strict axioms.

attribute-setup *axiom-inst* =
‹*Scan.succeed* (*Thm.rule-attribute* []
(*K* (*fn thm* => *thm RS* @{*thm vdash−properties*:*1*[*2*]})))›
Instantiate axiom as theorem.

Convenience methods and theorem sets for applying ”cqt:2”.

method *cqt-2-lambda-inst-prover* =
(*fast intro*: *AOT-instance-of-cqt-2-intro*)
method *cqt*:*2*[*lambda*] =

(*rule cqt:2*[*lambda*][*axiom-inst*]; *cqt-2-lambda-inst-prover*)
lemmas *cqt:2* =
cqt:2[*const-var*][*axiom-inst*] *cqt:2*[*lambda*][*axiom-inst*]
AOT-instance-of-cqt-2-intro
method *cqt:2* = (*safe intro*!: *cqt:2*)

AOT-theorem *vdash−properties:3*:
assumes ‹$\vdash_\Box \varphi$›
shows ‹$\Gamma \vdash \varphi$›
using *assms* **by** *blast*

AOT-theorem *vdash−properties:5*:
assumes ‹$\Gamma_1 \vdash \varphi$› **and** ‹$\Gamma_2 \vdash \varphi \to \psi$›
shows ‹$\Gamma_1, \Gamma_2 \vdash \psi$›
using *MP assms* **by** *blast*

AOT-theorem *vdash−properties:6*:
assumes ‹φ› **and** ‹$\varphi \to \psi$›
shows ‹ψ›
using *MP assms* **by** *blast*

AOT-theorem *vdash−properties:8*:
assumes ‹$\Gamma \vdash \varphi$› **and** ‹$\varphi \vdash \psi$›
shows ‹$\Gamma \vdash \psi$›
using *assms* **by** *argo*

AOT-theorem *vdash−properties:9*:
assumes ‹φ›
shows ‹$\psi \to \varphi$›
using *MP pl:1*[*axiom-inst*] *assms* **by** *blast*

AOT-theorem *vdash−properties:10*:
assumes ‹$\varphi \to \psi$› **and** ‹φ›
shows ‹ψ›
using *MP assms* **by** *blast*
lemmas $\to$*E* = *vdash−properties:10*

8.3 Two Fundamental Metarules: GEN and RN

AOT-theorem *rule−gen*:
assumes ‹*for arbitrary* α: $\varphi\{\alpha\}$›
shows ‹$\forall \alpha\ \varphi\{\alpha\}$›

using *assms* **by** (*metis AOT-var-of-term-inverse AOT-sem-denotes AOT-sem-forall*)
lemmas *GEN* = *rule−gen*

AOT-theorem *RN*[*prem*]:
assumes ‹$\Gamma \vdash_\Box \varphi$›
shows ‹$\Box\Gamma \vdash_\Box \Box\varphi$›
by (*meson AOT-sem-box assms image-iff*)
AOT-theorem *RN*:
assumes ‹$\vdash_\Box \varphi$›
shows ‹$\Box\varphi$›
using *RN*[*prem*] *assms* **by** *blast*

8.4 The Inferential Role of Definitions

AOT-axiom *df−rules−formulas*[*1*]:
assumes ‹$\varphi \equiv_{df} \psi$›
shows ‹$\varphi \to \psi$›

using *assms*
by (*auto simp*: *assms AOT-model-axiomI AOT-model-equiv-def AOT-sem-imp*)

AOT-axiom *df−rules−formulas*[*2*]:
assumes ‹$\varphi \equiv_{df} \psi$›
shows ‹$\psi \to \varphi$›

using *assms*
by (*auto simp*: *AOT-model-axiomI AOT-model-equiv-def AOT-sem-imp*)

AOT-theorem *df−rules−formulas*[*3*]:
assumes ‹$\varphi \equiv_{df} \psi$›
shows ‹$\varphi \to \psi$›
using *df−rules−formulas*[*1*][*axiom-inst*, *OF assms*].
AOT-theorem *df−rules−formulas*[*4*]:
assumes ‹$\varphi \equiv_{df} \psi$›
shows ‹$\psi \to \varphi$›
using *df−rules−formulas*[*2*][*axiom-inst*, *OF assms*].

AOT-axiom *df−rules−terms*[*1*]:
assumes ‹$\tau\{\alpha_1...\alpha_n\} =_{df} \sigma\{\alpha_1...\alpha_n\}$›
shows ‹$(\sigma\{\tau_1...\tau_n\}\downarrow \to \tau\{\tau_1...\tau_n\} = \sigma\{\tau_1...\tau_n\})$ &
$(\neg\sigma\{\tau_1...\tau_n\}\downarrow \to \neg\tau\{\tau_1...\tau_n\}\downarrow)$›

using *assms*
by (*simp add*: *AOT-model-axiomI AOT-sem-conj AOT-sem-imp AOT-sem-eq AOT-sem-not AOT-sem-denotes AOT-model-id-def*)
AOT-axiom *df−rules−terms*[*2*]:
assumes ‹$\tau =_{df} \sigma$›
shows ‹$(\sigma\downarrow \to \tau = \sigma)$ & $(\neg\sigma\downarrow \to \neg\tau\downarrow)$›
by (*metis df−rules−terms*[*1*] *case-unit-Unity assms*)

AOT-theorem *df−rules−terms*[*3*]:
assumes ‹$\tau\{\alpha_1...\alpha_n\} =_{df} \sigma\{\alpha_1...\alpha_n\}$›
shows ‹$(\sigma\{\tau_1...\tau_n\}\downarrow \to \tau\{\tau_1...\tau_n\} = \sigma\{\tau_1...\tau_n\})$ &
$(\neg\sigma\{\tau_1...\tau_n\}\downarrow \to \neg\tau\{\tau_1...\tau_n\}\downarrow)$›
using *df−rules−terms*[*1*][*axiom-inst*, *OF assms*].
AOT-theorem *df−rules−terms*[*4*]:
assumes ‹$\tau =_{df} \sigma$›
shows ‹$(\sigma\downarrow \to \tau = \sigma)$ & $(\neg\sigma\downarrow \to \neg\tau\downarrow)$›
using *df−rules−terms*[*2*][*axiom-inst*, *OF assms*].

8.5 The Theory of Negations and Conditionals

AOT-theorem *if−p−then−p*: ‹$\varphi \to \varphi$›
by (*meson pl*:*1*[*axiom-inst*] *pl*:*2*[*axiom-inst*] *MP*)

AOT-theorem *deduction−theorem*:
assumes ‹$\varphi \vdash \psi$›
shows ‹$\varphi \to \psi$›

using *assms* **by** (*simp add*: *AOT-sem-imp*)
lemmas *CP* = *deduction−theorem*
lemmas →*I* = *deduction−theorem*

AOT-theorem *ded−thm−cor*:*1*:
assumes ‹$\Gamma_1 \vdash \varphi \to \psi$› **and** ‹$\Gamma_2 \vdash \psi \to \chi$›
shows ‹$\Gamma_1, \Gamma_2 \vdash \varphi \to \chi$›
using →*E* →*I assms* **by** *blast*
AOT-theorem *ded−thm−cor*:*2*:
assumes ‹$\Gamma_1 \vdash \varphi \to (\psi \to \chi)$› **and** ‹$\Gamma_2 \vdash \psi$›
shows ‹$\Gamma_1, \Gamma_2 \vdash \varphi \to \chi$›
using →*E* →*I assms* **by** *blast*

AOT-theorem *ded−thm−cor*:*3*:

assumes ‹$\varphi \rightarrow \psi$› **and** ‹$\psi \rightarrow \chi$›
shows ‹$\varphi \rightarrow \chi$›
using $\rightarrow E$ $\rightarrow I$ *assms* **by** *blast*
declare *ded−thm−cor:3*[*trans*]
AOT-theorem *ded−thm−cor:4*:
assumes ‹$\varphi \rightarrow (\psi \rightarrow \chi)$› **and** ‹$\psi$›
shows ‹$\varphi \rightarrow \chi$›
using $\rightarrow E$ $\rightarrow I$ *assms* **by** *blast*

lemmas *Hypothetical Syllogism* = *ded−thm−cor:3*

AOT-theorem *useful−tautologies:1*: ‹$\neg\neg\varphi \rightarrow \varphi$›
by (*metis pl:3*[*axiom-inst*] $\rightarrow I$ *Hypothetical Syllogism*)
AOT-theorem *useful−tautologies:2*: ‹$\varphi \rightarrow \neg\neg\varphi$›
by (*metis pl:3*[*axiom-inst*] $\rightarrow I$ *ded−thm−cor:4*)
AOT-theorem *useful−tautologies:3*: ‹$\neg\varphi \rightarrow (\varphi \rightarrow \psi)$›
by (*meson ded−thm−cor:4 pl:3*[*axiom-inst*] $\rightarrow I$)
AOT-theorem *useful−tautologies:4*: ‹$(\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi)$›
by (*meson pl:3*[*axiom-inst*] *Hypothetical Syllogism* $\rightarrow I$)
AOT-theorem *useful−tautologies:5*: ‹$(\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\varphi)$›
by (*metis useful−tautologies:4 Hypothetical Syllogism* $\rightarrow I$)

AOT-theorem *useful−tautologies:6*: ‹$(\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\varphi)$›
by (*metis* $\rightarrow I$ *MP useful−tautologies:4*)

AOT-theorem *useful−tautologies:7*: ‹$(\neg\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \varphi)$›
by (*metis* $\rightarrow I$ *MP useful−tautologies:3 useful−tautologies:5*)

AOT-theorem *useful−tautologies:8*: ‹$\varphi \rightarrow (\neg\psi \rightarrow \neg(\varphi \rightarrow \psi))$›
by (*metis* $\rightarrow I$ *MP useful−tautologies:5*)

AOT-theorem *useful−tautologies:9*: ‹$(\varphi \rightarrow \psi) \rightarrow ((\neg\varphi \rightarrow \psi) \rightarrow \psi)$›
by (*metis* $\rightarrow I$ *MP useful−tautologies:6*)

AOT-theorem *useful−tautologies:10*: ‹$(\varphi \rightarrow \neg\psi) \rightarrow ((\varphi \rightarrow \psi) \rightarrow \neg\varphi)$›
by (*metis* $\rightarrow I$ *MP pl:3*[*axiom-inst*])

AOT-theorem *dn−i−e:1*:
assumes ‹φ›
shows ‹$\neg\neg\varphi$›
using *MP useful−tautologies:2 assms* **by** *blast*
lemmas $\neg\neg I$ = *dn−i−e:1*
AOT-theorem *dn−i−e:2*:
assumes ‹$\neg\neg\varphi$›
shows ‹φ›
using *MP useful−tautologies:1 assms* **by** *blast*
lemmas $\neg\neg E$ = *dn−i−e:2*

AOT-theorem *modus−tollens:1*:
assumes ‹$\varphi \rightarrow \psi$› **and** ‹$\neg\psi$›
shows ‹$\neg\varphi$›
using *MP useful−tautologies:5 assms* **by** *blast*
AOT-theorem *modus−tollens:2*:
assumes ‹$\varphi \rightarrow \neg\psi$› **and** ‹$\psi$›
shows ‹$\neg\varphi$›
using $\neg\neg I$ *modus−tollens:1 assms* **by** *blast*
lemmas *MT* = *modus−tollens:1 modus−tollens:2*

AOT-theorem *contraposition:1*[*1*]:
assumes ‹$\varphi \rightarrow \psi$›
shows ‹$\neg\psi \rightarrow \neg\varphi$›
using $\rightarrow I$ *MT*(*1*) *assms* **by** *blast*
AOT-theorem *contraposition:1*[*2*]:

assumes ‹¬ψ → ¬φ›
shows ‹φ → ψ›
using *→I ¬¬E MT(2) assms* **by** *blast*

AOT-theorem *contraposition:2*:
assumes ‹φ → ¬ψ›
shows ‹ψ → ¬φ›
using *→I MT(2) assms* **by** *blast*

AOT-theorem *reductio−aa:1*:
assumes ‹¬φ ⊢ ¬ψ› **and** ‹¬φ ⊢ ψ›
shows ‹φ›
using *→I ¬¬E MT(2) assms* **by** *blast*
AOT-theorem *reductio−aa:2*:
assumes ‹φ ⊢ ¬ψ› **and** ‹φ ⊢ ψ›
shows ‹¬φ›
using *reductio−aa:1 assms* **by** *blast*
lemmas *RAA = reductio−aa:1 reductio−aa:2*

AOT-theorem *exc−mid*: ‹φ ∨ ¬φ›
using *df−rules−formulas[4] if−p−then−p MP*
conventions:2 **by** *blast*

AOT-theorem *non−contradiction*: ‹¬(φ & ¬φ)›
using *df−rules−formulas[3] MT(2) useful−tautologies:2*
conventions:1 **by** *blast*

AOT-theorem *con−dis−taut:1*: ‹(φ & ψ) → φ›
by (*meson →I df−rules−formulas[3] MP RAA(1) conventions:1*)
AOT-theorem *con−dis−taut:2*: ‹(φ & ψ) → ψ›
by (*metis →I df−rules−formulas[3] MT(2) RAA(2)*
¬¬E conventions:1)
lemmas *Conjunction Simplification = con−dis−taut:1 con−dis−taut:2*

AOT-theorem *con−dis−taut:3*: ‹φ → (φ ∨ ψ)›
by (*meson contraposition:1[2] df−rules−formulas[4]*
MP →I conventions:2)
AOT-theorem *con−dis−taut:4*: ‹ψ → (φ ∨ ψ)›
using *Hypothetical Syllogism df−rules−formulas[4]*
pl:1[axiom-inst] conventions:2 **by** *blast*
lemmas *Disjunction Addition = con−dis−taut:3 con−dis−taut:4*

AOT-theorem *con−dis−taut:5*: ‹φ → (ψ → (φ & ψ))›
by (*metis contraposition:2 Hypothetical Syllogism →I*
df−rules−formulas[4] conventions:1)
lemmas *Adjunction = con−dis−taut:5*

AOT-theorem *con−dis−taut:6*: ‹(φ & φ) ≡ φ›
by (*metis Adjunction →I df−rules−formulas[4] MP*
Conjunction Simplification(1) conventions:3)
lemmas *Idempotence of & = con−dis−taut:6*

AOT-theorem *con−dis−taut:7*: ‹(φ ∨ φ) ≡ φ›
proof −
{
AOT-assume ‹φ ∨ φ›
AOT-hence ‹¬φ → φ›
using *conventions:2[THEN df−rules−formulas[3]] MP* **by** *blast*
AOT-hence ‹φ› **using** *if−p−then−p RAA(1) MP* **by** *blast*
}
moreover {
AOT-assume ‹φ›
AOT-hence ‹φ ∨ φ› **using** *Disjunction Addition(1) MP* **by** *blast*

```
  }
  ultimately AOT-show ‹(φ ∨ φ) ≡ φ›
    using conventions:3[THEN df−rules−formulas[4]] MP
    by (metis Adjunction →I)
qed
lemmas Idempotence of ∨ = con−dis−taut:7


AOT-theorem con−dis−i−e:1:
  assumes ‹φ› and ‹ψ›
  shows ‹φ & ψ›
  using Adjunction MP assms by blast
lemmas &I = con−dis−i−e:1

AOT-theorem con−dis−i−e:2:a:
  assumes ‹φ & ψ›
  shows ‹φ›
  using Conjunction Simplification(1) MP assms by blast
AOT-theorem con−dis−i−e:2:b:
  assumes ‹φ & ψ›
  shows ‹ψ›
  using Conjunction Simplification(2) MP assms by blast
lemmas &E = con−dis−i−e:2:a con−dis−i−e:2:b

AOT-theorem con−dis−i−e:3:a:
  assumes ‹φ›
  shows ‹φ ∨ ψ›
  using Disjunction Addition(1) MP assms by blast
AOT-theorem con−dis−i−e:3:b:
  assumes ‹ψ›
  shows ‹φ ∨ ψ›
  using Disjunction Addition(2) MP assms by blast
AOT-theorem con−dis−i−e:3:c:
  assumes ‹φ ∨ ψ› and ‹φ → χ› and ‹ψ → Θ›
  shows ‹χ ∨ Θ›
  by (metis con−dis−i−e:3:a Disjunction Addition(2)
          df−rules−formulas[3] MT(1) RAA(1)
          conventions:2 assms)
lemmas ∨I = con−dis−i−e:3:a con−dis−i−e:3:b con−dis−i−e:3:c

AOT-theorem con−dis−i−e:4:a:
  assumes ‹φ ∨ ψ› and ‹φ → χ› and ‹ψ → χ›
  shows ‹χ›
  by (metis MP RAA(2) df−rules−formulas[3] conventions:2 assms)
AOT-theorem con−dis−i−e:4:b:
  assumes ‹φ ∨ ψ› and ‹¬φ›
  shows ‹ψ›
  using con−dis−i−e:4:a RAA(1) →I assms by blast
AOT-theorem con−dis−i−e:4:c:
  assumes ‹φ ∨ ψ› and ‹¬ψ›
  shows ‹φ›
  using con−dis−i−e:4:a RAA(1) →I assms by blast
lemmas ∨E = con−dis−i−e:4:a con−dis−i−e:4:b con−dis−i−e:4:c

AOT-theorem raa−cor:1:
  assumes ‹¬φ ⊢ ψ & ¬ψ›
  shows ‹φ›
  using &E ∨E(3) ∨I(2) RAA(2) assms by blast
AOT-theorem raa−cor:2:
  assumes ‹φ ⊢ ψ & ¬ψ›
  shows ‹¬φ›
  using raa−cor:1 assms by blast
AOT-theorem raa−cor:3:
```

assumes ‹φ› **and** ‹$\neg\psi \vdash \neg\varphi$›
shows ‹ψ›
using *RAA assms* **by** *blast*
AOT-theorem *raa−cor:4*:
assumes ‹$\neg\varphi$› **and** ‹$\neg\psi \vdash \varphi$›
shows ‹ψ›
using *RAA assms* **by** *blast*
AOT-theorem *raa−cor:5*:
assumes ‹φ› **and** ‹$\psi \vdash \neg\varphi$›
shows ‹$\neg\psi$›
using *RAA assms* **by** *blast*
AOT-theorem *raa−cor:6*:
assumes ‹$\neg\varphi$› **and** ‹$\psi \vdash \varphi$›
shows ‹$\neg\psi$›
using *RAA assms* **by** *blast*

AOT-theorem *oth−class−taut:1:a*: ‹$(\varphi \rightarrow \psi) \equiv \neg(\varphi \,\&\, \neg\psi)$›
by (*rule conventions:3*[*THEN df−rules−formulas*[*4*], *THEN* →*E*])
(*metis* &*E* &*I raa−cor:3* →*I MP*)
AOT-theorem *oth−class−taut:1:b*: ‹$\neg(\varphi \rightarrow \psi) \equiv (\varphi \,\&\, \neg\psi)$›
by (*rule conventions:3*[*THEN df−rules−formulas*[*4*], *THEN* →*E*])
(*metis* &*E* &*I raa−cor:3* →*I MP*)
AOT-theorem *oth−class−taut:1:c*: ‹$(\varphi \rightarrow \psi) \equiv (\neg\varphi \vee \psi)$›
by (*rule conventions:3*[*THEN df−rules−formulas*[*4*], *THEN* →*E*])
(*metis* &*I* ∨*I*(*1*, *2*) ∨*E*(*3*) →*I MP raa−cor:1*)

AOT-theorem *oth−class−taut:2:a*: ‹$(\varphi \,\&\, \psi) \equiv (\psi \,\&\, \varphi)$›
by (*rule conventions:3*[*THEN df−rules−formulas*[*4*], *THEN* →*E*])
(*meson* &*I* &*E* →*I*)
lemmas *Commutativity of* & = *oth−class−taut:2:a*
AOT-theorem *oth−class−taut:2:b*: ‹$(\varphi \,\&\, (\psi \,\&\, \chi)) \equiv ((\varphi \,\&\, \psi) \,\&\, \chi)$›
by (*rule conventions:3*[*THEN df−rules−formulas*[*4*], *THEN* →*E*])
(*metis* &*I* &*E* →*I*)
lemmas *Associativity of* & = *oth−class−taut:2:b*
AOT-theorem *oth−class−taut:2:c*: ‹$(\varphi \vee \psi) \equiv (\psi \vee \varphi)$›
by (*rule conventions:3*[*THEN df−rules−formulas*[*4*], *THEN* →*E*])
(*metis* &*I* ∨*I*(*1*, *2*) ∨*E*(*1*) →*I*)
lemmas *Commutativity of* ∨ = *oth−class−taut:2:c*
AOT-theorem *oth−class−taut:2:d*: ‹$(\varphi \vee (\psi \vee \chi)) \equiv ((\varphi \vee \psi) \vee \chi)$›
by (*rule conventions:3*[*THEN df−rules−formulas*[*4*], *THEN* →*E*])
(*metis* &*I* ∨*I*(*1*, *2*) ∨*E*(*1*) →*I*)
lemmas *Associativity of* ∨ = *oth−class−taut:2:d*
AOT-theorem *oth−class−taut:2:e*: ‹$(\varphi \equiv \psi) \equiv (\psi \equiv \varphi)$›
by (*rule conventions:3*[*THEN df−rules−formulas*[*4*], *THEN* →*E*]; *rule* &*I*;
metis &*I df−rules−formulas*[*4*] *conventions:3* &*E*
Hypothetical Syllogism →*I df−rules−formulas*[*3*])
lemmas *Commutativity of* ≡ = *oth−class−taut:2:e*
AOT-theorem *oth−class−taut:2:f*: ‹$(\varphi \equiv (\psi \equiv \chi)) \equiv ((\varphi \equiv \psi) \equiv \chi)$›
using *conventions:3*[*THEN df−rules−formulas*[*4*]]
conventions:3[*THEN df−rules−formulas*[*3*]]
→*I* →*E* &*E* &*I*
by *metis*
lemmas *Associativity of* ≡ = *oth−class−taut:2:f*

AOT-theorem *oth−class−taut:3:a*: ‹$\varphi \equiv \varphi$›
using &*I vdash−properties:6 if−p−then−p*
df−rules−formulas[*4*] *conventions:3* **by** *blast*
AOT-theorem *oth−class−taut:3:b*: ‹$\varphi \equiv \neg\neg\varphi$›
using &*I useful−tautologies:1 useful−tautologies:2* →*E*
df−rules−formulas[*4*] *conventions:3* **by** *blast*
AOT-theorem *oth−class−taut:3:c*: ‹$\neg(\varphi \equiv \neg\varphi)$›
by (*metis* &*E* →*E RAA df−rules−formulas*[*3*] *conventions:3*)

AOT-theorem *oth−class−taut:4:a*: ‹$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$›
by (*metis* $\to E$ $\to I$)
AOT-theorem *oth−class−taut:4:b*: ‹$(\varphi \equiv \psi) \equiv (\neg\varphi \equiv \neg\psi)$›
using *conventions:3[THEN df−rules−formulas[4]]*
conventions:3[THEN df−rules−formulas[3]]
$\to I$ $\to E$ &*E* &*I* *RAA* **by** *metis*
AOT-theorem *oth−class−taut:4:c*: ‹$(\varphi \equiv \psi) \to ((\varphi \to \chi) \equiv (\psi \to \chi))$›
using *conventions:3[THEN df−rules−formulas[4]]*
conventions:3[THEN df−rules−formulas[3]]
$\to I$ $\to E$ &*E* &*I* **by** *metis*
AOT-theorem *oth−class−taut:4:d*: ‹$(\varphi \equiv \psi) \to ((\chi \to \varphi) \equiv (\chi \to \psi))$›
using *conventions:3[THEN df−rules−formulas[4]]*
conventions:3[THEN df−rules−formulas[3]]
$\to I$ $\to E$ &*E* &*I* **by** *metis*
AOT-theorem *oth−class−taut:4:e*: ‹$(\varphi \equiv \psi) \to ((\varphi \;\&\; \chi) \equiv (\psi \;\&\; \chi))$›
using *conventions:3[THEN df−rules−formulas[4]]*
conventions:3[THEN df−rules−formulas[3]]
$\to I$ $\to E$ &*E* &*I* **by** *metis*
AOT-theorem *oth−class−taut:4:f*: ‹$(\varphi \equiv \psi) \to ((\chi \;\&\; \varphi) \equiv (\chi \;\&\; \psi))$›
using *conventions:3[THEN df−rules−formulas[4]]*
conventions:3[THEN df−rules−formulas[3]]
$\to I$ $\to E$ &*E* &*I* **by** *metis*
AOT-theorem *oth−class−taut:4:g*: ‹$(\varphi \equiv \psi) \equiv ((\varphi \;\&\; \psi) \vee (\neg\varphi \;\&\; \neg\psi))$›
proof(*safe intro!*: *conventions:3[THEN df−rules−formulas[4], THEN* $\to E$*]*
&*I* $\to I$
dest!: *conventions:3[THEN df−rules−formulas[3], THEN* $\to E$*]*)
AOT-show ‹$\varphi \;\&\; \psi \vee (\neg\varphi \;\&\; \neg\psi)$› **if** ‹$(\varphi \to \psi) \;\&\; (\psi \to \varphi)$›
using &*E* $\vee I$ $\to E$ &*I* *raa−cor:1* $\to I$ $\vee E$ *that* **by** *metis*
next
AOT-show ‹ψ› **if** ‹$\varphi \;\&\; \psi \vee (\neg\varphi \;\&\; \neg\psi)$› **and** ‹$\varphi$›
using *that* $\vee E$ &*E* *raa−cor:3* **by** *blast*
next
AOT-show ‹φ› **if** ‹$\varphi \;\&\; \psi \vee (\neg\varphi \;\&\; \neg\psi)$› **and** ‹$\psi$›
using *that* $\vee E$ &*E* *raa−cor:3* **by** *blast*
qed
AOT-theorem *oth−class−taut:4:h*: ‹$\neg(\varphi \equiv \psi) \equiv ((\varphi \;\&\; \neg\psi) \vee (\neg\varphi \;\&\; \psi))$›
proof (*safe intro!*: *conventions:3[THEN df−rules−formulas[4], THEN* $\to E$*]*
&*I* $\to I$)
AOT-show ‹$\varphi \;\&\; \neg\psi \vee (\neg\varphi \;\&\; \psi)$› **if** ‹$\neg(\varphi \equiv \psi)$›
by (*metis that* &*I* $\vee I(1, 2)$ $\to I$ *MT(1) df−rules−formulas[4]*
raa−cor:3 conventions:3)
next
AOT-show ‹$\neg(\varphi \equiv \psi)$› **if** ‹$\varphi \;\&\; \neg\psi \vee (\neg\varphi \;\&\; \psi)$›
by (*metis that* &*E* $\vee E(2)$ $\to E$ *df−rules−formulas[3]*
raa−cor:3 conventions:3)
qed
AOT-theorem *oth−class−taut:5:a*: ‹$(\varphi \;\&\; \psi) \equiv \neg(\neg\varphi \vee \neg\psi)$›
using *conventions:3[THEN df−rules−formulas[4]]*
$\to I$ $\to E$ &*E* &*I* $\vee I$ $\vee E$ *RAA* **by** *metis*
AOT-theorem *oth−class−taut:5:b*: ‹$(\varphi \vee \psi) \equiv \neg(\neg\varphi \;\&\; \neg\psi)$›
using *conventions:3[THEN df−rules−formulas[4]]*
$\to I$ $\to E$ &*E* &*I* $\vee I$ $\vee E$ *RAA* **by** *metis*
AOT-theorem *oth−class−taut:5:c*: ‹$\neg(\varphi \;\&\; \psi) \equiv (\neg\varphi \vee \neg\psi)$›
using *conventions:3[THEN df−rules−formulas[4]]*
$\to I$ $\to E$ &*E* &*I* $\vee I$ $\vee E$ *RAA* **by** *metis*
AOT-theorem *oth−class−taut:5:d*: ‹$\neg(\varphi \vee \psi) \equiv (\neg\varphi \;\&\; \neg\psi)$›
using *conventions:3[THEN df−rules−formulas[4]]*
$\to I$ $\to E$ &*E* &*I* $\vee I$ $\vee E$ *RAA* **by** *metis*

lemmas *DeMorgan* = *oth−class−taut:5:c oth−class−taut:5:d*

AOT-theorem *oth−class−taut:6:a*:
‹$(\varphi \;\&\; (\psi \vee \chi)) \equiv ((\varphi \;\&\; \psi) \vee (\varphi \;\&\; \chi))$›

using *conventions*:*3*[*THEN df−rules−formulas*[*4*]]
$\rightarrow I$ $\rightarrow E$ &*E* &*I* $\vee I$ $\vee E$ *RAA* **by** *metis*
AOT-theorem *oth−class−taut*:*6*:*b*:
‹$(\varphi \vee (\psi \,\&\, \chi)) \equiv ((\varphi \vee \psi) \,\&\, (\varphi \vee \chi))$›
using *conventions*:*3*[*THEN df−rules−formulas*[*4*]]
$\rightarrow I$ $\rightarrow E$ &*E* &*I* $\vee I$ $\vee E$ *RAA* **by** *metis*

AOT-theorem *oth−class−taut*:*7*:*a*: ‹$((\varphi \,\&\, \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$›
by (*metis* &*I* $\rightarrow E$ $\rightarrow I$)
lemmas *Exportation* = *oth−class−taut*:*7*:*a*
AOT-theorem *oth−class−taut*:*7*:*b*: ‹$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \,\&\, \psi) \rightarrow \chi)$›
by (*metis* &*E* $\rightarrow E$ $\rightarrow I$)
lemmas *Importation* = *oth−class−taut*:*7*:*b*

AOT-theorem *oth−class−taut*:*8*:*a*:
‹$(\varphi \rightarrow (\psi \rightarrow \chi)) \equiv (\psi \rightarrow (\varphi \rightarrow \chi))$›
using *conventions*:*3*[*THEN df−rules−formulas*[*4*]] $\rightarrow I$ $\rightarrow E$ &*E* &*I*
by *metis*
lemmas *Permutation* = *oth−class−taut*:*8*:*a*
AOT-theorem *oth−class−taut*:*8*:*b*:
‹$(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \,\&\, \chi)))$›
by (*metis* &*I* $\rightarrow E$ $\rightarrow I$)
lemmas *Composition* = *oth−class−taut*:*8*:*b*
AOT-theorem *oth−class−taut*:*8*:*c*:
‹$(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi))$›
by (*metis* $\vee E(2)$ $\rightarrow E$ $\rightarrow I$ *RAA*(*1*))
AOT-theorem *oth−class−taut*:*8*:*d*:
‹$((\varphi \rightarrow \psi) \,\&\, (\chi \rightarrow \Theta)) \rightarrow ((\varphi \,\&\, \chi) \rightarrow (\psi \,\&\, \Theta))$›
by (*metis* &*E* &*I* $\rightarrow E$ $\rightarrow I$)
lemmas *Double Composition* = *oth−class−taut*:*8*:*d*
AOT-theorem *oth−class−taut*:*8*:*e*:
‹$((\varphi \,\&\, \psi) \equiv (\varphi \,\&\, \chi)) \equiv (\varphi \rightarrow (\psi \equiv \chi))$›
by (*metis conventions*:*3*[*THEN df−rules−formulas*[*4*]]
conventions:*3*[*THEN df−rules−formulas*[*3*]]
$\rightarrow I$ $\rightarrow E$ &*E* &*I*)
AOT-theorem *oth−class−taut*:*8*:*f*:
‹$((\varphi \,\&\, \psi) \equiv (\chi \,\&\, \psi)) \equiv (\psi \rightarrow (\varphi \equiv \chi))$›
by (*metis conventions*:*3*[*THEN df−rules−formulas*[*4*]]
conventions:*3*[*THEN df−rules−formulas*[*3*]]
$\rightarrow I$ $\rightarrow E$ &*E* &*I*)
AOT-theorem *oth−class−taut*:*8*:*g*:
‹$(\psi \equiv \chi) \rightarrow ((\varphi \vee \psi) \equiv (\varphi \vee \chi))$›
by (*metis conventions*:*3*[*THEN df−rules−formulas*[*4*]]
conventions:*3*[*THEN df−rules−formulas*[*3*]]
$\rightarrow I$ $\rightarrow E$ &*E* &*I* $\vee I$ $\vee E(1)$)
AOT-theorem *oth−class−taut*:*8*:*h*:
‹$(\psi \equiv \chi) \rightarrow ((\psi \vee \varphi) \equiv (\chi \vee \varphi))$›
by (*metis conventions*:*3*[*THEN df−rules−formulas*[*4*]]
conventions:*3*[*THEN df−rules−formulas*[*3*]]
$\rightarrow I$ $\rightarrow E$ &*E* &*I* $\vee I$ $\vee E(1)$)
AOT-theorem *oth−class−taut*:*8*:*i*:
‹$(\varphi \equiv (\psi \,\&\, \chi)) \rightarrow (\psi \rightarrow (\varphi \equiv \chi))$›
by (*metis conventions*:*3*[*THEN df−rules−formulas*[*4*]]
conventions:*3*[*THEN df−rules−formulas*[*3*]]
$\rightarrow I$ $\rightarrow E$ &*E* &*I*)

AOT-theorem *intro−elim*:*1*:
assumes ‹$\varphi \vee \psi$› **and** ‹$\varphi \equiv \chi$› **and** ‹$\psi \equiv \Theta$›
shows ‹$\chi \vee \Theta$›
by (*metis assms* $\vee I(1, 2)$ $\vee E(1)$ $\rightarrow I$ $\rightarrow E$ &*E*(*1*)
conventions:*3*[*THEN df−rules−formulas*[*3*]])

AOT-theorem *intro−elim*:*2*:

assumes ‹$\varphi \rightarrow \psi$› **and** ‹$\psi \rightarrow \varphi$›
shows ‹$\varphi \equiv \psi$›
by (*meson* &*I conventions*:*3 df−rules−formulas*[*4*] *MP assms*)
lemmas $\equiv I$ = *intro−elim*:*2*

AOT-theorem *intro−elim*:*3*:*a*:
assumes ‹$\varphi \equiv \psi$› **and** ‹φ›
shows ‹ψ›
by (*metis* ∨*I*(*1*) →*I* ∨*E*(*1*) *intro−elim*:*1 assms*)
AOT-theorem *intro−elim*:*3*:*b*:
assumes ‹$\varphi \equiv \psi$› **and** ‹ψ›
shows ‹φ›
using *intro−elim*:*3*:*a Commutativity of* ≡ *assms* **by** *blast*
AOT-theorem *intro−elim*:*3*:*c*:
assumes ‹$\varphi \equiv \psi$› **and** ‹$\neg\varphi$›
shows ‹$\neg\psi$›
using *intro−elim*:*3*:*b raa−cor*:*3 assms* **by** *blast*
AOT-theorem *intro−elim*:*3*:*d*:
assumes ‹$\varphi \equiv \psi$› **and** ‹$\neg\psi$›
shows ‹$\neg\varphi$›
using *intro−elim*:*3*:*a raa−cor*:*3 assms* **by** *blast*
AOT-theorem *intro−elim*:*3*:*e*:
assumes ‹$\varphi \equiv \psi$› **and** ‹$\psi \equiv \chi$›
shows ‹$\varphi \equiv \chi$›
by (*metis* ≡*I* →*I intro−elim*:*3*:*a intro−elim*:*3*:*b assms*)
declare *intro−elim*:*3*:*e*[*trans*]
AOT-theorem *intro−elim*:*3*:*f*:
assumes ‹$\varphi \equiv \psi$› **and** ‹$\varphi \equiv \chi$›
shows ‹$\chi \equiv \psi$›
by (*metis* ≡*I* →*I intro−elim*:*3*:*a intro−elim*:*3*:*b assms*)
lemmas ≡*E* = *intro−elim*:*3*:*a intro−elim*:*3*:*b intro−elim*:*3*:*c*
intro−elim:*3*:*d intro−elim*:*3*:*e intro−elim*:*3*:*f*

declare *Commutativity of* ≡[*THEN* ≡*E*(*1*), *sym*]

AOT-theorem *rule−eq−df*:*1*:
assumes ‹$\varphi \equiv_{df} \psi$›
shows ‹$\varphi \equiv \psi$›
by (*simp add*: ≡*I df−rules−formulas*[*3*] *df−rules−formulas*[*4*] *assms*)
lemmas ≡*Df* = *rule−eq−df*:*1*
AOT-theorem *rule−eq−df*:*2*:
assumes ‹$\varphi \equiv_{df} \psi$› **and** ‹φ›
shows ‹ψ›
using ≡*Df* ≡*E*(*1*) *assms* **by** *blast*
lemmas $\equiv_{df}E$ = *rule−eq−df*:*2*
AOT-theorem *rule−eq−df*:*3*:
assumes ‹$\varphi \equiv_{df} \psi$› **and** ‹ψ›
shows ‹φ›
using ≡*Df* ≡*E*(*2*) *assms* **by** *blast*
lemmas $\equiv_{df}I$ = *rule−eq−df*:*3*

AOT-theorem *df−simplify*:*1*:
assumes ‹$\varphi \equiv (\psi \& \chi)$› **and** ‹$\psi$›
shows ‹$\varphi \equiv \chi$›
by (*metis* &*E*(*2*) &*I* ≡*E*(*1*, *2*) ≡*I* →*I assms*)

AOT-theorem *df−simplify*:*2*:
assumes ‹$\varphi \equiv (\psi \& \chi)$› **and** ‹$\chi$›
shows ‹$\varphi \equiv \psi$›
by (*metis* &*E*(*1*) &*I* ≡*E*(*1*, *2*) ≡*I* →*I assms*)
lemmas ≡*S* = *df−simplify*:*1 df−simplify*:*2*

8.6 The Theory of Quantification

```
AOT-theorem rule−ui:1:
  assumes ‹∀ α φ{α}› and ‹τ↓›
  shows ‹φ{τ}›
  using →E cqt:1[axiom-inst] assms by blast
AOT-theorem rule−ui:2[const-var]:
  assumes ‹∀ α φ{α}›
  shows ‹φ{β}›
  by (simp add: rule−ui:1 cqt:2[const-var][axiom-inst] assms)
AOT-theorem rule−ui:2[lambda]:
  assumes ‹∀ F φ{F}› and ‹INSTANCE-OF-CQT-2(ψ)›
  shows ‹φ{[λν1...νn ψ{ν1...νn}]}›
  by (simp add: rule−ui:1 cqt:2[lambda][axiom-inst] assms)
AOT-theorem rule−ui:3:
  assumes ‹∀ α φ{α}›
  shows ‹φ{α}›
  by (simp add: rule−ui:2[const-var] assms)
lemmas ∀ E = rule−ui:1 rule−ui:2[const-var]
             rule−ui:2[lambda] rule−ui:3

AOT-theorem cqt−orig:1[const-var]: ‹∀ α φ{α} → φ{β}›
  by (simp add: ∀ E(2) →I)
AOT-theorem cqt−orig:1[lambda]:
  assumes ‹INSTANCE-OF-CQT-2(ψ)›
  shows ‹∀ F φ{F} → φ{[λν1...νn ψ{ν1...νn}]}›
  by (simp add: ∀ E(3) →I assms)
AOT-theorem cqt−orig:2: ‹∀ α (φ → ψ{α}) → (φ → ∀ α ψ{α})›
  by (metis →I GEN vdash−properties:6 ∀ E(4))
AOT-theorem cqt−orig:3: ‹∀ α φ{α} → φ{α}›
  using cqt−orig:1[const-var].

AOT-theorem universal:
  assumes ‹for arbitrary β: φ{β}›
  shows ‹∀ α φ{α}›
  using GEN assms .
lemmas ∀ I = universal
```

```
ML‹
fun get-instantiated-allI ctxt varname thm = let
val trm = Thm.concl-of thm
val trm =
  case trm of (@{const Trueprop} $ (@{const AOT-model-valid-in} $ - $ x)) => x
  | - => raise Term.TERM (Expected simple theorem., [trm])
fun extractVars (Const (const-name ‹AOT-term-of-var›, -) $ Var v) =
    (if fst (fst v) = fst varname then [Var v] else [])
  | extractVars (t1 $ t2) = extractVars t1 @ extractVars t2
  | extractVars (Abs (-, -, t)) = extractVars t
  | extractVars - = []
val vars = extractVars trm
val vars = fold Term.add-vars vars []
val var = hd vars
val trmty =
  case (snd var) of (Type (type-name ‹AOT-var›, [t])) => (t)
  | - => raise Term.TYPE (Expected variable type., [snd var], [Var var])
val trm = Abs (Term.string-of-vname (fst var), trmty, Term.abstract-over (
     Const (const-name ‹AOT-term-of-var›, Type (fun, [snd var, trmty]))
      $ Var var, trm))
val trm = Thm.cterm-of (Context.proof-of ctxt) trm
val ty = hd (Term.add-tvars (Thm.prop-of @{thm ∀ I}) [])
val typ = Thm.ctyp-of (Context.proof-of ctxt) trmty
val allthm = Drule.instantiate-normalize (TVars.make [(ty, typ)], Vars.empty) @{thm ∀ I}
```

```
val phi = hd (Term.add-vars (Thm.prop-of allthm) [])
val allthm = Drule.instantiate-normalize (TVars.empty, Vars.make [(phi,trm)]) allthm
in
allthm
end
›

attribute-setup ∀I =
  ‹Scan.lift (Scan.repeat1 Args.var) >> (fn args => Thm.rule-attribute []
  (fn ctxt => fn thm => fold (fn arg => fn thm =>
    thm RS get-instantiated-allI ctxt arg thm) args thm))›
  Quantify over a variable in a theorem using GEN.

attribute-setup unvarify =
  ‹Scan.lift (Scan.repeat1 Args.var) >> (fn args => Thm.rule-attribute []
  (fn ctxt => fn thm =>
    let
      fun get-inst-allI arg thm = thm RS get-instantiated-allI ctxt arg thm
      val thm = fold get-inst-allI args thm
      val thm = fold (K (fn thm => thm RS @{thm ∀E(1)})) args thm
    in
     thm
    end))›
  Generalize a statement about variables to a statement about denoting terms.



AOT-theorem cqt−basic:1: ‹∀α∀β φ{α,β} ≡ ∀β∀α φ{α,β}›
  by (metis ≡I ∀E(2) ∀I →I)

AOT-theorem cqt−basic:2:
  ‹∀α(φ{α} ≡ ψ{α}) ≡ (∀α(φ{α} → ψ{α}) & ∀α(ψ{α} → φ{α}))›
proof (rule ≡I; rule →I)
  AOT-assume ‹∀α(φ{α} ≡ ψ{α})›
  AOT-hence ‹φ{α} ≡ ψ{α}› for α using ∀E(2) by blast
  AOT-hence ‹φ{α} → ψ{α}› and ‹ψ{α} → φ{α}› for α
    using ≡E(1,2) →I by blast+
  AOT-thus ‹∀α(φ{α} → ψ{α}) & ∀α(ψ{α} → φ{α})›
    by (auto intro: &I ∀I)
next
  AOT-assume ‹∀α(φ{α} → ψ{α}) & ∀α(ψ{α} → φ{α})›
  AOT-hence ‹φ{α} → ψ{α}› and ‹ψ{α} → φ{α}› for α
    using ∀E(2) &E by blast+
  AOT-hence ‹φ{α} ≡ ψ{α}› for α
    using ≡I by blast
  AOT-thus ‹∀α(φ{α} ≡ ψ{α})› by (auto intro: ∀I)
qed

AOT-theorem cqt−basic:3: ‹∀α(φ{α} ≡ ψ{α}) → (∀α φ{α} ≡ ∀α ψ{α})›
proof(rule →I)
  AOT-assume ‹∀α(φ{α} ≡ ψ{α})›
  AOT-hence 1: ‹φ{α} ≡ ψ{α}› for α using ∀E(2) by blast
  {
    AOT-assume ‹∀α φ{α}›
    AOT-hence ‹∀α ψ{α}› using 1 ∀I ∀E(4) ≡E by metis
  }
  moreover {
    AOT-assume ‹∀α ψ{α}›
    AOT-hence ‹∀α φ{α}› using 1 ∀I ∀E(4) ≡E by metis
  }
  ultimately AOT-show ‹∀α φ{α} ≡ ∀α ψ{α}›
    using ≡I →I by auto
qed
```

AOT-theorem *cqt−basic:4*: ‹$\forall\alpha(\varphi\{\alpha\}$ & $\psi\{\alpha\}) \rightarrow (\forall\alpha\ \varphi\{\alpha\}$ & $\forall\alpha\ \psi\{\alpha\})$›
proof(*rule* →*I*)
  AOT-assume *0*: ‹$\forall\alpha(\varphi\{\alpha\}$ & $\psi\{\alpha\})$›
  AOT-have ‹$\varphi\{\alpha\}$› **and** ‹$\psi\{\alpha\}$› **for** α **using** ∀*E*(*2*) *0* &*E* **by** *blast+*
  AOT-thus ‹$\forall\alpha\ \varphi\{\alpha\}$ & $\forall\alpha\ \psi\{\alpha\}$›
    by (*auto intro*: ∀*I* &*I*)
qed

AOT-theorem *cqt−basic:5*: ‹$(\forall\alpha_1...\forall\alpha_n(\varphi\{\alpha_1...\alpha_n\})) \rightarrow \varphi\{\alpha_1...\alpha_n\}$›
  using *cqt−orig:3* **by** *blast*

AOT-theorem *cqt−basic:6*: ‹$\forall\alpha\forall\alpha\ \varphi\{\alpha\} \equiv \forall\alpha\ \varphi\{\alpha\}$›
  by (*meson* ≡*I* →*I* *GEN* *cqt−orig:1*[*const-var*])

AOT-theorem *cqt−basic:7*: ‹$(\varphi \rightarrow \forall\alpha\ \psi\{\alpha\}) \equiv \forall\alpha(\varphi \rightarrow \psi\{\alpha\})$›
  by (*metis* →*I* *vdash−properties:6* *rule−ui:3* ≡*I* *GEN*)

AOT-theorem *cqt−basic:8*: ‹$(\forall\alpha\ \varphi\{\alpha\} \vee \forall\alpha\ \psi\{\alpha\}) \rightarrow \forall\alpha\ (\varphi\{\alpha\} \vee \psi\{\alpha\})$›
  by (*simp add*: ∨*I*(*3*) →*I* *GEN* *cqt−orig:1*[*const-var*])

AOT-theorem *cqt−basic:9*:
  ‹$(\forall\alpha\ (\varphi\{\alpha\} \rightarrow \psi\{\alpha\})$ & $\forall\alpha\ (\psi\{\alpha\} \rightarrow \chi\{\alpha\})) \rightarrow \forall\alpha(\varphi\{\alpha\} \rightarrow \chi\{\alpha\})$›
proof −
  {
    AOT-assume ‹$\forall\alpha\ (\varphi\{\alpha\} \rightarrow \psi\{\alpha\})$›
    moreover AOT-assume ‹$\forall\alpha\ (\psi\{\alpha\} \rightarrow \chi\{\alpha\})$›
    ultimately AOT-have ‹$\varphi\{\alpha\} \rightarrow \psi\{\alpha\}$› **and** ‹$\psi\{\alpha\} \rightarrow \chi\{\alpha\}$› **for** α
      using ∀*E* **by** *blast+*
    AOT-hence ‹$\varphi\{\alpha\} \rightarrow \chi\{\alpha\}$› **for** α **by** (*metis* →*E* →*I*)
    AOT-hence ‹$\forall\alpha(\varphi\{\alpha\} \rightarrow \chi\{\alpha\})$› **using** ∀*I* **by** *fast*
  }
  thus *?thesis* **using** &*I* →*I* &*E* **by** *meson*
qed

AOT-theorem *cqt−basic:10*:
  ‹$(\forall\alpha(\varphi\{\alpha\} \equiv \psi\{\alpha\})$ & $\forall\alpha(\psi\{\alpha\} \equiv \chi\{\alpha\})) \rightarrow \forall\alpha\ (\varphi\{\alpha\} \equiv \chi\{\alpha\})$›
proof(*rule* →*I*; *rule* ∀*I*)
  fix β
  AOT-assume ‹$\forall\alpha(\varphi\{\alpha\} \equiv \psi\{\alpha\})$ & $\forall\alpha(\psi\{\alpha\} \equiv \chi\{\alpha\})$›
  AOT-hence ‹$\varphi\{\beta\} \equiv \psi\{\beta\}$› **and** ‹$\psi\{\beta\} \equiv \chi\{\beta\}$› **using** &*E* ∀*E* **by** *blast+*
  AOT-thus ‹$\varphi\{\beta\} \equiv \chi\{\beta\}$› **using** ≡*I* ≡*E* **by** *blast*
qed

AOT-theorem *cqt−basic:11*: ‹$\forall\alpha(\varphi\{\alpha\} \equiv \psi\{\alpha\}) \equiv \forall\alpha\ (\psi\{\alpha\} \equiv \varphi\{\alpha\})$›
proof (*rule* ≡*I*; *rule* →*I*)
  AOT-assume *0*: ‹$\forall\alpha(\varphi\{\alpha\} \equiv \psi\{\alpha\})$›
  {
    fix α
    AOT-have ‹$\varphi\{\alpha\} \equiv \psi\{\alpha\}$› **using** *0* ∀*E* **by** *blast*
    AOT-hence ‹$\psi\{\alpha\} \equiv \varphi\{\alpha\}$› **using** ≡*I* ≡*E* →*I* →*E* **by** *metis*
  }
  AOT-thus ‹$\forall\alpha(\psi\{\alpha\} \equiv \varphi\{\alpha\})$› **using** ∀*I* **by** *fast*
next
  AOT-assume *0*: ‹$\forall\alpha(\psi\{\alpha\} \equiv \varphi\{\alpha\})$›
  {
    fix α
    AOT-have ‹$\psi\{\alpha\} \equiv \varphi\{\alpha\}$› **using** *0* ∀*E* **by** *blast*
    AOT-hence ‹$\varphi\{\alpha\} \equiv \psi\{\alpha\}$› **using** ≡*I* ≡*E* →*I* →*E* **by** *metis*
  }
  AOT-thus ‹$\forall\alpha(\varphi\{\alpha\} \equiv \psi\{\alpha\})$› **using** ∀*I* **by** *fast*
qed

AOT-theorem *cqt−basic:12*: ‹$\forall\alpha\ \varphi\{\alpha\} \rightarrow \forall\alpha\ (\psi\{\alpha\} \rightarrow \varphi\{\alpha\})$›
by (*simp add*: $\forall E(2)$ $\rightarrow I$ *GEN*)

AOT-theorem *cqt−basic:13*: ‹$\forall\alpha\ \varphi\{\alpha\} \equiv \forall\beta\ \varphi\{\beta\}$›
using $\equiv I$ $\rightarrow I$ **by** *blast*

AOT-theorem *cqt−basic:14*:
‹$(\forall\alpha_1...\forall\alpha_n\ (\varphi\{\alpha_1...\alpha_n\} \rightarrow \psi\{\alpha_1...\alpha_n\})) \rightarrow$
$((\forall\alpha_1...\forall\alpha_n\ \varphi\{\alpha_1...\alpha_n\}) \rightarrow (\forall\alpha_1...\forall\alpha_n\ \psi\{\alpha_1...\alpha_n\}))$›
using *cqt:3*[*axiom-inst*] **by** *auto*

AOT-theorem *cqt−basic:15*:
‹$(\forall\alpha_1...\forall\alpha_n\ (\varphi \rightarrow \psi\{\alpha_1...\alpha_n\})) \rightarrow (\varphi \rightarrow (\forall\alpha_1...\forall\alpha_n\ \psi\{\alpha_1...\alpha_n\}))$›
using *cqt−orig:2* **by** *auto*

AOT-theorem *universal−cor*:
assumes ‹*for arbitrary* β: $\varphi\{\beta\}$›
shows ‹$\forall\alpha\ \varphi\{\alpha\}$›
using *GEN assms* **.**

AOT-theorem *existential:1*:
assumes ‹$\varphi\{\tau\}$› **and** ‹$\tau\downarrow$›
shows ‹$\exists\alpha\ \varphi\{\alpha\}$›
proof(*rule raa−cor:1*)
AOT-assume ‹$\neg\exists\alpha\ \varphi\{\alpha\}$›
AOT-hence ‹$\forall\alpha\ \neg\varphi\{\alpha\}$›
using $\equiv_{df}I$ *conventions:4* *RAA* &*I* **by** *blast*
AOT-hence ‹$\neg\varphi\{\tau\}$› **using** *assms*(*2*) $\forall E(1)$ $\rightarrow E$ **by** *blast*
AOT-thus ‹$\varphi\{\tau\}$ & $\neg\varphi\{\tau\}$› **using** *assms*(*1*) &*I* **by** *blast*
qed

AOT-theorem *existential:2*[*const-var*]:
assumes ‹$\varphi\{\beta\}$›
shows ‹$\exists\alpha\ \varphi\{\alpha\}$›
using *existential:1 cqt:2*[*const-var*][*axiom-inst*] *assms* **by** *blast*

AOT-theorem *existential:2*[*lambda*]:
assumes ‹$\varphi\{[\lambda\nu_1...\nu_n\ \psi\{\nu_1...\nu_n\}]\}$› **and** ‹*INSTANCE-OF-CQT-2*(ψ)›
shows ‹$\exists\alpha\ \varphi\{\alpha\}$›
using *existential:1 cqt:2*[*lambda*][*axiom-inst*] *assms* **by** *blast*
lemmas $\exists I$ = *existential:1 existential:2*[*const-var*]
existential:2[*lambda*]

AOT-theorem *instantiation*:
assumes ‹*for arbitrary* β: $\varphi\{\beta\} \vdash \psi$› **and** ‹$\exists\alpha\ \varphi\{\alpha\}$›
shows ‹ψ›
by (*metis* (*no-types*, *lifting*) $\equiv_{df}E$ *GEN raa−cor:3 conventions:4 assms*)
lemmas $\exists E$ = *instantiation*

AOT-theorem *cqt−further:1*: ‹$\forall\alpha\ \varphi\{\alpha\} \rightarrow \exists\alpha\ \varphi\{\alpha\}$›
using $\forall E(4)$ $\exists I(2)$ $\rightarrow I$ **by** *metis*

AOT-theorem *cqt−further:2*: ‹$\neg\forall\alpha\ \varphi\{\alpha\} \rightarrow \exists\alpha\ \neg\varphi\{\alpha\}$›
using $\forall I$ $\exists I(2)$ $\rightarrow I$ *RAA* **by** *metis*

AOT-theorem *cqt−further:3*: ‹$\forall\alpha\ \varphi\{\alpha\} \equiv \neg\exists\alpha\ \neg\varphi\{\alpha\}$›
using $\forall E(4)$ $\exists E$ $\rightarrow I$ *RAA*
by (*metis cqt−further:2* $\equiv I$ *modus−tollens:1*)

AOT-theorem *cqt−further:4*: ‹$\neg\exists\alpha\ \varphi\{\alpha\} \rightarrow \forall\alpha\ \neg\varphi\{\alpha\}$›
using $\forall I$ $\exists I(2)$$\rightarrow I$ *RAA* **by** *metis*

AOT-theorem *cqt−further:5*: ‹$\exists\alpha\ (\varphi\{\alpha\}\ \&\ \psi\{\alpha\}) \rightarrow (\exists\alpha\ \varphi\{\alpha\}\ \&\ \exists\alpha\ \psi\{\alpha\})$›

by (*metis* (*no-types*, *lifting*) &*E* &*I* ∃ *E* ∃ *I*(*2*) →*I*)

AOT-theorem *cqt−further*:*6*: ‹∃ α (φ{α} ∨ ψ{α}) → (∃ α φ{α} ∨ ∃ α ψ{α})›
by (*metis* (*mono-tags*, *lifting*) ∃ *E* ∃ *I*(*2*) ∨*E*(*3*) ∨*I*(*1*, *2*) →*I* *RAA*(*2*))

AOT-theorem *cqt−further*:*7*: ‹∃ α φ{α} ≡ ∃ β φ{β}›
by (*simp add*: *oth−class−taut*:*3*:*a*)

AOT-theorem *cqt−further*:*8*:
‹(∀ α φ{α} & ∀ α ψ{α}) → ∀ α (φ{α} ≡ ψ{α})›
by (*metis* (*mono-tags*, *lifting*) &*E* ≡*I* ∀ *E*(*2*) →*I* *GEN*)

AOT-theorem *cqt−further*:*9*:
‹(¬∃ α φ{α} & ¬∃ α ψ{α}) → ∀ α (φ{α} ≡ ψ{α})›
by (*metis* (*mono-tags*, *lifting*) &*E* ≡*I* ∃ *I*(*2*) →*I* *GEN* *raa−cor*:*4*)

AOT-theorem *cqt−further*:*10*:
‹(∃ α φ{α} & ¬∃ α ψ{α}) → ¬∀ α (φ{α} ≡ ψ{α})›
proof(*rule* →*I*; *rule* *raa−cor*:*2*)
AOT-assume *0*: ‹∃ α φ{α} & ¬∃ α ψ{α}›
then AOT-obtain α **where** ‹φ{α}› **using** ∃ *E* &*E*(*1*) **by** *metis*
moreover AOT-assume ‹∀ α (φ{α} ≡ ψ{α})›
ultimately AOT-have ‹ψ{α}› **using** ∀ *E*(*4*) ≡*E*(*1*) **by** *blast*
AOT-hence ‹∃ α ψ{α}› **using** ∃ *I* **by** *blast*
AOT-thus ‹∃ α ψ{α} & ¬∃ α ψ{α}› **using** *0* &*E*(*2*) &*I* **by** *blast*
qed

AOT-theorem *cqt−further*:*11*: ‹∃ α∃ β φ{α,β} ≡ ∃ β∃ α φ{α,β}›
using ≡*I* →*I* ∃ *I*(*2*) ∃ *E* **by** *metis*

8.7 Logical Existence, Identity, and Truth

AOT-theorem *log−prop−prop*:*1*: ‹[λ φ]↓›
using *cqt*:*2*[*lambda0*][*axiom-inst*] **by** *auto*

AOT-theorem *log−prop−prop*:*2*: ‹φ↓›
by (*rule* $\equiv_{df}$*I*[*OF existence*:*3*]) *cqt*:*2*[*lambda*]

AOT-theorem *exist−nec*: ‹τ↓ → □τ↓›
proof −
AOT-have ‹∀ β □β↓›
by (*simp add*: *GEN RN cqt*:*2*[*const-var*][*axiom-inst*])
AOT-thus ‹τ↓ → □τ↓›
using *cqt*:*1*[*axiom-inst*] →*E* **by** *blast*
qed

class *AOT-Term-id* = *AOT-Term* +
assumes *t*=*t−proper*:*1*[*AOT*]: ‹[*v* ⊨ τ = τ′ → τ↓]›
and *t*=*t−proper*:*2*[*AOT*]: ‹[*v* ⊨ τ = τ′ → τ′↓]›

instance κ :: *AOT-Term-id*
proof
AOT-modally-strict {
AOT-show ‹κ = κ′ → κ↓› **for** κ κ′
proof(*rule* →*I*)
AOT-assume ‹κ = κ′›
AOT-hence ‹*O*!κ ∨ *A*!κ›
by (*rule* ∨*I*(*3*)[*OF* $\equiv_{df}$*E*[*OF identity*:*1*]])
(*meson* →*I* ∨*I*(*1*) &*E*(*1*))+
AOT-thus ‹κ↓›
by (*rule* ∨*E*(*1*))

```
        (metis cqt:5:a[axiom-inst] →I →E &E(2))+
    qed
  }
next
  AOT-modally-strict {
    AOT-show ‹κ = κ' → κ'↓› for κ κ'
    proof(rule →I)
      AOT-assume ‹κ = κ'›
      AOT-hence ‹O!κ' ∨ A!κ'›
        by (rule ∨I(3)[OF ≡df E[OF identity:1]])
           (meson →I ∨I &E)+
      AOT-thus ‹κ'↓›
        by (rule ∨E(1))
           (metis cqt:5:a[axiom-inst] →I →E &E(2))+
    qed
  }
qed

instance rel :: (AOT-κs) AOT-Term-id
proof
  AOT-modally-strict {
    AOT-show ‹Π = Π' → Π↓› for Π Π' :: ‹<'a>›
    proof(rule →I)
      AOT-assume ‹Π = Π'›
      AOT-thus ‹Π↓› using ≡df E[OF identity:3[of Π Π']] &E by blast
    qed
  }
next
  AOT-modally-strict {
    AOT-show ‹Π = Π' → Π'↓› for Π Π' :: ‹<'a>›
    proof(rule →I)
      AOT-assume ‹Π = Π'›
      AOT-thus ‹Π'↓› using ≡df E[OF identity:3[of Π Π']] &E by blast
    qed
  }
qed

instance o :: AOT-Term-id
proof
  AOT-modally-strict {
    fix φ ψ
    AOT-show ‹φ = ψ → φ↓›
    proof(rule →I)
      AOT-assume ‹φ = ψ›
      AOT-thus ‹φ↓› using ≡df E[OF identity:4[of φ ψ]] &E by blast
    qed
  }
next
  AOT-modally-strict {
    fix φ ψ
    AOT-show ‹φ = ψ → ψ↓›
    proof(rule →I)
      AOT-assume ‹φ = ψ›
      AOT-thus ‹ψ↓› using ≡df E[OF identity:4[of φ ψ]] &E by blast
    qed
  }
qed

instance prod :: (AOT-Term-id, AOT-Term-id) AOT-Term-id
proof
  AOT-modally-strict {
    fix τ τ' :: ‹'a×'b›
    AOT-show ‹τ = τ' → τ↓›
```

proof (*induct* τ; *induct* τ'; *rule* $\rightarrow I$)
fix τ_1 $\tau_1{}'$ :: $'a$ **and** τ_2 $\tau_2{}'$ :: $'b$
AOT-assume ‹«(τ_1, τ_2)» = «($\tau_1{}'$, $\tau_2{}'$)»›
AOT-hence ‹($\tau_1 = \tau_1{}'$) & ($\tau_2 = \tau_2{}'$)› **by** (*metis* $\equiv_{df}E$ *tuple-identity-1*)
AOT-hence ‹$\tau_1\downarrow$› **and** ‹$\tau_2\downarrow$›
using *t=t−proper:1* &*E* *vdash−properties:10* **by** *blast+*
AOT-thus ‹«(τ_1, τ_2)»$\downarrow$› **by** (*metis* $\equiv_{df}I$ &*I* *tuple-denotes*)
qed
}
next
AOT-modally-strict {
fix τ τ' :: ‹$'a\times'b$›
AOT-show ‹$\tau = \tau' \rightarrow \tau'\downarrow$›
proof (*induct* τ; *induct* τ'; *rule* $\rightarrow I$)
fix τ_1 $\tau_1{}'$ :: $'a$ **and** τ_2 $\tau_2{}'$ :: $'b$
AOT-assume ‹«(τ_1, τ_2)» = «($\tau_1{}'$, $\tau_2{}'$)»›
AOT-hence ‹($\tau_1 = \tau_1{}'$) & ($\tau_2 = \tau_2{}'$)› **by** (*metis* $\equiv_{df}E$ *tuple-identity-1*)
AOT-hence ‹$\tau_1{}'\downarrow$› **and** ‹$\tau_2{}'\downarrow$›
using *t=t−proper:2* &*E* *vdash−properties:10* **by** *blast+*
AOT-thus ‹«($\tau_1{}'$, $\tau_2{}'$)»$\downarrow$› **by** (*metis* $\equiv_{df}I$ &*I* *tuple-denotes*)
qed
}
qed

AOT-register-type-constraints
Term: ‹-::*AOT-Term-id*› ‹-::*AOT-Term-id*›
AOT-register-type-constraints
Individual: ‹κ› ‹-::{*AOT-κs*, *AOT-Term-id*}›
AOT-register-type-constraints
Relation: ‹<-::{*AOT-κs*, *AOT-Term-id*}>›

AOT-theorem *id−rel−nec−equiv:1*:
‹$\Pi = \Pi' \rightarrow \Box\forall x_1...\forall x_n$ ($[\Pi]x_1...x_n \equiv [\Pi']x_1...x_n$)›
proof(*rule* $\rightarrow I$)
AOT-assume *assumption*: ‹$\Pi = \Pi'$›
AOT-hence ‹$\Pi\downarrow$› **and** ‹$\Pi'\downarrow$›
using *t=t−proper:1 t=t−proper:2 MP* **by** *blast+*
moreover AOT-have ‹$\forall F\forall G$ ($F = G \rightarrow ((\Box\forall x_1...\forall x_n ([F]x_1...x_n \equiv [F]x_1...x_n)) \rightarrow$
$\Box\forall x_1...\forall x_n ([F]x_1...x_n \equiv [G]x_1...x_n)))$›
apply (*rule GEN*)+ **using** *l−identity*[*axiom-inst*] **by** *force*
ultimately AOT-have ‹$\Pi = \Pi' \rightarrow ((\Box\forall x_1...\forall x_n ([\Pi]x_1...x_n \equiv [\Pi]x_1...x_n)) \rightarrow$
$\Box\forall x_1...\forall x_n ([\Pi]x_1...x_n \equiv [\Pi']x_1...x_n))$›
using $\forall E(1)$ **by** *blast*
AOT-hence ‹$(\Box\forall x_1...\forall x_n ([\Pi]x_1...x_n \equiv [\Pi]x_1...x_n)) \rightarrow$
$\Box\forall x_1...\forall x_n ([\Pi]x_1...x_n \equiv [\Pi']x_1...x_n)$›
using *assumption* $\rightarrow E$ **by** *blast*
moreover AOT-have ‹$\Box\forall x_1...\forall x_n ([\Pi]x_1...x_n \equiv [\Pi]x_1...x_n)$›
by (*simp add*: *RN oth−class−taut:3:a universal−cor*)
ultimately AOT-show ‹$\Box\forall x_1...\forall x_n ([\Pi]x_1...x_n \equiv [\Pi']x_1...x_n)$›
using $\rightarrow E$ **by** *blast*
qed

AOT-theorem *id−rel−nec−equiv:2*: ‹$\varphi = \psi \rightarrow \Box(\varphi \equiv \psi)$›
proof(*rule* $\rightarrow I$)
AOT-assume *assumption*: ‹$\varphi = \psi$›
AOT-hence ‹$\varphi\downarrow$› **and** ‹$\psi\downarrow$›
using *t=t−proper:1 t=t−proper:2 MP* **by** *blast+*
moreover AOT-have ‹$\forall p\forall q$ ($p = q \rightarrow ((\Box(p \equiv p) \rightarrow \Box(p \equiv q))))$›
apply (*rule GEN*)+ **using** *l−identity*[*axiom-inst*] **by** *force*
ultimately AOT-have ‹$\varphi = \psi \rightarrow (\Box(\varphi \equiv \varphi) \rightarrow \Box(\varphi \equiv \psi))$›
using $\forall E(1)$ **by** *blast*
AOT-hence ‹$\Box(\varphi \equiv \varphi) \rightarrow \Box(\varphi \equiv \psi)$›

```
    using assumption →E by blast
  moreover AOT-have ‹□(φ ≡ φ)›
    by (simp add: RN oth−class−taut:3:a universal−cor)
  ultimately AOT-show ‹□(φ ≡ ψ)›
    using →E by blast
qed

AOT-theorem rule=E:
  assumes ‹φ{τ}› and ‹τ = σ›
  shows ‹φ{σ}›
proof −
  AOT-have ‹τ↓› and ‹σ↓›
    using assms(2) t=t−proper:1 t=t−proper:2 →E by blast+
  moreover AOT-have ‹∀α∀β(α = β → (φ{α} → φ{β}))›
    apply (rule GEN)+ using l−identity[axiom-inst] by blast
  ultimately AOT-have ‹τ = σ → (φ{τ} → φ{σ})›
    using ∀E(1) by blast
  AOT-thus ‹φ{σ}› using assms →E by blast
qed

AOT-theorem propositions−lemma:1: ‹[λ φ] = φ›
proof −
  AOT-have ‹φ↓› by (simp add: log−prop−prop:2)
  moreover AOT-have ‹∀p [λ p] = p›
    using lambda−predicates:3[zero][axiom-inst] ∀I by fast
  ultimately AOT-show ‹[λ φ] = φ›
    using ∀E by blast
qed

AOT-theorem propositions−lemma:2: ‹[λ φ] ≡ φ›
proof −
  AOT-have ‹[λ φ] ≡ [λ φ]› by (simp add: oth−class−taut:3:a)
  AOT-thus ‹[λ φ] ≡ φ› using propositions−lemma:1 rule=E by blast
qed
```

propositions-lemma:3 through propositions-lemma:5 hold implicitly

```
AOT-theorem propositions−lemma:6: ‹(φ ≡ ψ) ≡ ([λ φ] ≡ [λ ψ])›
  by (metis ≡E(1) ≡E(5) Associativity of ≡ propositions−lemma:2)
```

dr-alphabetic-rules holds implicitly

```
AOT-theorem oa−exist:1: ‹O!↓›
proof −
  AOT-have ‹[λx ◇[E!]x]↓› by cqt:2[lambda]
  AOT-hence 1: ‹O! = [λx ◇[E!]x]›
    using df−rules−terms[4][OF oa:1, THEN &E(1)] →E by blast
  AOT-show ‹O!↓› using t=t−proper:1[THEN →E, OF 1] by simp
qed

AOT-theorem oa−exist:2: ‹A!↓›
proof −
  AOT-have ‹[λx ¬◇[E!]x]↓› by cqt:2[lambda]
  AOT-hence 1: ‹A! = [λx ¬◇[E!]x]›
    using df−rules−terms[4][OF oa:2, THEN &E(1)] →E by blast
  AOT-show ‹A!↓› using t=t−proper:1[THEN →E, OF 1] by simp
qed

AOT-theorem oa−exist:3: ‹O!x ∨ A!x›
proof(rule raa−cor:1)
  AOT-assume ‹¬(O!x ∨ A!x)›
  AOT-hence A: ‹¬O!x› and B: ‹¬A!x›
    using Disjunction Addition(1) modus−tollens:1
        ∨I(2) raa−cor:5 by blast+
  AOT-have C: ‹O! = [λx ◇[E!]x]›
```

by (*rule df−rules−terms*[*4*][*OF oa*:*1*, *THEN* &*E*(*1*), *THEN* →*E*]) *cqt*:*2*
AOT-have *D*: ‹$A! = [\lambda x\ \neg\Diamond[E!]x]$›
by (*rule df−rules−terms*[*4*][*OF oa*:*2*, *THEN* &*E*(*1*), *THEN* →*E*]) *cqt*:*2*
AOT-have *E*: ‹$\neg[\lambda x\ \Diamond[E!]x]x$›
using *A C rule*=*E* **by** *fast*
AOT-have *F*: ‹$\neg[\lambda x\ \neg\Diamond[E!]x]x$›
using *B D rule*=*E* **by** *fast*
AOT-have *G*: ‹$[\lambda x\ \Diamond[E!]x]x \equiv \Diamond[E!]x$›
by (*rule lambda−predicates*:*2*[*axiom-inst*, *THEN* →*E*]) *cqt*:*2*
AOT-have *H*: ‹$[\lambda x\ \neg\Diamond[E!]x]x \equiv \neg\Diamond[E!]x$›
by (*rule lambda−predicates*:*2*[*axiom-inst*, *THEN* →*E*]) *cqt*:*2*
AOT-show ‹$\neg\Diamond[E!]x\ \&\ \neg\neg\Diamond[E!]x$› **using** *G E* ≡*E H F* ≡*E* &*I* **by** *metis*
qed

AOT-theorem *p−identity−thm2*:*1*: ‹$F = G \equiv \Box\forall x(x[F] \equiv x[G])$›
proof −
AOT-have ‹$F = G \equiv F\downarrow\ \&\ G\downarrow\ \&\ \Box\forall x(x[F] \equiv x[G])$›
using *identity*:*2 df−rules−formulas*[*3*] *df−rules−formulas*[*4*]
→*E* &*E* ≡*I* →*I* **by** *blast*
moreover AOT-have ‹$F\downarrow$› **and** ‹$G\downarrow$›
by (*auto simp*: *cqt*:*2*[*const-var*][*axiom-inst*])
ultimately AOT-show ‹$F = G \equiv \Box\forall x(x[F] \equiv x[G])$›
using ≡*S*(*1*) &*I* **by** *blast*
qed

AOT-theorem *p−identity−thm2*:*2*[*2*]:
‹$F = G \equiv \forall y_1([\lambda x\ [F]xy_1] = [\lambda x\ [G]xy_1]\ \&\ [\lambda x\ [F]y_1x] = [\lambda x\ [G]y_1x])$›
proof −
AOT-have ‹$F = G \equiv F\downarrow\ \&\ G\downarrow\ \&$
$\forall y_1([\lambda x\ [F]xy_1] = [\lambda x\ [G]xy_1]\ \&\ [\lambda x\ [F]y_1x] = [\lambda x\ [G]y_1x])$›
using *identity*:*3*[*2*] *df−rules−formulas*[*3*] *df−rules−formulas*[*4*]
→*E* &*E* ≡*I* →*I* **by** *blast*
moreover AOT-have ‹$F\downarrow$› **and** ‹$G\downarrow$›
by (*auto simp*: *cqt*:*2*[*const-var*][*axiom-inst*])
ultimately show *?thesis*
using ≡*S*(*1*) &*I* **by** *blast*
qed

AOT-theorem *p−identity−thm2*:*2*[*3*]:
‹$F = G \equiv \forall y_1\forall y_2([\lambda x\ [F]xy_1y_2] = [\lambda x\ [G]xy_1y_2]\ \&$
$[\lambda x\ [F]y_1xy_2] = [\lambda x\ [G]y_1xy_2]\ \&$
$[\lambda x\ [F]y_1y_2x] = [\lambda x\ [G]y_1y_2x])$›
proof −
AOT-have ‹$F = G \equiv F\downarrow\ \&\ G\downarrow\ \&\ \forall y_1\forall y_2([\lambda x\ [F]xy_1y_2] = [\lambda x\ [G]xy_1y_2]\ \&$
$[\lambda x\ [F]y_1xy_2] = [\lambda x\ [G]y_1xy_2]\ \&$
$[\lambda x\ [F]y_1y_2x] = [\lambda x\ [G]y_1y_2x])$›
using *identity*:*3*[*3*] *df−rules−formulas*[*3*] *df−rules−formulas*[*4*]
→*E* &*E* ≡*I* →*I* **by** *blast*
moreover AOT-have ‹$F\downarrow$› **and** ‹$G\downarrow$›
by (*auto simp*: *cqt*:*2*[*const-var*][*axiom-inst*])
ultimately show *?thesis*
using ≡*S*(*1*) &*I* **by** *blast*
qed

AOT-theorem *p−identity−thm2*:*2*[*4*]:
‹$F = G \equiv \forall y_1\forall y_2\forall y_3([\lambda x\ [F]xy_1y_2y_3] = [\lambda x\ [G]xy_1y_2y_3]\ \&$
$[\lambda x\ [F]y_1xy_2y_3] = [\lambda x\ [G]y_1xy_2y_3]\ \&$
$[\lambda x\ [F]y_1y_2xy_3] = [\lambda x\ [G]y_1y_2xy_3]\ \&$
$[\lambda x\ [F]y_1y_2y_3x] = [\lambda x\ [G]y_1y_2y_3x])$›
proof −
AOT-have ‹$F = G \equiv F\downarrow\ \&\ G\downarrow\ \&\ \forall y_1\forall y_2\forall y_3([\lambda x\ [F]xy_1y_2y_3] = [\lambda x\ [G]xy_1y_2y_3]\ \&$
$[\lambda x\ [F]y_1xy_2y_3] = [\lambda x\ [G]y_1xy_2y_3]\ \&$
$[\lambda x\ [F]y_1y_2xy_3] = [\lambda x\ [G]y_1y_2xy_3]\ \&$

$[\lambda x\ [F]y_1y_2y_3x] = [\lambda x\ [G]y_1y_2y_3x])$›
using *identity*:*3*[*4*] *df−rules−formulas*[*3*] *df−rules−formulas*[*4*]
→*E* &*E* ≡*I* →*I* **by** *blast*
moreover AOT-have ‹$F\downarrow$› **and** ‹$G\downarrow$›
by (*auto simp*: *cqt*:*2*[*const-var*][*axiom-inst*])
ultimately show *?thesis*
using ≡*S*(*1*) &*I* **by** *blast*
qed

AOT-theorem *p−identity−thm2*:*2*:
‹$F = G \equiv \forall x_1...\forall x_n$ «*AOT-sem-proj-id* x_1x_n ($\lambda\ \tau$. «$[F]\tau$») ($\lambda\ \tau$. «$[G]\tau$»)»›
proof −
AOT-have ‹$F = G \equiv F\downarrow\ \&\ G\downarrow\ \&$
$\forall x_1...\forall x_n$ «*AOT-sem-proj-id* x_1x_n ($\lambda\ \tau$. «$[F]\tau$») ($\lambda\ \tau$. «$[G]\tau$»)»›
using *identity*:*3* *df−rules−formulas*[*3*] *df−rules−formulas*[*4*]
→*E* &*E* ≡*I* →*I* **by** *blast*
moreover AOT-have ‹$F\downarrow$› **and** ‹$G\downarrow$›
by (*auto simp*: *cqt*:*2*[*const-var*][*axiom-inst*])
ultimately show *?thesis*
using ≡*S*(*1*) &*I* **by** *blast*
qed

AOT-theorem *p−identity−thm2*:*3*:
‹$p = q \equiv [\lambda x\ p] = [\lambda x\ q]$›
proof −
AOT-have ‹$p = q \equiv p\downarrow\ \&\ q\downarrow\ \&\ [\lambda x\ p] = [\lambda x\ q]$›
using *identity*:*4* *df−rules−formulas*[*3*] *df−rules−formulas*[*4*]
→*E* &*E* ≡*I* →*I* **by** *blast*
moreover AOT-have ‹$p\downarrow$› **and** ‹$q\downarrow$›
by (*auto simp*: *cqt*:*2*[*const-var*][*axiom-inst*])
ultimately show *?thesis*
using ≡*S*(*1*) &*I* **by** *blast*
qed

class *AOT-Term-id-2* = *AOT-Term-id* + **assumes** *id−eq*:*1*: ‹$[v \models \alpha = \alpha]$›

instance κ :: *AOT-Term-id-2*
proof
AOT-modally-strict {
fix *x*
{
AOT-assume ‹$O!x$›
moreover AOT-have ‹$\Box\forall F([F]x \equiv [F]x)$›
using *RN GEN oth−class−taut*:*3*:*a* **by** *fast*
ultimately AOT-have ‹$O!x\ \&\ O!x\ \&\ \Box\forall F([F]x \equiv [F]x)$› **using** &*I* **by** *simp*
}
moreover {
AOT-assume ‹$A!x$›
moreover AOT-have ‹$\Box\forall F(x[F] \equiv x[F])$›
using *RN GEN oth−class−taut*:*3*:*a* **by** *fast*
ultimately AOT-have ‹$A!x\ \&\ A!x\ \&\ \Box\forall F(x[F] \equiv x[F])$› **using** &*I* **by** *simp*
}
ultimately AOT-have ‹$(O!x\ \&\ O!x\ \&\ \Box\forall F([F]x \equiv [F]x)) \vee$
$(A!x\ \&\ A!x\ \&\ \Box\forall F(x[F] \equiv x[F]))$›
using *oa−exist*:*3* ∨*I*(*1*) ∨*I*(*2*) ∨*E*(*3*) *raa−cor*:*1* **by** *blast*
AOT-thus ‹$x = x$›
using *identity*:*1*[*THEN df−rules−formulas*[*4*]] →*E* **by** *blast*
}
qed

instance *rel* :: ({*AOT-κs*,*AOT-Term-id-2*}) *AOT-Term-id-2*
proof
AOT-modally-strict {

```
    fix F :: <'a> AOT-var
    AOT-have 0: ‹[λx_1...x_n [F]x_1...x_n] = F›
      by (simp add: lambda−predicates:3[axiom-inst])
    AOT-have ‹[λx_1...x_n [F]x_1...x_n]↓›
      by cqt:2[lambda]
    AOT-hence ‹[λx_1...x_n [F]x_1...x_n] = [λx_1...x_n [F]x_1...x_n]›
      using lambda−predicates:1[axiom-inst] →E by blast
    AOT-show ‹F = F› using rule=E 0 by force
  }
qed

instance o :: AOT-Term-id-2
proof
  AOT-modally-strict {
    fix p
    AOT-have 0: ‹[λ p] = p›
      by (simp add: lambda−predicates:3[zero][axiom-inst])
    AOT-have ‹[λ p]↓›
      by (rule cqt:2[lambda0][axiom-inst])
    AOT-hence ‹[λ p] = [λ p]›
      using lambda−predicates:1[zero][axiom-inst] →E by blast
    AOT-show ‹p = p› using rule=E 0 by force
  }
qed

instance prod :: (AOT-Term-id-2, AOT-Term-id-2) AOT-Term-id-2
proof
  AOT-modally-strict {
    fix α :: ‹('a×'b) AOT-var›
    AOT-show ‹α = α›
    proof (induct)
      AOT-show ‹τ = τ› if ‹τ↓› for τ :: ‹'a×'b›
        using that
      proof (induct τ)
        fix τ_1 :: 'a and τ_2 :: 'b
        AOT-assume ‹«(τ_1,τ_2)»↓›
        AOT-hence ‹τ_1↓› and ‹τ_2↓›
          using ≡_df E &E tuple-denotes by blast+
        AOT-hence ‹τ_1 = τ_1› and ‹τ_2 = τ_2›
          using id−eq:1[unvarify α] by blast+
        AOT-thus ‹«(τ_1, τ_2)» = «(τ_1, τ_2)»›
          by (metis ≡_df I &I tuple-identity-1)
      qed
    qed
  }
qed

AOT-register-type-constraints
  Term: ‹-::AOT-Term-id-2› ‹-::AOT-Term-id-2›
AOT-register-type-constraints
  Individual: ‹κ› ‹-::{AOT-κs, AOT-Term-id-2}›
AOT-register-type-constraints
  Relation: ‹<-::{AOT-κs, AOT-Term-id-2}>›

AOT-theorem id−eq:2: ‹α = β → β = α›
  by (meson rule=E deduction−theorem)

AOT-theorem id−eq:3: ‹α = β & β = γ → α = γ›
  using rule=E →I &E by blast

AOT-theorem id−eq:4: ‹α = β ≡ ∀γ (α = γ ≡ β = γ)›
proof (rule ≡I; rule →I)
  AOT-assume 0: ‹α = β›
```

AOT-hence *1*: ‹$\beta = \alpha$› **using** *id−eq:2* →*E* **by** *blast*
AOT-show ‹$\forall \gamma\ (\alpha = \gamma \equiv \beta = \gamma)$›
by (*rule GEN*) (*metis* ≡*I* →*I 0 1 rule*=*E*)
next
AOT-assume ‹$\forall \gamma\ (\alpha = \gamma \equiv \beta = \gamma)$›
AOT-hence ‹$\alpha = \alpha \equiv \beta = \alpha$› **using** ∀*E*(*2*) **by** *blast*
AOT-hence ‹$\alpha = \alpha \rightarrow \beta = \alpha$› **using** ≡*E*(*1*) →*I* **by** *blast*
AOT-hence ‹$\beta = \alpha$› **using** *id−eq:1* →*E* **by** *blast*
AOT-thus ‹$\alpha = \beta$› **using** *id−eq:2* →*E* **by** *blast*
qed

AOT-theorem *rule*=*I*:*1*:
assumes ‹$\tau\downarrow$›
shows ‹$\tau = \tau$›
proof −
AOT-have ‹$\forall \alpha\ (\alpha = \alpha)$›
by (*rule GEN*) (*metis id−eq:1*)
AOT-thus ‹$\tau = \tau$› **using** *assms* ∀*E* **by** *blast*
qed

AOT-theorem *rule*=*I*:*2*[*const-var*]: $\alpha = \alpha$
using *id−eq:1*.

AOT-theorem *rule*=*I*:*2*[*lambda*]:
assumes ‹*INSTANCE-OF-CQT-2*(φ)›
shows $[\lambda\nu_1...\nu_n\ \varphi\{\nu_1...\nu_n\}] = [\lambda\nu_1...\nu_n\ \varphi\{\nu_1...\nu_n\}]$
proof −
AOT-have ‹$\forall \alpha\ (\alpha = \alpha)$›
by (*rule GEN*) (*metis id−eq:1*)
moreover AOT-have ‹$[\lambda\nu_1...\nu_n\ \varphi\{\nu_1...\nu_n\}]\downarrow$›
using *assms* **by** (*rule cqt:2*[*lambda*][*axiom-inst*])
ultimately AOT-show ‹$[\lambda\nu_1...\nu_n\ \varphi\{\nu_1...\nu_n\}] = [\lambda\nu_1...\nu_n\ \varphi\{\nu_1...\nu_n\}]$›
using *assms* ∀*E* **by** *blast*
qed

lemmas =*I* = *rule*=*I*:*1* *rule*=*I*:*2*[*const-var*] *rule*=*I*:*2*[*lambda*]

AOT-theorem *rule−id−df*:*1*:
assumes ‹$\tau\{\alpha_1...\alpha_n\} =_{df} \sigma\{\alpha_1...\alpha_n\}$› **and** ‹$\sigma\{\tau_1...\tau_n\}\downarrow$›
shows ‹$\tau\{\tau_1...\tau_n\} = \sigma\{\tau_1...\tau_n\}$›
proof −
AOT-have ‹$\sigma\{\tau_1...\tau_n\}\downarrow \rightarrow \tau\{\tau_1...\tau_n\} = \sigma\{\tau_1...\tau_n\}$›
using *df−rules−terms*[*3*] *assms*(*1*) &*E* **by** *blast*
AOT-thus ‹$\tau\{\tau_1...\tau_n\} = \sigma\{\tau_1...\tau_n\}$›
using *assms*(*2*) →*E* **by** *blast*
qed

AOT-theorem *rule−id−df*:*1*[*zero*]:
assumes ‹$\tau =_{df} \sigma$› **and** ‹$\sigma\downarrow$›
shows ‹$\tau = \sigma$›
proof −
AOT-have ‹$\sigma\downarrow \rightarrow \tau = \sigma$›
using *df−rules−terms*[*4*] *assms*(*1*) &*E* **by** *blast*
AOT-thus ‹$\tau = \sigma$›
using *assms*(*2*) →*E* **by** *blast*
qed

AOT-theorem *rule−id−df*:*2*:*a*:
assumes ‹$\tau\{\alpha_1...\alpha_n\} =_{df} \sigma\{\alpha_1...\alpha_n\}$› **and** ‹$\sigma\{\tau_1...\tau_n\}\downarrow$› **and** ‹$\varphi\{\tau\{\tau_1...\tau_n\}\}$›
shows ‹$\varphi\{\sigma\{\tau_1...\tau_n\}\}$›
proof −
AOT-have ‹$\tau\{\tau_1...\tau_n\} = \sigma\{\tau_1...\tau_n\}$› **using** *rule−id−df*:*1* *assms*(*1*,*2*) **by** *blast*
AOT-thus ‹$\varphi\{\sigma\{\tau_1...\tau_n\}\}$› **using** *assms*(*3*) *rule*=*E* **by** *blast*

qed

AOT-theorem *rule−id−df:2:a*[*2*]:
  assumes ‹τ{«(α_1,α_2)»} $=_{df}$ σ{«(α_1,α_2)»}›
    and ‹σ{«(τ_1,τ_2)»}↓›
      and ‹φ{τ{«(τ_1,τ_2)»}}›
  shows ‹φ{σ{«(τ_1::′*a*::*AOT-Term-id-2*,τ_2::′*b*::*AOT-Term-id-2*)»}}›
proof −
  AOT-have ‹τ{«(τ_1,τ_2)»} = σ{«(τ_1,τ_2)»}›
    using *rule−id−df:1 assms*(*1*,*2*) **by** *auto*
  AOT-thus ‹φ{σ{«(τ_1,τ_2)»}}› **using** *assms*(*3*) *rule=E* **by** *blast*
qed

AOT-theorem *rule−id−df:2:a*[*zero*]:
  assumes ‹τ $=_{df}$ σ› **and** ‹σ↓› **and** ‹φ{τ}›
  shows ‹φ{σ}›
proof −
  AOT-have ‹τ = σ› **using** *rule−id−df:1*[*zero*] *assms*(*1*,*2*) **by** *blast*
  AOT-thus ‹φ{σ}› **using** *assms*(*3*) *rule=E* **by** *blast*
qed

lemmas $=_{df}$*E* = *rule−id−df:2:a rule−id−df:2:a*[*zero*]

AOT-theorem *rule−id−df:2:b*:
  assumes ‹τ{α_1...α_n} $=_{df}$ σ{α_1...α_n}› **and** ‹σ{τ_1...τ_n}↓› **and** ‹φ{σ{τ_1...τ_n}}›
  shows ‹φ{τ{τ_1...τ_n}}›
proof −
  AOT-have ‹τ{τ_1...τ_n} = σ{τ_1...τ_n}›
    using *rule−id−df:1 assms*(*1*,*2*) **by** *blast*
  AOT-hence ‹σ{τ_1...τ_n} = τ{τ_1...τ_n}›
    using *rule=E =I*(*1*) *t=t−proper:1* →*E* **by** *fast*
  AOT-thus ‹φ{τ{τ_1...τ_n}}› **using** *assms*(*3*) *rule=E* **by** *blast*
qed

AOT-theorem *rule−id−df:2:b*[*2*]:
  assumes ‹τ{«(α_1,α_2)»} $=_{df}$ σ{«(α_1,α_2)»}›
      and ‹σ{«(τ_1,τ_2)»}↓›
      and ‹φ{σ{«(τ_1,τ_2)»}}›
  shows ‹φ{τ{«(τ_1::′*a*::*AOT-Term-id-2*,τ_2::′*b*::*AOT-Term-id-2*)»}}›
proof −
  AOT-have ‹τ{«(τ_1,τ_2)»} = σ{«(τ_1,τ_2)»}›
    using *=I*(*1*) *rule−id−df:2:a*[*2*] *RAA*(*1*) *assms*(*1*,*2*) →*I* **by** *metis*
  AOT-hence ‹σ{«(τ_1,τ_2)»} = τ{«(τ_1,τ_2)»}›
    using *rule=E =I*(*1*) *t=t−proper:1* →*E* **by** *fast*
  AOT-thus ‹φ{τ{«(τ_1,τ_2)»}}› **using** *assms*(*3*) *rule=E* **by** *blast*
qed

AOT-theorem *rule−id−df:2:b*[*zero*]:
  assumes ‹τ $=_{df}$ σ› **and** ‹σ↓› **and** ‹φ{σ}›
  shows ‹φ{τ}›
proof −
  AOT-have ‹τ = σ› **using** *rule−id−df:1*[*zero*] *assms*(*1*,*2*) **by** *blast*
  AOT-hence ‹σ = τ›
    using *rule=E =I*(*1*) *t=t−proper:1* →*E* **by** *fast*
  AOT-thus ‹φ{τ}› **using** *assms*(*3*) *rule=E* **by** *blast*
qed

lemmas $=_{df}$*I* = *rule−id−df:2:b rule−id−df:2:b*[*zero*]

AOT-theorem *free−thms:1*: ‹τ↓ ≡ ∃ β (β = τ)›
  by (*metis* ∃ *E rule=I:1 t=t−proper:2* →*I* ∃ *I*(*1*) ≡*I* →*E*)

AOT-theorem *free−thms:2*: ‹∀ α φ{α} → (∃ β (β = τ) → φ{τ})›

by $(metis\ \exists\ E\ rule{=}E\ cqt{:}2[const\text{-}var][axiom\text{-}inst]\ {\rightarrow}I\ \forall\ E(1))$

AOT-theorem $free{-}thms{:}3[const\text{-}var]$: ‹$\exists\,\beta\ (\beta = \alpha)$›
by $(meson\ \exists\,I(2)\ id{-}eq{:}1)$

AOT-theorem $free{-}thms{:}3[lambda]$:
assumes ‹$INSTANCE\text{-}OF\text{-}CQT\text{-}2(\varphi)$›
shows ‹$\exists\,\beta\ (\beta = [\lambda\nu_1...\nu_n\ \varphi\{\nu_1...\nu_n\}])$›
by $(meson\ {=}I(3)\ assms\ cqt{:}2[lambda][axiom\text{-}inst]\ existential{:}1)$

AOT-theorem $free{-}thms{:}4[rel]$:
‹$([\Pi]\kappa_1...\kappa_n \vee \kappa_1...\kappa_n[\Pi]) \rightarrow \exists\,\beta\ (\beta = \Pi)$›
by $(metis\ rule{=}I{:}1\ \&E(1)\ \vee E(1)\ cqt{:}5{:}a[axiom\text{-}inst]$
$cqt{:}5{:}b[axiom\text{-}inst]\ {\rightarrow}I\ \exists\,I(1))$

AOT-theorem $free{-}thms{:}4[vars]$:
‹$([\Pi]\kappa_1...\kappa_n \vee \kappa_1...\kappa_n[\Pi]) \rightarrow \exists\,\beta_1...\exists\,\beta_n\ (\beta_1...\beta_n = \kappa_1...\kappa_n)$›
by $(metis\ rule{=}I{:}1\ \&E(2)\ \vee E(1)\ cqt{:}5{:}a[axiom\text{-}inst]$
$cqt{:}5{:}b[axiom\text{-}inst]\ {\rightarrow}I\ \exists\,I(1))$

AOT-theorem $free{-}thms{:}4[1,rel]$:
‹$([\Pi]\kappa \vee \kappa[\Pi]) \rightarrow \exists\,\beta\ (\beta = \Pi)$›
by $(metis\ rule{=}I{:}1\ \&E(1)\ \vee E(1)\ cqt{:}5{:}a[axiom\text{-}inst]$
$cqt{:}5{:}b[axiom\text{-}inst]\ {\rightarrow}I\ \exists\,I(1))$
AOT-theorem $free{-}thms{:}4[1,1]$:
‹$([\Pi]\kappa \vee \kappa[\Pi]) \rightarrow \exists\,\beta\ (\beta = \kappa)$›
by $(metis\ rule{=}I{:}1\ \&E(2)\ \vee E(1)\ cqt{:}5{:}a[axiom\text{-}inst]$
$cqt{:}5{:}b[axiom\text{-}inst]\ {\rightarrow}I\ \exists\,I(1))$

AOT-theorem $free{-}thms{:}4[2,rel]$:
‹$([\Pi]\kappa_1\kappa_2 \vee \kappa_1\kappa_2[\Pi]) \rightarrow \exists\,\beta\ (\beta = \Pi)$›
by $(metis\ rule{=}I{:}1\ \&E(1)\ \vee E(1)\ cqt{:}5{:}a[2][axiom\text{-}inst]$
$cqt{:}5{:}b[2][axiom\text{-}inst]\ {\rightarrow}I\ \exists\,I(1))$
AOT-theorem $free{-}thms{:}4[2,1]$:
‹$([\Pi]\kappa_1\kappa_2 \vee \kappa_1\kappa_2[\Pi]) \rightarrow \exists\,\beta\ (\beta = \kappa_1)$›
by $(metis\ rule{=}I{:}1\ \&E\ \vee E(1)\ cqt{:}5{:}a[2][axiom\text{-}inst]$
$cqt{:}5{:}b[2][axiom\text{-}inst]\ {\rightarrow}I\ \exists\,I(1))$
AOT-theorem $free{-}thms{:}4[2,2]$:
‹$([\Pi]\kappa_1\kappa_2 \vee \kappa_1\kappa_2[\Pi]) \rightarrow \exists\,\beta\ (\beta = \kappa_2)$›
by $(metis\ rule{=}I{:}1\ \&E(2)\ \vee E(1)\ cqt{:}5{:}a[2][axiom\text{-}inst]$
$cqt{:}5{:}b[2][axiom\text{-}inst]\ {\rightarrow}I\ \exists\,I(1))$
AOT-theorem $free{-}thms{:}4[3,rel]$:
‹$([\Pi]\kappa_1\kappa_2\kappa_3 \vee \kappa_1\kappa_2\kappa_3[\Pi]) \rightarrow \exists\,\beta\ (\beta = \Pi)$›
by $(metis\ rule{=}I{:}1\ \&E(1)\ \vee E(1)\ cqt{:}5{:}a[3][axiom\text{-}inst]$
$cqt{:}5{:}b[3][axiom\text{-}inst]\ {\rightarrow}I\ \exists\,I(1))$
AOT-theorem $free{-}thms{:}4[3,1]$:
‹$([\Pi]\kappa_1\kappa_2\kappa_3 \vee \kappa_1\kappa_2\kappa_3[\Pi]) \rightarrow \exists\,\beta\ (\beta = \kappa_1)$›
by $(metis\ rule{=}I{:}1\ \&E\ \vee E(1)\ cqt{:}5{:}a[3][axiom\text{-}inst]$
$cqt{:}5{:}b[3][axiom\text{-}inst]\ {\rightarrow}I\ \exists\,I(1))$
AOT-theorem $free{-}thms{:}4[3,2]$:
‹$([\Pi]\kappa_1\kappa_2\kappa_3 \vee \kappa_1\kappa_2\kappa_3[\Pi]) \rightarrow \exists\,\beta\ (\beta = \kappa_2)$›
by $(metis\ rule{=}I{:}1\ \&E\ \vee E(1)\ cqt{:}5{:}a[3][axiom\text{-}inst]$
$cqt{:}5{:}b[3][axiom\text{-}inst]\ {\rightarrow}I\ \exists\,I(1))$
AOT-theorem $free{-}thms{:}4[3,3]$:
‹$([\Pi]\kappa_1\kappa_2\kappa_3 \vee \kappa_1\kappa_2\kappa_3[\Pi]) \rightarrow \exists\,\beta\ (\beta = \kappa_3)$›
by $(metis\ rule{=}I{:}1\ \&E(2)\ \vee E(1)\ cqt{:}5{:}a[3][axiom\text{-}inst]$
$cqt{:}5{:}b[3][axiom\text{-}inst]\ {\rightarrow}I\ \exists\,I(1))$
AOT-theorem $free{-}thms{:}4[4,rel]$:
‹$([\Pi]\kappa_1\kappa_2\kappa_3\kappa_4 \vee \kappa_1\kappa_2\kappa_3\kappa_4[\Pi]) \rightarrow \exists\,\beta\ (\beta = \Pi)$›
by $(metis\ rule{=}I{:}1\ \&E(1)\ \vee E(1)\ cqt{:}5{:}a[4][axiom\text{-}inst]$
$cqt{:}5{:}b[4][axiom\text{-}inst]\ {\rightarrow}I\ \exists\,I(1))$
AOT-theorem $free{-}thms{:}4[4,1]$:
‹$([\Pi]\kappa_1\kappa_2\kappa_3\kappa_4 \vee \kappa_1\kappa_2\kappa_3\kappa_4[\Pi]) \rightarrow \exists\,\beta\ (\beta = \kappa_1)$›

by (*metis rule*=*I*:*1* &*E* ∨*E*(*1*) *cqt*:*5*:*a*[*4*][*axiom-inst*]
cqt:*5*:*b*[*4*][*axiom-inst*] →*I* ∃*I*(*1*))
AOT-theorem *free*−*thms*:*4*[*4*,*2*]:
‹$([\Pi]\kappa_1\kappa_2\kappa_3\kappa_4 \vee \kappa_1\kappa_2\kappa_3\kappa_4[\Pi]) \rightarrow \exists\beta\ (\beta = \kappa_2)$›
by (*metis rule*=*I*:*1* &*E* ∨*E*(*1*) *cqt*:*5*:*a*[*4*][*axiom-inst*]
cqt:*5*:*b*[*4*][*axiom-inst*] →*I* ∃*I*(*1*))
AOT-theorem *free*−*thms*:*4*[*4*,*3*]:
‹$([\Pi]\kappa_1\kappa_2\kappa_3\kappa_4 \vee \kappa_1\kappa_2\kappa_3\kappa_4[\Pi]) \rightarrow \exists\beta\ (\beta = \kappa_3)$›
by (*metis rule*=*I*:*1* &*E* ∨*E*(*1*) *cqt*:*5*:*a*[*4*][*axiom-inst*]
cqt:*5*:*b*[*4*][*axiom-inst*] →*I* ∃*I*(*1*))
AOT-theorem *free*−*thms*:*4*[*4*,*4*]:
‹$([\Pi]\kappa_1\kappa_2\kappa_3\kappa_4 \vee \kappa_1\kappa_2\kappa_3\kappa_4[\Pi]) \rightarrow \exists\beta\ (\beta = \kappa_4)$›
by (*metis rule*=*I*:*1* &*E*(*2*) ∨*E*(*1*) *cqt*:*5*:*a*[*4*][*axiom-inst*]
cqt:*5*:*b*[*4*][*axiom-inst*] →*I* ∃*I*(*1*))

AOT-theorem *ex*:*1*:*a*: ‹$\forall\alpha\ \alpha\downarrow$›
by (*rule GEN*) (*fact cqt*:*2*[*const-var*][*axiom-inst*])
AOT-theorem *ex*:*1*:*b*: ‹$\forall\alpha\exists\beta(\beta = \alpha)$›
by (*rule GEN*) (*fact free*−*thms*:*3*[*const-var*])

AOT-theorem *ex*:*2*:*a*: ‹$\Box\alpha\downarrow$›
by (*rule RN*) (*fact cqt*:*2*[*const-var*][*axiom-inst*])
AOT-theorem *ex*:*2*:*b*: ‹$\Box\exists\beta(\beta = \alpha)$›
by (*rule RN*) (*fact free*−*thms*:*3*[*const-var*])

AOT-theorem *ex*:*3*:*a*: ‹$\Box\forall\alpha\ \alpha\downarrow$›
by (*rule RN*) (*fact ex*:*1*:*a*)
AOT-theorem *ex*:*3*:*b*: ‹$\Box\forall\alpha\exists\beta(\beta = \alpha)$›
by (*rule RN*) (*fact ex*:*1*:*b*)

AOT-theorem *ex*:*4*:*a*: ‹$\forall\alpha\ \Box\alpha\downarrow$›
by (*rule GEN*; *rule RN*) (*fact cqt*:*2*[*const-var*][*axiom-inst*])
AOT-theorem *ex*:*4*:*b*: ‹$\forall\alpha\Box\exists\beta(\beta = \alpha)$›
by (*rule GEN*; *rule RN*) (*fact free*−*thms*:*3*[*const-var*])

AOT-theorem *ex*:*5*:*a*: ‹$\Box\forall\alpha\ \Box\alpha\downarrow$›
by (*rule RN*) (*simp add*: *ex*:*4*:*a*)
AOT-theorem *ex*:*5*:*b*: ‹$\Box\forall\alpha\Box\exists\beta(\beta = \alpha)$›
by (*rule RN*) (*simp add*: *ex*:*4*:*b*)

AOT-theorem *all*−*self*=:*1*: ‹$\Box\forall\alpha(\alpha = \alpha)$›
by (*rule RN*; *rule GEN*) (*fact id*−*eq*:*1*)
AOT-theorem *all*−*self*=:*2*: ‹$\forall\alpha\Box(\alpha = \alpha)$›
by (*rule GEN*; *rule RN*) (*fact id*−*eq*:*1*)

AOT-theorem *id*−*nec*:*1*: ‹$\alpha = \beta \rightarrow \Box(\alpha = \beta)$›
proof(*rule* →*I*)
AOT-assume ‹$\alpha = \beta$›
moreover AOT-have ‹$\Box(\alpha = \alpha)$›
by (*rule RN*) (*fact id*−*eq*:*1*)
ultimately AOT-show ‹$\Box(\alpha = \beta)$› **using** *rule*=*E* **by** *fast*
qed

AOT-theorem *id*−*nec*:*2*: ‹$\tau = \sigma \rightarrow \Box(\tau = \sigma)$›
proof(*rule* →*I*)
AOT-assume *asm*: ‹$\tau = \sigma$›
moreover AOT-have ‹$\tau\downarrow$›
using *calculation t*=*t*−*proper*:*1* →*E* **by** *blast*
moreover AOT-have ‹$\Box(\tau = \tau)$›
using *calculation all*−*self*=:*2* ∀*E*(*1*) **by** *blast*
ultimately AOT-show ‹$\Box(\tau = \sigma)$› **using** *rule*=*E* **by** *fast*
qed

AOT-theorem *term−out:1*: ‹$\varphi\{\alpha\} \equiv \exists\beta\ (\beta = \alpha\ \&\ \varphi\{\beta\})$›
proof (*rule* ≡*I*; *rule* →*I*)
  AOT-assume *asm*: ‹$\varphi\{\alpha\}$›
  AOT-show ‹$\exists\beta\ (\beta = \alpha\ \&\ \varphi\{\beta\})$›
    by (*rule* ∃*I*(*2*)[**where** *β*=*α*]; *rule* &*I*)
      (*auto simp*: *id−eq*:*1 asm*)
next
  AOT-assume *0*: ‹$\exists\beta\ (\beta = \alpha\ \&\ \varphi\{\beta\})$›
  AOT-obtain *β* **where** ‹$\beta = \alpha\ \&\ \varphi\{\beta\}$›
    using ∃*E*[*rotated*, *OF 0*] **by** *blast*
  AOT-thus ‹$\varphi\{\alpha\}$› **using** &*E rule*=*E* **by** *blast*
qed

AOT-theorem *term−out:2*: ‹$\tau\downarrow \rightarrow (\varphi\{\tau\} \equiv \exists\alpha(\alpha = \tau\ \&\ \varphi\{\alpha\}))$›
proof(*rule* →*I*)
  AOT-assume ‹$\tau\downarrow$›
  moreover AOT-have ‹$\forall\alpha\ (\varphi\{\alpha\} \equiv \exists\beta\ (\beta = \alpha\ \&\ \varphi\{\beta\}))$›
    by (*rule GEN*) (*fact term−out*:*1*)
  ultimately AOT-show ‹$\varphi\{\tau\} \equiv \exists\alpha(\alpha = \tau\ \&\ \varphi\{\alpha\})$›
    using ∀ *E* **by** *blast*
qed

AOT-theorem *term−out*:*3*:
  ‹$(\varphi\{\alpha\}\ \&\ \forall\beta(\varphi\{\beta\} \rightarrow \beta = \alpha)) \equiv \forall\beta(\varphi\{\beta\} \equiv \beta = \alpha)$›
  apply (*rule* ≡*I*; *rule* →*I*)
   apply (*frule* &*E*(*1*))
   apply (*drule* &*E*(*2*))
   apply (*rule GEN*; *rule* ≡*I*; *rule* →*I*)
  using *rule−ui*:*2*[*const-var*] *vdash−properties*:*5*
    apply *blast*
   apply (*meson rule*=*E id−eq*:*1*)
  apply (*rule* &*I*)
  using *id−eq*:*1* ≡*E*(*2*) *rule−ui*:*3*
   apply *blast*
  apply (*rule GEN*; *rule* →*I*)
  using ≡*E*(*1*) *rule−ui*:*2*[*const-var*]
  by *blast*

AOT-theorem *term−out*:*4*:
  ‹$(\varphi\{\beta\}\ \&\ \forall\alpha(\varphi\{\alpha\} \rightarrow \alpha = \beta)) \equiv \forall\alpha(\varphi\{\alpha\} \equiv \alpha = \beta)$›
  using *term−out*:*3* **.**

AOT-define *AOT-exists-unique* :: ‹$\alpha \Rightarrow \varphi \Rightarrow \varphi$› *uniqueness*:*1*:
  ‹«*AOT-exists-unique* φ» $\equiv_{df} \exists\alpha\ (\varphi\{\alpha\}\ \&\ \forall\beta\ (\varphi\{\beta\} \rightarrow \beta = \alpha))$›
syntax (*input*) *-AOT-exists-unique* :: ‹$\alpha \Rightarrow \varphi \Rightarrow \varphi$› (‹∃!- -› [*1*,*40*])
syntax (**output**) *-AOT-exists-unique* :: ‹$\alpha \Rightarrow \varphi \Rightarrow \varphi$› (‹∃!-′(-′)› [*1*,*40*])
AOT-syntax-print-translations
  -AOT-exists-unique τ φ <= *CONST AOT-exists-unique* (*-abs τ φ*)
syntax
   -AOT-exists-unique-ellipse :: ‹*id-position* ⇒ *id-position* ⇒ φ ⇒ φ›
   (‹∃!-...∃!- -› [*1*,*40*])
parse-ast-translation‹
[(***syntax-const*** ‹*-AOT-exists-unique-ellipse*›,
  fn ctx => *fn* [*a*,*b*,*c*] => *Ast.mk-appl* (*Ast.Constant AOT-exists-unique*)
  [*parseEllipseList -AOT-vars ctx* [*a*,*b*],*c*]),
 (***syntax-const*** ‹*-AOT-exists-unique*›,
  AOT-restricted-binder
    const-name ‹*AOT-exists-unique*›
    const-syntax ‹*AOT-conj*›)]›
print-translation‹*AOT-syntax-print-translations* [
  AOT-preserve-binder-abs-tr′

const-syntax ‹*AOT-exists-unique*›
syntax-const ‹-*AOT-exists-unique*›
(***syntax-const*** ‹-*AOT-exists-unique-ellipse*›, *true*)
const-name ‹*AOT-conj*›,
AOT-binder-trans
@{*theory*}
@{*binding AOT-exists-unique-binder*}
syntax-const ‹-*AOT-exists-unique*›
]›

context *AOT-meta-syntax*
begin
notation *AOT-exists-unique* (**binder** ‹∃ **!**› *20*)
end
context *AOT-no-meta-syntax*
begin
no-notation *AOT-exists-unique* (**binder** ‹∃ **!**› *20*)
end

AOT-theorem *uniqueness:2*: ‹$\exists !\alpha\ \varphi\{\alpha\} \equiv \exists\alpha\forall\beta(\varphi\{\beta\} \equiv \beta = \alpha)$›
proof(*rule* ≡*I*; *rule* →*I*)
AOT-assume ‹$\exists !\alpha\ \varphi\{\alpha\}$›
AOT-hence ‹$\exists\alpha\ (\varphi\{\alpha\}\ \&\ \forall\beta\ (\varphi\{\beta\} \rightarrow \beta = \alpha))$›
using *uniqueness:1* $\equiv_{df}E$ **by** *blast*
then AOT-obtain α **where** ‹$\varphi\{\alpha\}\ \&\ \forall\beta\ (\varphi\{\beta\} \rightarrow \beta = \alpha)$›
using *instantiation*[*rotated*] **by** *blast*
AOT-hence ‹$\forall\beta(\varphi\{\beta\} \equiv \beta = \alpha)$›
using *term−out:3* ≡*E* **by** *blast*
AOT-thus ‹$\exists\alpha\forall\beta(\varphi\{\beta\} \equiv \beta = \alpha)$›
using ∃*I* **by** *fast*
next
AOT-assume ‹$\exists\alpha\forall\beta(\varphi\{\beta\} \equiv \beta = \alpha)$›
then AOT-obtain α **where** ‹$\forall\beta\ (\varphi\{\beta\} \equiv \beta = \alpha)$›
using *instantiation*[*rotated*] **by** *blast*
AOT-hence ‹$\varphi\{\alpha\}\ \&\ \forall\beta\ (\varphi\{\beta\} \rightarrow \beta = \alpha)$›
using *term−out:3* ≡*E* **by** *blast*
AOT-hence ‹$\exists\alpha\ (\varphi\{\alpha\}\ \&\ \forall\beta\ (\varphi\{\beta\} \rightarrow \beta = \alpha))$›
using ∃*I* **by** *fast*
AOT-thus ‹$\exists !\alpha\ \varphi\{\alpha\}$›
using *uniqueness:1* $\equiv_{df}I$ **by** *blast*
qed

AOT-theorem *uni−most*: ‹$\exists !\alpha\ \varphi\{\alpha\} \rightarrow \forall\beta\forall\gamma((\varphi\{\beta\}\ \&\ \varphi\{\gamma\}) \rightarrow \beta = \gamma)$›
proof(*rule* →*I*; *rule GEN*; *rule GEN*; *rule* →*I*)
fix β γ
AOT-assume ‹$\exists !\alpha\ \varphi\{\alpha\}$›
AOT-hence ‹$\exists\alpha\forall\beta(\varphi\{\beta\} \equiv \beta = \alpha)$›
using *uniqueness:2* ≡*E* **by** *blast*
then AOT-obtain α **where** ‹$\forall\beta(\varphi\{\beta\} \equiv \beta = \alpha)$›
using *instantiation*[*rotated*] **by** *blast*
moreover AOT-assume ‹$\varphi\{\beta\}\ \&\ \varphi\{\gamma\}$›
ultimately AOT-have ‹$\beta = \alpha$› **and** ‹$\gamma = \alpha$›
using ∀*E*(*2*) &*E* ≡*E*(*1*,*2*) **by** *blast+*
AOT-thus ‹$\beta = \gamma$›
by (*metis rule=E id−eq:2* →*E*)
qed

AOT-theorem *nec−exist−!*: ‹$\forall\alpha(\varphi\{\alpha\} \rightarrow \Box\varphi\{\alpha\}) \rightarrow (\exists !\alpha\ \varphi\{\alpha\} \rightarrow \exists !\alpha\ \Box\varphi\{\alpha\})$›
proof (*rule* →*I*; *rule* →*I*)
AOT-assume *a*: ‹$\forall\alpha(\varphi\{\alpha\} \rightarrow \Box\varphi\{\alpha\})$›
AOT-assume ‹$\exists !\alpha\ \varphi\{\alpha\}$›
AOT-hence ‹$\exists\alpha\ (\varphi\{\alpha\}\ \&\ \forall\beta\ (\varphi\{\beta\} \rightarrow \beta = \alpha))$›

using *uniqueness:1* $\equiv_{df}$*E* **by** *blast*
then AOT-obtain α **where** ξ: ‹φ{α} & ∀β (φ{β} → β = α)›
using *instantiation*[*rotated*] **by** *blast*
AOT-have ‹□φ{α}›
using ξ *a* &*E* ∀*E* →*E* **by** *fast*
moreover AOT-have ‹∀β (□φ{β} → β = α)›
apply (*rule GEN*; *rule* →*I*)
using ξ[*THEN* &*E*(*2*), *THEN* ∀*E*(*2*), *THEN* →*E*]
qml:*2*[*axiom-inst*, *THEN* →*E*] **by** *blast*
ultimately AOT-have ‹(□φ{α} & ∀β (□φ{β} → β = α))›
using &*I* **by** *blast*
AOT-thus ‹∃!α □φ{α}›
using *uniqueness:1* $\equiv_{df}$*I* ∃*I* **by** *fast*
qed

8.8 The Theory of Actuality and Descriptions

AOT-theorem *act−cond*: ‹𝓐(φ → ψ) → (𝓐φ → 𝓐ψ)›
using →*I* ≡*E*(*1*) *logic−actual−nec*:*2*[*axiom-inst*] **by** *blast*

AOT-theorem *nec−imp−act*: ‹□φ → 𝓐φ›
by (*metis act−cond contraposition*:*1*[*2*] ≡*E*(*4*)
qml:*2*[*THEN act-closure*, *axiom-inst*]
qml−act:*2*[*axiom-inst*] *RAA*(*1*) →*E* →*I*)

AOT-theorem *act−conj−act*:*1*: ‹𝓐(𝓐φ → φ)›
using →*I* ≡*E*(*2*) *logic−actual−nec*:*2*[*axiom-inst*]
logic−actual−nec:*4*[*axiom-inst*] **by** *blast*

AOT-theorem *act−conj−act*:*2*: ‹𝓐(φ → 𝓐φ)›
by (*metis* →*I* ≡*E*(*2*, *4*) *logic−actual−nec*:*2*[*axiom-inst*]
logic−actual−nec:*4*[*axiom-inst*] *RAA*(*1*))

AOT-theorem *act−conj−act*:*3*: ‹(𝓐φ & 𝓐ψ) → 𝓐(φ & ψ)›
proof −
AOT-have ‹□(φ → (ψ → (φ & ψ)))›
by (*rule RN*) (*fact Adjunction*)
AOT-hence ‹𝓐(φ → (ψ → (φ & ψ)))›
using *nec−imp−act* →*E* **by** *blast*
AOT-hence ‹𝓐φ → 𝓐(ψ → (φ & ψ))›
using *act−cond* →*E* **by** *blast*
moreover AOT-have ‹𝓐(ψ → (φ & ψ)) → (𝓐ψ → 𝓐(φ & ψ))›
by (*fact act−cond*)
ultimately AOT-have ‹𝓐φ → (𝓐ψ → 𝓐(φ & ψ))›
using →*I* →*E* **by** *metis*
AOT-thus ‹(𝓐φ & 𝓐ψ) → 𝓐(φ & ψ)›
by (*metis Importation* →*E*)
qed

AOT-theorem *act−conj−act*:*4*: ‹𝓐(𝓐φ ≡ φ)›
proof −
AOT-have ‹(𝓐(𝓐φ → φ) & 𝓐(φ → 𝓐φ)) → 𝓐((𝓐φ → φ) & (φ → 𝓐φ))›
by (*fact act−conj−act*:*3*)
moreover AOT-have ‹𝓐(𝓐φ → φ) & 𝓐(φ → 𝓐φ)›
using &*I act−conj−act*:*1 act−conj−act*:*2* **by** *simp*
ultimately AOT-have ζ: ‹𝓐((𝓐φ → φ) & (φ → 𝓐φ))›
using →*E* **by** *blast*
AOT-have ‹𝓐(((𝓐φ → φ) & (φ → 𝓐φ)) → (𝓐φ ≡ φ))›
using *conventions*:*3*[*THEN df−rules−formulas*[*2*],
THEN act-closure, *axiom-inst*] **by** *blast*
AOT-hence ‹𝓐((𝓐φ → φ) & (φ → 𝓐φ)) → 𝓐(𝓐φ ≡ φ)›
using *act−cond* →*E* **by** *blast*
AOT-thus ‹𝓐(𝓐φ ≡ φ)› **using** ζ →*E* **by** *blast*

qed

inductive *arbitrary-actualization* **for** φ **where**
‹*arbitrary-actualization* φ «$\boldsymbol{\mathcal{A}}\varphi$»›
| ‹*arbitrary-actualization* φ «$\boldsymbol{\mathcal{A}}\psi$»› **if** ‹*arbitrary-actualization* φ ψ›
declare *arbitrary-actualization.cases*[*AOT*]
arbitrary-actualization.induct[*AOT*]
arbitrary-actualization.simps[*AOT*]
arbitrary-actualization.intros[*AOT*]
syntax *arbitrary-actualization* :: ‹φ' ⇒ φ' ⇒ *AOT-prop*›
(‹*ARBITRARY'-ACTUALIZATION'*(-,-')›)

notepad
begin
AOT-modally-strict {
fix φ
AOT-have ‹*ARBITRARY-ACTUALIZATION*($\boldsymbol{\mathcal{A}}\varphi \equiv \varphi$, $\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}\varphi \equiv \varphi)$)›
using *AOT-PLM.arbitrary-actualization.intros* **by** *metis*
AOT-have ‹*ARBITRARY-ACTUALIZATION*($\boldsymbol{\mathcal{A}}\varphi \equiv \varphi$, $\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}\varphi \equiv \varphi)$)›
using *AOT-PLM.arbitrary-actualization.intros* **by** *metis*
AOT-have ‹*ARBITRARY-ACTUALIZATION*($\boldsymbol{\mathcal{A}}\varphi \equiv \varphi$, $\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}\varphi \equiv \varphi)$)›
using *AOT-PLM.arbitrary-actualization.intros* **by** *metis*
}
end

AOT-theorem *closure−act*:*1*:
assumes ‹*ARBITRARY-ACTUALIZATION*($\boldsymbol{\mathcal{A}}\varphi \equiv \varphi$, ψ)›
shows ‹ψ›
using *assms* **proof**(*induct*)
case *1*
AOT-show ‹$\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}\varphi \equiv \varphi)$›
by (*simp add*: *act−conj−act*:*4*)
next
case (*2* ψ)
AOT-thus ‹$\boldsymbol{\mathcal{A}}\psi$›
by (*metis arbitrary-actualization.simps* ≡*E*(*1*)
logic−actual−nec:*4*[*axiom-inst*])
qed

AOT-theorem *closure−act*:*2*: ‹$\forall\alpha$ $\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}\varphi\{\alpha\} \equiv \varphi\{\alpha\})$›
by (*simp add*: *act−conj−act*:*4* ∀*I*)

AOT-theorem *closure−act*:*3*: ‹$\boldsymbol{\mathcal{A}}\forall\alpha$ $\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}\varphi\{\alpha\} \equiv \varphi\{\alpha\})$›
by (*metis* (*no-types*, *lifting*) *act−conj−act*:*4* ≡*E*(*1*,*2*) ∀*I*
logic−actual−nec:*3*[*axiom-inst*]
logic−actual−nec:*4*[*axiom-inst*])

AOT-theorem *closure−act*:*4*: ‹$\boldsymbol{\mathcal{A}}\forall\alpha_1...\forall\alpha_n$ $\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}\varphi\{\alpha_1...\alpha_n\} \equiv \varphi\{\alpha_1...\alpha_n\})$›
using *closure−act*:*3* **.**

AOT-act-theorem *RA*[*1*]:
assumes ‹⊢ φ›
shows ‹⊢ $\boldsymbol{\mathcal{A}}\varphi$›
— While this proof is rejected in PLM, we merely state it as modally-fragile rule, which addresses the concern in PLM.
using ¬¬*E* *assms* ≡*E*(*3*) *logic−actual*[*act-axiom-inst*]
logic−actual−nec:*1*[*axiom-inst*] *modus−tollens*:*2* **by** *blast*
AOT-theorem *RA*[*2*]:
assumes ‹$\vdash_\Box$ φ›
shows ‹$\vdash_\Box$ $\boldsymbol{\mathcal{A}}\varphi$›
— This rule is in fact a consequence of RN and does not require an appeal to the semantics itself.

using *RN assms nec−imp−act vdash−properties*:*5* **by** *blast*
AOT-theorem *RA*[*3*]:
assumes ‹Γ $\vdash_\Box$ φ›
shows ‹**𝒜**Γ $\vdash_\Box$ **𝒜**φ›

This rule is only derivable from the semantics, but apparently no proof actually relies on it. If this turns out to be required, it is valid to derive it from the semantics just like RN, but we refrain from doing so, unless necessary.

oops — discard the rule

AOT-act-theorem *ANeg*:*1*: ‹¬**𝒜**φ ≡ ¬φ›
by (*simp add*: *RA*[*1*] *contraposition*:*1*[*1*] *deduction−theorem*
≡*I logic−actual*[*act-axiom-inst*])

AOT-act-theorem *ANeg*:*2*: ‹¬**𝒜**¬φ ≡ φ›
using *ANeg*:*1* ≡*I* ≡*E*(*5*) *useful−tautologies*:*1*
useful−tautologies:*2* **by** *blast*

AOT-theorem *Act−Basic*:*1*: ‹**𝒜**φ ∨ **𝒜**¬φ›
by (*meson* ∨*I*(*1*,*2*) ≡*E*(*2*) *logic−actual−nec*:*1*[*axiom-inst*] *raa−cor*:*1*)

AOT-theorem *Act−Basic*:*2*: ‹**𝒜**(φ & ψ) ≡ (**𝒜**φ & **𝒜**ψ)›
proof (*rule* ≡*I*; *rule* →*I*)
AOT-assume ‹**𝒜**(φ & ψ)›
moreover AOT-have ‹**𝒜**((φ & ψ) → φ)›
by (*simp add*: *RA*[*2*] *Conjunction Simplification*(*1*))
moreover AOT-have ‹**𝒜**((φ & ψ) → ψ)›
by (*simp add*: *RA*[*2*] *Conjunction Simplification*(*2*))
ultimately AOT-show ‹**𝒜**φ & **𝒜**ψ›
using *act−cond*[*THEN* →*E*, *THEN* →*E*] &*I* **by** *metis*
next
AOT-assume ‹**𝒜**φ & **𝒜**ψ›
AOT-thus ‹**𝒜**(φ & ψ)›
using *act−conj−act*:*3 vdash−properties*:*6* **by** *blast*
qed

AOT-theorem *Act−Basic*:*3*: ‹**𝒜**(φ ≡ ψ) ≡ (**𝒜**(φ → ψ) & **𝒜**(ψ → φ))›
proof (*rule* ≡*I*; *rule* →*I*)
AOT-assume ‹**𝒜**(φ ≡ ψ)›
moreover AOT-have ‹**𝒜**((φ ≡ ψ) → (φ → ψ))›
by (*simp add*: *RA*[*2*] *deduction−theorem* ≡*E*(*1*))
moreover AOT-have ‹**𝒜**((φ ≡ ψ) → (ψ → φ))›
by (*simp add*: *RA*[*2*] *deduction−theorem* ≡*E*(*2*))
ultimately AOT-show ‹**𝒜**(φ → ψ) & **𝒜**(ψ → φ)›
using *act−cond*[*THEN* →*E*, *THEN* →*E*] &*I* **by** *metis*
next
AOT-assume ‹**𝒜**(φ → ψ) & **𝒜**(ψ → φ)›
AOT-hence ‹**𝒜**((φ → ψ) & (ψ → φ))›
by (*metis act−conj−act*:*3 vdash−properties*:*10*)
moreover AOT-have ‹**𝒜**(((φ → ψ) & (ψ → φ)) → (φ ≡ ψ))›
by (*simp add*: *conventions*:*3 RA*[*2*] *df−rules−formulas*[*2*]
vdash−properties:*1*[*2*])
ultimately AOT-show ‹**𝒜**(φ ≡ ψ)›
using *act−cond*[*THEN* →*E*, *THEN* →*E*] **by** *metis*
qed

AOT-theorem *Act−Basic*:*4*: ‹(**𝒜**(φ → ψ) & **𝒜**(ψ → φ)) ≡ (**𝒜**φ ≡ **𝒜**ψ)›
proof (*rule* ≡*I*; *rule* →*I*)
AOT-assume *0*: ‹**𝒜**(φ → ψ) & **𝒜**(ψ → φ)›
AOT-show ‹**𝒜**φ ≡ **𝒜**ψ›
using *0* &*E act−cond*[*THEN* →*E*, *THEN* →*E*] ≡*I* →*I* **by** *metis*
next
AOT-assume ‹**𝒜**φ ≡ **𝒜**ψ›

```
  AOT-thus ‹𝓐(φ → ψ) & 𝓐(ψ → φ)›
    by (metis →I logic−actual−nec:2[axiom-inst] ≡E(1,2) &I)
qed

AOT-theorem Act−Basic:5: ‹𝓐(φ ≡ ψ) ≡ (𝓐φ ≡ 𝓐ψ)›
  using Act−Basic:3 Act−Basic:4 ≡E(5) by blast

AOT-theorem Act−Basic:6: ‹𝓐φ ≡ □𝓐φ›
  by (simp add: ≡I qml:2[axiom-inst] qml−act:1[axiom-inst])

AOT-theorem Act−Basic:7: ‹𝓐□φ → □𝓐φ›
  by (metis Act−Basic:6 →I →E ≡E(1,2) nec−imp−act
          qml−act:2[axiom-inst])

AOT-theorem Act−Basic:8: ‹□φ → □𝓐φ›
  using Hypothetical Syllogism nec−imp−act qml−act:1[axiom-inst] by blast

AOT-theorem Act−Basic:9: ‹𝓐(φ ∨ ψ) ≡ (𝓐φ ∨ 𝓐ψ)›
proof (rule ≡I; rule →I)
  AOT-assume ‹𝓐(φ ∨ ψ)›
  AOT-thus ‹𝓐φ ∨ 𝓐ψ›
  proof (rule raa−cor:3)
    AOT-assume ‹¬(𝓐φ ∨ 𝓐ψ)›
    AOT-hence ‹¬𝓐φ & ¬𝓐ψ›
      by (metis ≡E(1) oth−class−taut:5:d)
    AOT-hence ‹𝓐¬φ & 𝓐¬ψ›
      using logic−actual−nec:1[axiom-inst, THEN ≡E(2)] &E &I by metis
    AOT-hence ‹𝓐(¬φ & ¬ψ)›
      using ≡E Act−Basic:2 by metis
    moreover AOT-have ‹𝓐((¬φ & ¬ψ) ≡ ¬(φ ∨ ψ))›
      using RA[2] ≡E(6) oth−class−taut:3:a oth−class−taut:5:d by blast
    moreover AOT-have ‹𝓐(¬φ & ¬ψ) ≡ 𝓐(¬(φ ∨ ψ))›
      using calculation(2) by (metis Act−Basic:5 ≡E(1))
    ultimately AOT-have ‹𝓐(¬(φ ∨ ψ))› using ≡E by blast
    AOT-thus ‹¬𝓐(φ ∨ ψ)›
      using logic−actual−nec:1[axiom-inst, THEN ≡E(1)] by auto
  qed
next
  AOT-assume ‹𝓐φ ∨ 𝓐ψ›
  AOT-thus ‹𝓐(φ ∨ ψ)›
    by (meson RA[2] act−cond ∨I(1) ∨E(1) Disjunction Addition(1,2))
qed

AOT-theorem Act−Basic:10: ‹𝓐∃α φ{α} ≡ ∃α 𝓐φ{α}›
proof −
  AOT-have ϑ: ‹¬𝓐∀α ¬φ{α} ≡ ¬∀α 𝓐¬φ{α}›
    by (rule oth−class−taut:4:b[THEN ≡E(1)])
       (metis logic−actual−nec:3[axiom-inst])
  AOT-have ξ: ‹¬∀α 𝓐¬φ{α} ≡ ¬∀α ¬𝓐φ{α}›
    by (rule oth−class−taut:4:b[THEN ≡E(1)])
       (rule logic−actual−nec:1[THEN universal-closure,
              axiom-inst, THEN cqt−basic:3[THEN →E]])
  AOT-have ‹𝓐(∃α φ{α}) ≡ 𝓐(¬∀α ¬φ{α})›
    using conventions:4[THEN df−rules−formulas[1],
                         THEN act-closure, axiom-inst]
          conventions:4[THEN df−rules−formulas[2],
                         THEN act-closure, axiom-inst]
    Act−Basic:4[THEN ≡E(1)] &I Act−Basic:5[THEN ≡E(2)] by metis
  also AOT-have ‹... ≡ ¬𝓐∀α ¬φ{α}›
    by (simp add: logic−actual−nec:1 vdash−properties:1[2])
  also AOT-have ‹... ≡ ¬∀α 𝓐 ¬φ{α}› using ϑ by blast
  also AOT-have ‹... ≡ ¬∀α ¬𝓐 φ{α}› using ξ by blast
  also AOT-have ‹... ≡ ∃α 𝓐 φ{α}›
```

using *conventions:4[THEN ≡Df]* **by** (*metis ≡E(6) oth−class−taut:3:a*)
finally AOT-show ‹$\mathcal{A}\exists\alpha\ \varphi\{\alpha\} \equiv \exists\alpha\ \mathcal{A}\varphi\{\alpha\}$› **.**
qed

AOT-theorem *Act−Basic:11*:
‹$\mathcal{A}\forall\alpha(\varphi\{\alpha\} \equiv \psi\{\alpha\}) \equiv \forall\alpha(\mathcal{A}\varphi\{\alpha\} \equiv \mathcal{A}\psi\{\alpha\})$›
proof(*rule ≡I*; *rule →I*)
AOT-assume ‹$\mathcal{A}\forall\alpha(\varphi\{\alpha\} \equiv \psi\{\alpha\})$›
AOT-hence ‹$\forall\alpha\mathcal{A}(\varphi\{\alpha\} \equiv \psi\{\alpha\})$›
using *logic−actual−nec:3[axiom-inst, THEN ≡E(1)]* **by** *blast*
AOT-hence ‹$\mathcal{A}(\varphi\{\alpha\} \equiv \psi\{\alpha\})$› **for** α **using** $\forall E$ **by** *blast*
AOT-hence ‹$\mathcal{A}\varphi\{\alpha\} \equiv \mathcal{A}\psi\{\alpha\}$› **for** α **by** (*metis Act−Basic:5 ≡E(1)*)
AOT-thus ‹$\forall\alpha(\mathcal{A}\varphi\{\alpha\} \equiv \mathcal{A}\psi\{\alpha\})$› **by** (*rule ∀I*)
next
AOT-assume ‹$\forall\alpha(\mathcal{A}\varphi\{\alpha\} \equiv \mathcal{A}\psi\{\alpha\})$›
AOT-hence ‹$\mathcal{A}\varphi\{\alpha\} \equiv \mathcal{A}\psi\{\alpha\}$› **for** α **using** $\forall E$ **by** *blast*
AOT-hence ‹$\mathcal{A}(\varphi\{\alpha\} \equiv \psi\{\alpha\})$› **for** α **by** (*metis Act−Basic:5 ≡E(2)*)
AOT-hence ‹$\forall\alpha\ \mathcal{A}(\varphi\{\alpha\} \equiv \psi\{\alpha\})$› **by** (*rule ∀I*)
AOT-thus ‹$\mathcal{A}\forall\alpha(\varphi\{\alpha\} \equiv \psi\{\alpha\})$›
using *logic−actual−nec:3[axiom-inst, THEN ≡E(2)]* **by** *fast*
qed

AOT-act-theorem *act−quant−uniq*:
‹$\forall\beta(\mathcal{A}\varphi\{\beta\} \equiv \beta = \alpha) \equiv \forall\beta(\varphi\{\beta\} \equiv \beta = \alpha)$›
proof(*rule ≡I*; *rule →I*)
AOT-assume ‹$\forall\beta(\mathcal{A}\varphi\{\beta\} \equiv \beta = \alpha)$›
AOT-hence ‹$\mathcal{A}\varphi\{\beta\} \equiv \beta = \alpha$› **for** β **using** $\forall E$ **by** *blast*
AOT-hence ‹$\varphi\{\beta\} \equiv \beta = \alpha$› **for** β
using *≡I →I RA[1] ≡E(1,2) logic−actual[act-axiom-inst] →E*
by *metis*
AOT-thus ‹$\forall\beta(\varphi\{\beta\} \equiv \beta = \alpha)$› **by** (*rule ∀I*)
next
AOT-assume ‹$\forall\beta(\varphi\{\beta\} \equiv \beta = \alpha)$›
AOT-hence ‹$\varphi\{\beta\} \equiv \beta = \alpha$› **for** β **using** $\forall E$ **by** *blast*
AOT-hence ‹$\mathcal{A}\varphi\{\beta\} \equiv \beta = \alpha$› **for** β
using *≡I →I RA[1] ≡E(1,2) logic−actual[act-axiom-inst] →E*
by *metis*
AOT-thus ‹$\forall\beta(\mathcal{A}\varphi\{\beta\} \equiv \beta = \alpha)$› **by** (*rule ∀I*)
qed

AOT-act-theorem *fund−cont−desc*: ‹$x = \iota x(\varphi\{x\}) \equiv \forall z(\varphi\{z\} \equiv z = x)$›
using *descriptions[axiom-inst] act−quant−uniq ≡E(5)* **by** *fast*

AOT-act-theorem *hintikka*: ‹$x = \iota x(\varphi\{x\}) \equiv (\varphi\{x\}\ \&\ \forall z\ (\varphi\{z\} \rightarrow z = x))$›
using *Commutativity of ≡[THEN ≡E(1)] term−out:3*
fund−cont−desc ≡E(5) **by** *blast*

locale *russell-axiom* =
fixes ψ
assumes *ψ-denotes-asm*: $[v \models \psi\{\kappa\}] \Longrightarrow [v \models \kappa\downarrow]$
begin
AOT-act-theorem *russell−axiom*:
‹$\psi\{\iota x\ \varphi\{x\}\} \equiv \exists x(\varphi\{x\}\ \&\ \forall z(\varphi\{z\} \rightarrow z = x)\ \&\ \psi\{x\})$›
proof −
AOT-have *b*: ‹$\forall x\ (x = \iota x\ \varphi\{x\} \equiv (\varphi\{x\}\ \&\ \forall z(\varphi\{z\} \rightarrow z = x)))$›
using *hintikka ∀I* **by** *fast*
show *?thesis*
proof(*rule ≡I*; *rule →I*)
AOT-assume *c*: ‹$\psi\{\iota x\ \varphi\{x\}\}$›
AOT-hence *d*: ‹$\iota x\ \varphi\{x\}\downarrow$›
using *ψ-denotes-asm* **by** *blast*

```
    AOT-hence ‹∃ y (y = ιx φ{x})›
      by (metis rule=I:1 existential:1)
    then AOT-obtain a where a-def: ‹a = ιx φ{x}›
      using instantiation[rotated] by blast
    moreover AOT-have ‹a = ιx φ{x} ≡ (φ{a} & ∀ z(φ{z} → z = a))›
      using b ∀ E by blast
    ultimately AOT-have ‹φ{a} & ∀ z(φ{z} → z = a)›
      using ≡E by blast
    moreover AOT-have ‹ψ{a}›
    proof –
      AOT-have 1: ‹∀ x∀ y(x = y → y = x)›
        by (simp add: id–eq:2 universal–cor)
      AOT-have ‹a = ιx φ{x} →  ιx φ{x} = a›
        by (rule ∀ E(1)[where τ=«ιx φ{x}»]; rule ∀ E(2)[where β=a])
           (auto simp: 1 d universal–cor)
      AOT-thus ‹ψ{a}›
        using a-def c rule=E →E by blast
    qed
    ultimately AOT-have ‹φ{a} & ∀ z(φ{z} → z = a) & ψ{a}› by (rule &I)
    AOT-thus ‹∃ x(φ{x} & ∀ z(φ{z} → z = x) & ψ{x})› by (rule ∃ I)
  next
    AOT-assume ‹∃ x(φ{x} & ∀ z(φ{z} → z = x) & ψ{x})›
    then AOT-obtain b where g: ‹φ{b} & ∀ z(φ{z} → z = b) & ψ{b}›
      using instantiation[rotated] by blast
    AOT-hence h: ‹b = ιx φ{x} ≡ (φ{b} & ∀ z(φ{z} → z = b))›
      using b ∀ E by blast
    AOT-have ‹φ{b} & ∀ z(φ{z} → z = b)› and j: ‹ψ{b}›
      using g &E by blast+
    AOT-hence ‹b = ιx φ{x}› using h ≡E by blast
    AOT-thus ‹ψ{ιx φ{x}}› using j rule=E by blast
  qed
qed
end

interpretation russell–axiom[exe,1]: russell-axiom ‹λ κ . «[Π]κ»›
  by standard (metis cqt:5:a[1][axiom-inst, THEN →E] &E(2))
interpretation russell–axiom[exe,2,1,1]: russell-axiom ‹λ κ . «[Π]κκ′»›
  by standard (metis cqt:5:a[2][axiom-inst, THEN →E] &E)
interpretation russell–axiom[exe,2,1,2]: russell-axiom ‹λ κ . «[Π]κ′κ»›
  by standard (metis cqt:5:a[2][axiom-inst, THEN →E] &E(2))
interpretation russell–axiom[exe,2,2]: russell-axiom ‹λ κ . «[Π]κκ»›
  by standard (metis cqt:5:a[2][axiom-inst, THEN →E] &E(2))
interpretation russell–axiom[exe,3,1,1]: russell-axiom ‹λ κ . «[Π]κκ′κ′′»›
  by standard (metis cqt:5:a[3][axiom-inst, THEN →E] &E)
interpretation russell–axiom[exe,3,1,2]: russell-axiom ‹λ κ . «[Π]κ′κκ′′»›
  by standard (metis cqt:5:a[3][axiom-inst, THEN →E] &E)
interpretation russell–axiom[exe,3,1,3]: russell-axiom ‹λ κ . «[Π]κ′κ′′κ»›
  by standard (metis cqt:5:a[3][axiom-inst, THEN →E] &E(2))
interpretation russell–axiom[exe,3,2,1]: russell-axiom ‹λ κ . «[Π]κκκ′»›
  by standard (metis cqt:5:a[3][axiom-inst, THEN →E] &E)
interpretation russell–axiom[exe,3,2,2]: russell-axiom ‹λ κ . «[Π]κκ′κ»›
  by standard (metis cqt:5:a[3][axiom-inst, THEN →E] &E(2))
interpretation russell–axiom[exe,3,2,3]: russell-axiom ‹λ κ . «[Π]κ′κκ»›
  by standard (metis cqt:5:a[3][axiom-inst, THEN →E] &E(2))
interpretation russell–axiom[exe,3,3]: russell-axiom ‹λ κ . «[Π]κκκ»›
  by standard (metis cqt:5:a[3][axiom-inst, THEN →E] &E(2))

interpretation russell–axiom[enc,1]: russell-axiom ‹λ κ . «κ[Π]»›
  by standard (metis cqt:5:b[1][axiom-inst, THEN →E] &E(2))
interpretation russell–axiom[enc,2,1]: russell-axiom ‹λ κ . «κκ′[Π]»›
  by standard (metis cqt:5:b[2][axiom-inst, THEN →E] &E)
interpretation russell–axiom[enc,2,2]: russell-axiom ‹λ κ . «κ′κ[Π]»›
  by standard (metis cqt:5:b[2][axiom-inst, THEN →E] &E(2))
```

```
interpretation russell−axiom[enc,2,3]: russell-axiom ‹λ κ . «κκ[Π]»›
  by standard (metis cqt:5:b[2][axiom-inst, THEN →E] &E(2))
interpretation russell−axiom[enc,3,1,1]: russell-axiom ‹λ κ . «κκ'κ''[Π]»›
  by standard (metis cqt:5:b[3][axiom-inst, THEN →E] &E)
interpretation russell−axiom[enc,3,1,2]: russell-axiom ‹λ κ . «κ'κκ''[Π]»›
  by standard (metis cqt:5:b[3][axiom-inst, THEN →E] &E)
interpretation russell−axiom[enc,3,1,3]: russell-axiom ‹λ κ . «κ'κ''κ[Π]»›
  by standard (metis cqt:5:b[3][axiom-inst, THEN →E] &E(2))
interpretation russell−axiom[enc,3,2,1]: russell-axiom ‹λ κ . «κκκ'[Π]»›
  by standard (metis cqt:5:b[3][axiom-inst, THEN →E] &E)
interpretation russell−axiom[enc,3,2,2]: russell-axiom ‹λ κ . «κκ'κ[Π]»›
  by standard (metis cqt:5:b[3][axiom-inst, THEN →E] &E(2))
interpretation russell−axiom[enc,3,2,3]: russell-axiom ‹λ κ . «κ'κκ[Π]»›
  by standard (metis cqt:5:b[3][axiom-inst, THEN →E] &E(2))
interpretation russell−axiom[enc,3,3]: russell-axiom ‹λ κ . «κκκ[Π]»›
  by standard (metis cqt:5:b[3][axiom-inst, THEN →E] &E(2))

AOT-act-theorem !−exists:1: ‹ιx φ{x}↓ ≡ ∃!x φ{x}›
proof(rule ≡I; rule →I)
  AOT-assume ‹ιx φ{x}↓›
  AOT-hence ‹∃ y (y = ιx φ{x})› by (metis rule=I:1 existential:1)
  then AOT-obtain a where ‹a = ιx φ{x}›
    using instantiation[rotated] by blast
  AOT-hence ‹φ{a} & ∀ z (φ{z} → z = a)›
    using hintikka ≡E by blast
  AOT-hence ‹∃ x (φ{x} & ∀ z (φ{z} → z = x))›
    by (rule ∃ I)
  AOT-thus ‹∃ !x φ{x}›
    using uniqueness:1[THEN ≡df I] by blast
next
  AOT-assume ‹∃ !x φ{x}›
  AOT-hence ‹∃ x (φ{x} & ∀ z (φ{z} → z = x))›
    using uniqueness:1[THEN ≡df E] by blast
  then AOT-obtain b where ‹φ{b} & ∀ z (φ{z} → z = b)›
    using instantiation[rotated] by blast
  AOT-hence ‹b = ιx φ{x}›
    using hintikka ≡E by blast
  AOT-thus ‹ιx φ{x}↓›
    by (metis t=t−proper:2 vdash−properties:6)
qed

AOT-act-theorem !−exists:2: ‹∃ y(y=ιx φ{x}) ≡ ∃ !x φ{x}›
  using !−exists:1 free−thms:1 ≡E(6) by blast

AOT-act-theorem y−in:1: ‹x = ιx φ{x} → φ{x}›
  using &E(1) →I hintikka ≡E(1) by blast


AOT-act-theorem y−in:2: ‹z = ιx φ{x} → φ{z}› using y−in:1.

AOT-act-theorem y−in:3: ‹ιx φ{x}↓ → φ{ιx φ{x}}›
proof(rule →I)
  AOT-assume ‹ιx φ{x}↓›
  AOT-hence ‹∃ y (y = ιx φ{x})›
    by (metis rule=I:1 existential:1)
  then AOT-obtain a where ‹a = ιx φ{x}›
    using instantiation[rotated] by blast
  moreover AOT-have ‹φ{a}›
    using calculation hintikka ≡E(1) &E by blast
  ultimately AOT-show ‹φ{ιx φ{x}}› using rule=E by blast
qed

AOT-act-theorem y−in:4: ‹∃ y (y = ιx φ{x}) → φ{ιx φ{x}}›
```

```
  using y−in:3[THEN →E] free−thms:1[THEN ≡E(2)] →I by blast

AOT-theorem act−quant−nec:
  ‹∀β (𝒜φ{β} ≡ β = α) ≡ ∀β(𝒜𝒜φ{β} ≡ β = α)›
proof(rule ≡I; rule →I)
  AOT-assume ‹∀β (𝒜φ{β} ≡ β = α)›
  AOT-hence ‹𝒜φ{β} ≡ β = α› for β using ∀E by blast
  AOT-hence ‹𝒜𝒜φ{β} ≡ β = α› for β
    by (metis Act−Basic:5 act−conj−act:4 ≡E(1) ≡E(5))
  AOT-thus ‹∀β(𝒜𝒜φ{β} ≡ β = α)›
    by (rule ∀I)
next
  AOT-assume ‹∀β(𝒜𝒜φ{β} ≡ β = α)›
  AOT-hence ‹𝒜𝒜φ{β} ≡ β = α› for β using ∀E by blast
  AOT-hence ‹𝒜φ{β} ≡ β = α› for β
    by (metis Act−Basic:5 act−conj−act:4 ≡E(1) ≡E(6))
  AOT-thus ‹∀β (𝒜φ{β} ≡ β = α)›
    by (rule ∀I)
qed

AOT-theorem equi−desc−descA:1: ‹x = ιx φ{x} ≡ x = ιx(𝒜φ{x})›
proof −
  AOT-have ‹x = ιx φ{x} ≡ ∀z (𝒜φ{z} ≡ z = x)›
    using descriptions[axiom-inst] by blast
  also AOT-have ‹... ≡ ∀z (𝒜𝒜φ{z} ≡ z = x)›
  proof(rule ≡I; rule →I; rule ∀I)
    AOT-assume ‹∀z (𝒜φ{z} ≡ z = x)›
    AOT-hence ‹𝒜φ{a} ≡ a = x› for a
      using ∀E by blast
    AOT-thus ‹𝒜𝒜φ{a} ≡ a = x› for a
      by (metis Act−Basic:5 act−conj−act:4 ≡E(1) ≡E(5))
  next
    AOT-assume ‹∀z (𝒜𝒜φ{z} ≡ z = x)›
    AOT-hence ‹𝒜𝒜φ{a} ≡ a = x› for a
      using ∀E by blast
    AOT-thus ‹𝒜φ{a} ≡ a = x› for a
      by (metis Act−Basic:5 act−conj−act:4 ≡E(1) ≡E(6))
  qed
  also AOT-have ‹... ≡ x = ιx(𝒜φ{x})›
    using Commutativity of ≡[THEN ≡E(1)] descriptions[axiom-inst] by fast
  finally show ?thesis .
qed

AOT-theorem equi−desc−descA:2: ‹ιx φ{x}↓ → ιx φ{x} = ιx(𝒜φ{x})›
proof(rule →I)
  AOT-assume ‹ιx φ{x}↓›
  AOT-hence ‹∃y (y = ιx φ{x})›
    by (metis rule=I:1 existential:1)
  then AOT-obtain a where ‹a = ιx φ{x}›
    using instantiation[rotated] by blast
  moreover AOT-have ‹a = ιx(𝒜φ{x})›
    using calculation equi−desc−descA:1[THEN ≡E(1)] by blast
  ultimately AOT-show ‹ιx φ{x} = ιx(𝒜φ{x})›
    using rule=E by fast
qed

AOT-theorem nec−hintikka−scheme:
  ‹x = ιx φ{x} ≡ 𝒜φ{x} & ∀z(𝒜φ{z} → z = x)›
proof −
  AOT-have ‹x = ιx φ{x} ≡ ∀z(𝒜φ{z} ≡ z = x)›
    using descriptions[axiom-inst] by blast
  also AOT-have ‹. . . ≡ (𝒜φ{x} & ∀z(𝒜φ{z} → z = x))›
```

```
    using Commutativity of ≡[THEN ≡E(1)] term−out:3 by fast
  finally show ?thesis.
qed

AOT-theorem equiv−desc−eq:1:
  ‹𝒜∀x(φ{x} ≡ ψ{x}) → ∀x (x = ιx φ{x} ≡ x = ιx ψ{x})›
proof(rule →I; rule ∀I)
  fix β
  AOT-assume ‹𝒜∀x(φ{x} ≡ ψ{x})›
  AOT-hence ‹𝒜(φ{x} ≡ ψ{x})› for x
    using logic−actual−nec:3[axiom-inst, THEN ≡E(1)] ∀E(2) by blast
  AOT-hence 0: ‹𝒜φ{x} ≡ 𝒜ψ{x}› for x
    by (metis Act−Basic:5 ≡E(1))
  AOT-have ‹β = ιx φ{x} ≡ 𝒜φ{β} & ∀z(𝒜φ{z} → z = β)›
    using nec−hintikka−scheme by blast
  also AOT-have ‹... ≡ 𝒜ψ{β} & ∀z(𝒜ψ{z} → z = β)›
  proof (rule ≡I; rule →I)
    AOT-assume 1: ‹𝒜φ{β} & ∀z(𝒜φ{z} → z = β)›
    AOT-hence ‹𝒜φ{z} → z = β› for z
      using &E ∀E by blast
    AOT-hence ‹𝒜ψ{z} → z = β› for z
      using 0 ≡E →I →E by metis
    AOT-hence ‹∀z(𝒜ψ{z} → z = β)›
      using ∀I by fast
    moreover AOT-have ‹𝒜ψ{β}›
      using &E 0[THEN ≡E(1)] 1 by blast
    ultimately AOT-show ‹𝒜ψ{β} & ∀z(𝒜ψ{z} → z = β)›
      using &I by blast
  next
    AOT-assume 1: ‹𝒜ψ{β} & ∀z(𝒜ψ{z} → z = β)›
    AOT-hence ‹𝒜ψ{z} → z = β› for z
      using &E ∀E by blast
    AOT-hence ‹𝒜φ{z} → z = β› for z
      using 0 ≡E →I →E by metis
    AOT-hence ‹∀z(𝒜φ{z} → z = β)›
      using ∀I by fast
    moreover AOT-have ‹𝒜φ{β}›
      using &E 0[THEN ≡E(2)] 1 by blast
    ultimately AOT-show ‹𝒜φ{β} & ∀z(𝒜φ{z} → z = β)›
      using &I by blast
  qed
  also AOT-have ‹... ≡ β = ιx ψ{x}›
    using Commutativity of ≡[THEN ≡E(1)] nec−hintikka−scheme by blast
  finally AOT-show ‹β = ιx φ{x} ≡ β = ιx ψ{x}› .
qed

AOT-theorem equiv−desc−eq:2:
  ‹ιx φ{x}↓ & 𝒜∀x(φ{x} ≡ ψ{x}) → ιx φ{x} = ιx ψ{x}›
proof(rule →I)
  AOT-assume ‹ιx φ{x}↓ & 𝒜∀x(φ{x} ≡ ψ{x})›
  AOT-hence 0: ‹∃y (y = ιx φ{x})› and
            1: ‹∀x (x = ιx φ{x} ≡ x = ιx ψ{x})›
    using &E free−thms:1[THEN ≡E(1)] equiv−desc−eq:1 →E by blast+
  then AOT-obtain a where ‹a = ιx φ{x}›
    using instantiation[rotated] by blast
  moreover AOT-have ‹a = ιx ψ{x}›
    using calculation 1 ∀E ≡E(1) by fast
  ultimately AOT-show ‹ιx φ{x} = ιx ψ{x}›
    using rule=E by fast
qed

AOT-theorem equiv−desc−eq:3:
  ‹ιx φ{x}↓ & □∀x(φ{x} ≡ ψ{x}) → ιx φ{x} = ιx ψ{x}›
```

using $\rightarrow I$ *equiv−desc−eq*:*2*[*THEN* $\rightarrow E$, *OF* &*I*] &*E*
nec−imp−act[*THEN* $\rightarrow E$] **by** *metis*

AOT-theorem *equiv−desc−eq*:*4*: ‹$\iota x\ \varphi\{x\}\downarrow \rightarrow \Box\iota x\ \varphi\{x\}\downarrow$›
proof(*rule* $\rightarrow I$)
AOT-assume ‹$\iota x\ \varphi\{x\}\downarrow$›
AOT-hence ‹$\exists\, y\ (y = \iota x\ \varphi\{x\})$›
by (*metis rule*=*I*:*1 existential*:*1*)
then AOT-obtain *a* **where** ‹$a = \iota x\ \varphi\{x\}$›
using *instantiation*[*rotated*] **by** *blast*
AOT-thus ‹$\Box\iota x\ \varphi\{x\}\downarrow$›
using *ex*:*2*:*a rule*=*E* **by** *fast*
qed

AOT-theorem *equiv−desc−eq*:*5*: ‹$\iota x\ \varphi\{x\}\downarrow \rightarrow \exists\, y\ \Box(y = \iota x\ \varphi\{x\})$›
proof(*rule* $\rightarrow I$)
AOT-assume ‹$\iota x\ \varphi\{x\}\downarrow$›
AOT-hence ‹$\exists\, y\ (y = \iota x\ \varphi\{x\})$›
by (*metis rule*=*I*:*1 existential*:*1*)
then AOT-obtain *a* **where** ‹$a = \iota x\ \varphi\{x\}$›
using *instantiation*[*rotated*] **by** *blast*
AOT-hence ‹$\Box(a = \iota x\ \varphi\{x\})$›
by (*metis id−nec*:*2 vdash−properties*:*10*)
AOT-thus ‹$\exists\, y\ \Box(y = \iota x\ \varphi\{x\})$›
by (*rule* $\exists\, I$)
qed

AOT-act-theorem *equiv−desc−eq2*:*1*:
‹$\forall\, x\ (\varphi\{x\} \equiv \psi\{x\}) \rightarrow \forall\, x\ (x = \iota x\ \varphi\{x\} \equiv x = \iota x\ \psi\{x\})$›
using $\rightarrow I$ *logic−actual*[*act-axiom-inst*, *THEN* $\rightarrow E$]
equiv−desc−eq:*1*[*THEN* $\rightarrow E$]
RA[*1*] *deduction−theorem* **by** *blast*

AOT-act-theorem *equiv−desc−eq2*:*2*:
‹$\iota x\ \varphi\{x\}\downarrow\ \&\ \forall\, x\ (\varphi\{x\} \equiv \psi\{x\}) \rightarrow \iota x\ \varphi\{x\} = \iota x\ \psi\{x\}$›
using $\rightarrow I$ *logic−actual*[*act-axiom-inst*, *THEN* $\rightarrow E$]
equiv−desc−eq:*2*[*THEN* $\rightarrow E$, *OF* &*I*]
RA[*1*] *deduction−theorem* &*E* **by** *metis*

context *russell-axiom*
begin
AOT-theorem *nec−russell−axiom*:
‹$\psi\{\iota x\ \varphi\{x\}\} \equiv \exists\, x(\mathcal{A}\varphi\{x\}\ \&\ \forall\, z(\mathcal{A}\varphi\{z\} \rightarrow z = x)\ \&\ \psi\{x\})$›
proof −
AOT-have *b*: ‹$\forall\, x\ (x = \iota x\ \varphi\{x\} \equiv (\mathcal{A}\varphi\{x\}\ \&\ \forall\, z(\mathcal{A}\varphi\{z\} \rightarrow z = x)))$›
using *nec−hintikka−scheme* $\forall\, I$ **by** *fast*
show *?thesis*
proof(*rule* $\equiv I$; *rule* $\rightarrow I$)
AOT-assume *c*: ‹$\psi\{\iota x\ \varphi\{x\}\}$›
AOT-hence *d*: ‹$\iota x\ \varphi\{x\}\downarrow$›
using *ψ-denotes-asm* **by** *blast*
AOT-hence ‹$\exists\, y\ (y = \iota x\ \varphi\{x\})$›
by (*metis rule*=*I*:*1 existential*:*1*)
then AOT-obtain *a* **where** *a-def*: ‹$a = \iota x\ \varphi\{x\}$›
using *instantiation*[*rotated*] **by** *blast*
moreover AOT-have ‹$a = \iota x\ \varphi\{x\} \equiv (\mathcal{A}\varphi\{a\}\ \&\ \forall\, z(\mathcal{A}\varphi\{z\} \rightarrow z = a))$›
using *b* $\forall\, E$ **by** *blast*
ultimately AOT-have ‹$\mathcal{A}\varphi\{a\}\ \&\ \forall\, z(\mathcal{A}\varphi\{z\} \rightarrow z = a)$›
using $\equiv E$ **by** *blast*
moreover AOT-have ‹$\psi\{a\}$›
proof −
AOT-have *1*: ‹$\forall\, x\forall\, y(x = y \rightarrow y = x)$›

```
      by (simp add: id−eq:2 universal−cor)
    AOT-have ‹a = ιx φ{x} → ιx φ{x} = a›
      by (rule ∀E(1)[where τ=«ιx φ{x}»]; rule ∀E(2)[where β=a])
        (auto simp: d universal−cor 1)
    AOT-thus ‹ψ{a}›
      using a-def c rule=E →E by metis
  qed
  ultimately AOT-have ‹𝓐φ{a} & ∀z(𝓐φ{z} → z = a) & ψ{a}›
    by (rule &I)
  AOT-thus ‹∃x(𝓐φ{x} & ∀z(𝓐φ{z} → z = x) & ψ{x})›
    by (rule ∃I)
next
  AOT-assume ‹∃x(𝓐φ{x} & ∀z(𝓐φ{z} → z = x) & ψ{x})›
  then AOT-obtain b where g: ‹𝓐φ{b} & ∀z(𝓐φ{z} → z = b) & ψ{b}›
    using instantiation[rotated] by blast
  AOT-hence h: ‹b = ιx φ{x} ≡ (𝓐φ{b} & ∀z(𝓐φ{z} → z = b))›
    using b ∀E by blast
  AOT-have ‹𝓐φ{b} & ∀z(𝓐φ{z} → z = b)› and j: ‹ψ{b}›
    using g &E by blast+
  AOT-hence ‹b = ιx φ{x}›
    using h ≡E by blast
  AOT-thus ‹ψ{ιx φ{x}}›
    using j rule=E by blast
qed
qed
end

AOT-theorem actual−desc:1: ‹ιx φ{x}↓ ≡ ∃!x 𝓐φ{x}›
proof (rule ≡I; rule →I)
  AOT-assume ‹ιx φ{x}↓›
  AOT-hence ‹∃y (y = ιx φ{x})›
    by (metis rule=I:1 existential:1)
  then AOT-obtain a where ‹a = ιx φ{x}›
    using instantiation[rotated] by blast
  moreover AOT-have ‹a = ιx φ{x} ≡ ∀z(𝓐φ{z} ≡ z = a)›
    using descriptions[axiom-inst] by blast
  ultimately AOT-have ‹∀z(𝓐φ{z} ≡ z = a)›
    using ≡E by blast
  AOT-hence ‹∃x∀z(𝓐φ{z} ≡ z = x)› by (rule ∃I)
  AOT-thus ‹∃!x 𝓐φ{x}›
    using uniqueness:2[THEN ≡E(2)] by fast
next
  AOT-assume ‹∃!x 𝓐φ{x}›
  AOT-hence ‹∃x∀z(𝓐φ{z} ≡ z = x)›
    using uniqueness:2[THEN ≡E(1)] by fast
  then AOT-obtain a where ‹∀z(𝓐φ{z} ≡ z = a)›
    using instantiation[rotated] by blast
  moreover AOT-have ‹a = ιx φ{x} ≡ ∀z(𝓐φ{z} ≡ z = a)›
    using descriptions[axiom-inst] by blast
  ultimately AOT-have ‹a = ιx φ{x}›
    using ≡E by blast
  AOT-thus ‹ιx φ{x}↓›
    by (metis t=t−proper:2 vdash−properties:6)
qed

AOT-theorem actual−desc:2: ‹x = ιx φ{x} → 𝓐φ{x}›
  using &E(1) contraposition:1[2] ≡E(1) nec−hintikka−scheme
      reductio−aa:2 vdash−properties:9 by blast


AOT-theorem actual−desc:3: ‹z = ιx φ{x} → 𝓐φ{z}›
  using actual−desc:2.
```

AOT-theorem *actual−desc:4*: ‹$\iota x\ \varphi\{x\}\downarrow \to \mathcal{A}\varphi\{\iota x\ \varphi\{x\}\}$›
proof(*rule* →*I*)
  AOT-assume ‹$\iota x\ \varphi\{x\}\downarrow$›
  AOT-hence ‹$\exists\, y\ (y = \iota x\ \varphi\{x\})$› **by** (*metis rule*=*I*:*1 existential*:*1*)
  then AOT-obtain *a* **where** ‹$a = \iota x\ \varphi\{x\}$› **using** *instantiation*[*rotated*] **by** *blast*
  AOT-thus ‹$\mathcal{A}\varphi\{\iota x\ \varphi\{x\}\}$›
    using *actual−desc:2 rule*=*E* →*E* **by** *fast*
qed

AOT-theorem *actual−desc:5*: ‹$\iota x\ \varphi\{x\} = \iota x\ \psi\{x\} \to \mathcal{A}\forall\, x(\varphi\{x\} \equiv \psi\{x\})$›
proof(*rule* →*I*)
  AOT-assume *0*: ‹$\iota x\ \varphi\{x\} = \iota x\ \psi\{x\}$›
  AOT-hence *φ-down*: ‹$\iota x\ \varphi\{x\}\downarrow$› **and** *ψ-down*: ‹$\iota x\ \psi\{x\}\downarrow$›
    using *t*=*t*−*proper*:*1 t*=*t*−*proper*:*2 vdash*−*properties*:*6* **by** *blast*+
  AOT-hence ‹$\exists\, y\ (y = \iota x\ \varphi\{x\})$› **and** ‹$\exists\, y\ (y = \iota x\ \psi\{x\})$›
    by (*metis rule*=*I*:*1 existential*:*1*)+
  then AOT-obtain *a* **and** *b* **where** *a-eq*: ‹$a = \iota x\ \varphi\{x\}$› **and** *b-eq*: ‹$b = \iota x\ \psi\{x\}$›
    using *instantiation*[*rotated*] **by** *metis*

  AOT-have ‹$\forall\,\alpha\forall\,\beta\ (\alpha = \beta \to \beta = \alpha)$›
    by (*rule* ∀*I*; *rule* ∀*I*; *rule id*−*eq*:*2*)
  AOT-hence ‹$\forall\,\beta\ (\iota x\ \varphi\{x\} = \beta \to \beta = \iota x\ \varphi\{x\})$›
    using ∀*E φ-down* **by** *blast*
  AOT-hence ‹$\iota x\ \varphi\{x\} = \iota x\ \psi\{x\} \to \iota x\ \psi\{x\} = \iota x\ \varphi\{x\}$›
    using ∀*E ψ-down* **by** *blast*
  AOT-hence *1*: ‹$\iota x\ \psi\{x\} = \iota x\ \varphi\{x\}$› **using** *0*
    →*E* **by** *blast*

  AOT-have ‹$\mathcal{A}\varphi\{x\} \equiv \mathcal{A}\psi\{x\}$› **for** *x*
  proof(*rule* ≡*I*; *rule* →*I*)
    AOT-assume ‹$\mathcal{A}\varphi\{x\}$›
    moreover AOT-have ‹$\mathcal{A}\varphi\{x\} \to x = a$› **for** *x*
      using *nec*−*hintikka*−*scheme*[*THEN* ≡*E*(*1*), *OF a-eq*, *THEN* &*E*(*2*)]
        ∀*E* **by** *blast*
    ultimately AOT-have ‹$x = a$›
      using →*E* **by** *blast*
    AOT-hence ‹$x = \iota x\ \varphi\{x\}$›
      using *a-eq rule*=*E* **by** *blast*
    AOT-hence ‹$x = \iota x\ \psi\{x\}$›
      using *0 rule*=*E* **by** *blast*
    AOT-thus ‹$\mathcal{A}\psi\{x\}$›
      by (*metis actual*−*desc*:*3 vdash*−*properties*:*6*)
  next
    AOT-assume ‹$\mathcal{A}\psi\{x\}$›
    moreover AOT-have ‹$\mathcal{A}\psi\{x\} \to x = b$› **for** *x*
      using *nec*−*hintikka*−*scheme*[*THEN* ≡*E*(*1*), *OF b-eq*, *THEN* &*E*(*2*)]
        ∀*E* **by** *blast*
    ultimately AOT-have ‹$x = b$›
      using →*E* **by** *blast*
    AOT-hence ‹$x = \iota x\ \psi\{x\}$›
      using *b-eq rule*=*E* **by** *blast*
    AOT-hence ‹$x = \iota x\ \varphi\{x\}$›
      using *1 rule*=*E* **by** *blast*
    AOT-thus ‹$\mathcal{A}\varphi\{x\}$›
      by (*metis actual*−*desc*:*3 vdash*−*properties*:*6*)
  qed
  AOT-hence ‹$\mathcal{A}(\varphi\{x\} \equiv \psi\{x\})$› **for** *x*
    by (*metis Act*−*Basic*:*5* ≡*E*(*2*))
  AOT-hence ‹$\forall\, x\ \mathcal{A}(\varphi\{x\} \equiv \psi\{x\})$›
    by (*rule* ∀*I*)
  AOT-thus ‹$\mathcal{A}\forall\, x\ (\varphi\{x\} \equiv \psi\{x\})$›
    using *logic*−*actual*−*nec*:*3*[*axiom-inst*, *THEN* ≡*E*(*2*)] **by** *fast*
qed

```
AOT-theorem !box−desc:1: ‹∃!x □φ{x} → ∀ y (y = ιx φ{x} → φ{y})›
proof(rule →I)
  AOT-assume ‹∃!x □φ{x}›
  AOT-hence ζ: ‹∃ x (□φ{x} & ∀ z (□φ{z} → z = x))›
    using uniqueness:1[THEN ≡df E] by blast
  then AOT-obtain b where ϑ: ‹□φ{b} & ∀ z (□φ{z} → z = b)›
    using instantiation[rotated] by blast
  AOT-show ‹∀ y (y = ιx φ{x} → φ{y})›
  proof(rule GEN; rule →I)
    fix y
    AOT-assume ‹y = ιx φ{x}›
    AOT-hence ‹𝓐φ{y} & ∀ z (𝓐φ{z} → z = y)›
      using nec−hintikka−scheme[THEN ≡E(1)] by blast
    AOT-hence ‹𝓐φ{b} → b = y›
      using &E ∀ E by blast
    moreover AOT-have ‹𝓐φ{b}›
      using ϑ[THEN &E(1)]  by (metis nec−imp−act →E)
    ultimately AOT-have ‹b = y›
      using →E by blast
    moreover AOT-have ‹φ{b}›
      using ϑ[THEN &E(1)]  by (metis qml:2[axiom-inst] →E)
    ultimately AOT-show ‹φ{y}›
      using rule=E by blast
  qed
qed
```

```
AOT-theorem !box−desc:2:
  ‹∀ x (φ{x} → □φ{x}) → (∃!x φ{x} → ∀ y (y = ιx φ{x} → φ{y}))›
proof(rule →I; rule →I)
  AOT-assume ‹∀ x (φ{x} → □φ{x})›
  moreover AOT-assume ‹∃!x φ{x}›
  ultimately AOT-have ‹∃!x □φ{x}›
    using nec−exist−![THEN →E, THEN →E] by blast
  AOT-thus ‹∀ y (y = ιx φ{x} → φ{y})›
    using !box−desc:1 →E by blast
qed
```

```
AOT-theorem dr−alphabetic−thm: ‹ιν φ{ν}↓ → ιν φ{ν} = ιμ φ{μ}›
  by (simp add: rule=I:1 →I)
```

8.9 The Theory of Necessity

```
AOT-theorem RM:1[prem]:
  assumes ‹Γ ⊢□ φ → ψ›
  shows ‹□Γ ⊢□ □φ → □ψ›
proof −
  AOT-have ‹□Γ ⊢□ □(φ → ψ)›
    using RN[prem] assms by blast
  AOT-thus ‹□Γ ⊢□ □φ → □ψ›
    by (metis qml:1[axiom-inst] →E)
qed
```

```
AOT-theorem RM:1:
  assumes ‹⊢□ φ → ψ›
  shows ‹⊢□ □φ → □ψ›
  using RM:1[prem] assms by blast
```

```
lemmas RM = RM:1
```

```
AOT-theorem RM:2[prem]:
  assumes ‹Γ ⊢□ φ → ψ›
```

shows ‹$\Box\Gamma \vdash_\Box \Diamond\varphi \to \Diamond\psi$›
proof −
AOT-have ‹$\Gamma \vdash_\Box \neg\psi \to \neg\varphi$›
using *assms*
by (*simp add*: *contraposition*:*1*[*1*])
AOT-hence ‹$\Box\Gamma \vdash_\Box \Box\neg\psi \to \Box\neg\varphi$›
using *RM*:*1*[*prem*] **by** *blast*
AOT-thus ‹$\Box\Gamma \vdash_\Box \Diamond\varphi \to \Diamond\psi$›
by (*meson* $\equiv_{df}E$ $\equiv_{df}I$ *conventions*:*5* $\to I$ *modus−tollens*:*1*)
qed

AOT-theorem *RM*:*2*:
assumes ‹$\vdash_\Box \varphi \to \psi$›
shows ‹$\vdash_\Box \Diamond\varphi \to \Diamond\psi$›
using *RM*:*2*[*prem*] *assms* **by** *blast*

lemmas *RM*$\Diamond$ = *RM*:*2*

AOT-theorem *RM*:*3*[*prem*]:
assumes ‹$\Gamma \vdash_\Box \varphi \equiv \psi$›
shows ‹$\Box\Gamma \vdash_\Box \Box\varphi \equiv \Box\psi$›
proof −
AOT-have ‹$\Gamma \vdash_\Box \varphi \to \psi$› **and** ‹$\Gamma \vdash_\Box \psi \to \varphi$›
using *assms* $\equiv E$ $\to I$ **by** *metis+*
AOT-hence ‹$\Box\Gamma \vdash_\Box \Box\varphi \to \Box\psi$› **and** ‹$\Box\Gamma \vdash_\Box \Box\psi \to \Box\varphi$›
using *RM*:*1*[*prem*] **by** *metis+*
AOT-thus ‹$\Box\Gamma \vdash_\Box \Box\varphi \equiv \Box\psi$›
by (*simp add*: $\equiv I$)
qed

AOT-theorem *RM*:*3*:
assumes ‹$\vdash_\Box \varphi \equiv \psi$›
shows ‹$\vdash_\Box \Box\varphi \equiv \Box\psi$›
using *RM*:*3*[*prem*] *assms* **by** *blast*

lemmas *RE* = *RM*:*3*

AOT-theorem *RM*:*4*[*prem*]:
assumes ‹$\Gamma \vdash_\Box \varphi \equiv \psi$›
shows ‹$\Box\Gamma \vdash_\Box \Diamond\varphi \equiv \Diamond\psi$›
proof −
AOT-have ‹$\Gamma \vdash_\Box \varphi \to \psi$› **and** ‹$\Gamma \vdash_\Box \psi \to \varphi$›
using *assms* $\equiv E$ $\to I$ **by** *metis+*
AOT-hence ‹$\Box\Gamma \vdash_\Box \Diamond\varphi \to \Diamond\psi$› **and** ‹$\Box\Gamma \vdash_\Box \Diamond\psi \to \Diamond\varphi$›
using *RM*:*2*[*prem*] **by** *metis+*
AOT-thus ‹$\Box\Gamma \vdash_\Box \Diamond\varphi \equiv \Diamond\psi$›
by (*simp add*: $\equiv I$)
qed

AOT-theorem *RM*:*4*:
assumes ‹$\vdash_\Box \varphi \equiv \psi$›
shows ‹$\vdash_\Box \Diamond\varphi \equiv \Diamond\psi$›
using *RM*:*4*[*prem*] *assms* **by** *blast*

lemmas *RE*$\Diamond$ = *RM*:*4*

AOT-theorem *KBasic*:*1*: ‹$\Box\varphi \to \Box(\psi \to \varphi)$›
by (*simp add*: *RM pl*:*1*[*axiom-inst*])

AOT-theorem *KBasic*:*2*: ‹$\Box\neg\varphi \to \Box(\varphi \to \psi)$›
by (*simp add*: *RM useful−tautologies*:*3*)

AOT-theorem *KBasic*:*3*: ‹$\Box(\varphi \ \& \ \psi) \equiv (\Box\varphi \ \& \ \Box\psi)$›

proof (*rule* ≡*I*; *rule* →*I*)
  AOT-assume ‹□(φ & ψ)›
  AOT-thus ‹□φ & □ψ›
    by (*meson RM* &*I Conjunction Simplification*(*1*, *2*) →*E*)
next
  AOT-have ‹□φ → □(ψ → (φ & ψ))›
    by (*simp add*: *RM*:*1 Adjunction*)
  AOT-hence ‹□φ → (□ψ → □(φ & ψ))›
    by (*metis Hypothetical Syllogism qml*:*1*[*axiom-inst*])
  moreover AOT-assume ‹□φ & □ψ›
  ultimately AOT-show ‹□(φ & ψ)›
    using →*E* &*E* **by** *blast*
qed

AOT-theorem *KBasic*:*4*: ‹□(φ ≡ ψ) ≡ (□(φ → ψ) & □(ψ → φ))›
proof −
  AOT-have ϑ: ‹□((φ → ψ) & (ψ → φ)) ≡ (□(φ → ψ) & □(ψ → φ))›
    by (*fact KBasic*:*3*)
  AOT-modally-strict {
    AOT-have ‹(φ ≡ ψ) ≡ ((φ → ψ) & (ψ → φ))›
      by (*fact conventions*:*3*[*THEN* ≡*Df*])
  }
  AOT-hence ξ: ‹□(φ ≡ ψ) ≡ □((φ → ψ) & (ψ → φ))›
    by (*rule RE*)
  with ξ **and** ϑ **AOT-show** ‹□(φ ≡ ψ) ≡ (□(φ → ψ) & □(ψ → φ))›
    using ≡*E*(*5*) **by** *blast*
qed

AOT-theorem *KBasic*:*5*: ‹(□(φ → ψ) & □(ψ → φ)) → (□φ ≡ □ψ)›
proof −
  AOT-have ‹□(φ → ψ) → (□φ → □ψ)›
    by (*fact qml*:*1*[*axiom-inst*])
  moreover AOT-have ‹□(ψ → φ) → (□ψ → □φ)›
    by (*fact qml*:*1*[*axiom-inst*])
  ultimately AOT-have ‹(□(φ → ψ) & □(ψ → φ)) → ((□φ → □ψ) & (□ψ → □φ))›
    by (*metis* &*I MP Double Composition*)
  moreover AOT-have ‹((□φ → □ψ) & (□ψ → □φ)) → (□φ ≡ □ψ)›
    using *conventions*:*3*[*THEN* $\equiv_{df}$*I*] →*I* **by** *blast*
  ultimately AOT-show ‹(□(φ → ψ) & □(ψ → φ)) → (□φ ≡ □ψ)›
    by (*metis Hypothetical Syllogism*)
qed

AOT-theorem *KBasic*:*6*: ‹□(φ ≡ ψ) → (□φ ≡ □ψ)›
  using *KBasic*:*4 KBasic*:*5 deduction*−*theorem* ≡*E*(*1*) →*E* **by** *blast*
AOT-theorem *KBasic*:*7*: ‹((□φ & □ψ) ∨ (□¬φ & □¬ψ)) → □(φ ≡ ψ)›
proof (*rule* →*I*; *drule* ∨*E*(*1*); (*rule* →*I*)?)
  AOT-assume ‹□φ & □ψ›
  AOT-hence ‹□φ› **and** ‹□ψ› **using** &*E* **by** *blast*+
  AOT-hence ‹□(φ → ψ)› **and** ‹□(ψ → φ)› **using** *KBasic*:*1* →*E* **by** *blast*+
  AOT-hence ‹□(φ → ψ) & □(ψ → φ)› **using** &*I* **by** *blast*
  AOT-thus ‹□(φ ≡ ψ)› **by** (*metis KBasic*:*4* ≡*E*(*2*))
next
  AOT-assume ‹□¬φ & □¬ψ›
  AOT-hence *0*: ‹□(¬φ & ¬ψ)› **using** *KBasic*:*3*[*THEN* ≡*E*(*2*)] **by** *blast*
  AOT-modally-strict {
    AOT-have ‹(¬φ & ¬ψ) → (φ ≡ ψ)›
      by (*metis* &*E*(*1*) &*E*(*2*) *deduction*−*theorem* ≡*I reductio*−*aa*:*1*)
  }
  AOT-hence ‹□(¬φ & ¬ψ) → □(φ ≡ ψ)›
    by (*rule RM*)
  AOT-thus ‹□(φ ≡ ψ)› **using** *0* →*E* **by** *blast*
qed(*auto*)

```
AOT-theorem KBasic:8: ‹□(φ & ψ) → □(φ ≡ ψ)›
  by (meson RM:1 &E(1) &E(2) deduction−theorem ≡I)
AOT-theorem KBasic:9: ‹□(¬φ & ¬ψ) → □(φ ≡ ψ)›
  by (metis RM:1 &E(1) &E(2) deduction−theorem ≡I raa−cor:4)
AOT-theorem KBasic:10: ‹□φ ≡ □¬¬φ›
  by (simp add: RM:3 oth−class−taut:3:b)
AOT-theorem KBasic:11: ‹¬□φ ≡ ◊¬φ›
proof (rule ≡I; rule →I)
  AOT-show ‹◊¬φ› if ‹¬□φ›
    using that ≡_df I conventions:5 KBasic:10 ≡E(3) by blast
next
  AOT-show ‹¬□φ› if ‹◊¬φ›
    using ≡_df E conventions:5 KBasic:10 ≡E(4) that by blast
qed
AOT-theorem KBasic:12: ‹□φ ≡ ¬◊¬φ›
proof (rule ≡I; rule →I)
  AOT-show ‹¬◊¬φ› if ‹□φ›
    using ¬¬I KBasic:11 ≡E(3) that by blast
next
  AOT-show ‹□φ› if ‹¬◊¬φ›
  using KBasic:11 ≡E(1) reductio−aa:1 that by blast
qed
AOT-theorem KBasic:13: ‹□(φ → ψ) → (◊φ → ◊ψ)›
proof −
  AOT-have ‹φ → ψ ⊢□ φ → ψ› by blast
  AOT-hence ‹□(φ → ψ) ⊢□ ◊φ → ◊ψ›
    using RM:2[prem] by blast
  AOT-thus ‹□(φ → ψ) → (◊φ → ◊ψ)› using →I by blast
qed
lemmas K◊ = KBasic:13
AOT-theorem KBasic:14: ‹◊□φ ≡ ¬□◊¬φ›
  by (meson RE◊ KBasic:11 KBasic:12 ≡E(6) oth−class−taut:3:a)
AOT-theorem KBasic:15: ‹(□φ ∨ □ψ) → □(φ ∨ ψ)›
proof −
  AOT-modally-strict {
    AOT-have ‹φ → (φ ∨ ψ)› and ‹ψ → (φ ∨ ψ)›
      by (auto simp: Disjunction Addition(1) Disjunction Addition(2))
  }
  AOT-hence ‹□φ → □(φ ∨ ψ)› and ‹□ψ → □(φ ∨ ψ)›
    using RM by blast+
  AOT-thus ‹(□φ ∨ □ψ) → □(φ ∨ ψ)›
    by (metis ∨E(1) deduction−theorem)
qed

AOT-theorem KBasic:16: ‹(□φ & ◊ψ) → ◊(φ & ψ)›
  by (meson KBasic:13 RM:1 Adjunction Hypothetical Syllogism
          Importation →E)

AOT-theorem rule−sub−lem:1:a:
  assumes ‹⊢□ □(ψ ≡ χ)›
  shows ‹⊢□ ¬ψ ≡ ¬χ›
  using qml:2[axiom-inst, THEN →E, OF assms]
       ≡E(1) oth−class−taut:4:b by blast

AOT-theorem rule−sub−lem:1:b:
  assumes ‹⊢□ □(ψ ≡ χ)›
  shows ‹⊢□ (ψ → Θ) ≡ (χ → Θ)›
  using qml:2[axiom-inst, THEN →E, OF assms]
  using oth−class−taut:4:c vdash−properties:6 by blast

AOT-theorem rule−sub−lem:1:c:
  assumes ‹⊢□ □(ψ ≡ χ)›
  shows ‹⊢□ (Θ → ψ) ≡ (Θ → χ)›
```

using *qml*:*2*[*axiom-inst*, *THEN* →*E*, *OF assms*]
using *oth−class−taut*:*4*:*d vdash−properties*:*6* **by** *blast*

AOT-theorem *rule−sub−lem*:*1*:*d*:
assumes ‹*for arbitrary* α: $\vdash_\Box$ □(ψ{α} ≡ χ{α})›
shows ‹$\vdash_\Box$ ∀α ψ{α} ≡ ∀α χ{α}›
proof −
AOT-modally-strict {
AOT-have ‹∀α (ψ{α} ≡ χ{α})›
using *qml*:*2*[*axiom-inst*, *THEN* →*E*, *OF assms*] ∀*I* **by** *fast*
AOT-hence *0*: ‹ψ{α} ≡ χ{α}› **for** α **using** ∀*E* **by** *blast*
AOT-show ‹∀α ψ{α} ≡ ∀α χ{α}›
proof (*rule* ≡*I*; *rule* →*I*)
AOT-assume ‹∀α ψ{α}›
AOT-hence ‹ψ{α}› **for** α **using** ∀*E* **by** *blast*
AOT-hence ‹χ{α}› **for** α **using** *0* ≡*E* **by** *blast*
AOT-thus ‹∀α χ{α}› **by** (*rule* ∀*I*)
next
AOT-assume ‹∀α χ{α}›
AOT-hence ‹χ{α}› **for** α **using** ∀*E* **by** *blast*
AOT-hence ‹ψ{α}› **for** α **using** *0* ≡*E* **by** *blast*
AOT-thus ‹∀α ψ{α}› **by** (*rule* ∀*I*)
qed
}
qed

AOT-theorem *rule−sub−lem*:*1*:*e*:
assumes ‹$\vdash_\Box$ □(ψ ≡ χ)›
shows ‹$\vdash_\Box$ [λ ψ] ≡ [λ χ]›
using *qml*:*2*[*axiom-inst*, *THEN* →*E*, *OF assms*]
using ≡*E*(*1*) *propositions−lemma*:*6* **by** *blast*

AOT-theorem *rule−sub−lem*:*1*:*f*:
assumes ‹$\vdash_\Box$ □(ψ ≡ χ)›
shows ‹$\vdash_\Box$ **𝓐**ψ ≡ **𝓐**χ›
using *qml*:*2*[*axiom-inst*, *THEN* →*E*, *OF assms*, *THEN RA*[*2*]]
by (*metis Act−Basic*:*5* ≡*E*(*1*))

AOT-theorem *rule−sub−lem*:*1*:*g*:
assumes ‹$\vdash_\Box$ □(ψ ≡ χ)›
shows ‹$\vdash_\Box$ □ψ ≡ □χ›
using *KBasic*:*6 assms vdash−properties*:*6* **by** *blast*

Note that instead of deriving *rule−sub−lem*:*2*, *rule−sub−lem*:*3*, *rule−sub−lem*:*4*, and *rule−sub−nec*, we construct substitution methods instead.

class *AOT-subst* =
fixes *AOT-subst* :: (′*a* ⇒ o) ⇒ *bool*
and *AOT-subst-cond* :: ′*a* ⇒ ′*a* ⇒ *bool*
assumes *AOT-subst*:
AOT-subst φ ⟹ *AOT-subst-cond* ψ χ ⟹ [*v* ⊨ «φ ψ» ≡ «φ χ»]

named-theorems *AOT-substI*

instantiation o :: *AOT-subst*
begin

inductive *AOT-subst*-o **where**
AOT-subst-o-*id*[*AOT-substI*]:
‹*AOT-subst*-o (λφ. φ)›
| *AOT-subst*-o-*const*[*AOT-substI*]:
‹*AOT-subst*-o (λφ. ψ)›
| *AOT-subst*-o-*not*[*AOT-substI*]:
‹*AOT-subst*-o Θ ⟹ *AOT-subst*-o (λ φ. «¬Θ{φ}»)›

| *AOT-subst-o-imp*[*AOT-substI*]:
‹*AOT-subst-o* Θ ⟹ *AOT-subst-o* Ξ ⟹ *AOT-subst-o* (λ φ. «Θ{φ} → Ξ{φ}»)›
| *AOT-subst-o-lambda0*[*AOT-substI*]:
‹*AOT-subst-o* Θ ⟹ *AOT-subst-o* (λ φ. (*AOT-lambda0* (Θ φ)))›
| *AOT-subst-o-act*[*AOT-substI*]:
‹*AOT-subst-o* Θ ⟹ *AOT-subst-o* (λ φ. «𝓐Θ{φ}»)›
| *AOT-subst-o-box*[*AOT-substI*]:
‹*AOT-subst-o* Θ ⟹ *AOT-subst-o* (λ φ. «□Θ{φ}»)›
| *AOT-subst-o-by-def*[*AOT-substI*]:
‹(⋀ ψ . *AOT-model-equiv-def* (Θ ψ) (Ξ ψ)) ⟹
AOT-subst-o Ξ ⟹ *AOT-subst-o* Θ›

definition *AOT-subst-cond-o* **where**
‹*AOT-subst-cond-o* ≡ λ ψ χ . ∀ v . [v ⊨ ψ ≡ χ]›

instance
proof
fix ψ χ :: o **and** φ :: ‹o ⇒ o›
assume *cond*: ‹*AOT-subst-cond* ψ χ›
assume ‹*AOT-subst* φ›
moreover **AOT-have** ‹⊢□ ψ ≡ χ›
using *cond* **unfolding** *AOT-subst-cond-o-def* **by** *blast*
ultimately **AOT-show** ‹⊢□ φ{ψ} ≡ φ{χ}›
proof (*induct arbitrary*: ψ χ)
case *AOT-subst-o-id*
thus *?case*
using ≡*E*(*2*) *oth−class−taut*:*4*:*b* *rule−sub−lem*:*1*:*a* **by** *blast*
next
case (*AOT-subst-o-const* ψ)
thus *?case*
by (*simp add*: *oth−class−taut*:*3*:*a*)
next
case (*AOT-subst-o-not* Θ)
thus *?case*
by (*simp add*: *RN rule−sub−lem*:*1*:*a*)
next
case (*AOT-subst-o-imp* Θ Ξ)
thus *?case*
by (*meson RN* ≡*E*(*5*) *rule−sub−lem*:*1*:*b* *rule−sub−lem*:*1*:*c*)
next
case (*AOT-subst-o-lambda0* Θ)
thus *?case*
by (*simp add*: *RN rule−sub−lem*:*1*:*e*)
next
case (*AOT-subst-o-act* Θ)
thus *?case*
by (*simp add*: *RN rule−sub−lem*:*1*:*f*)
next
case (*AOT-subst-o-box* Θ)
thus *?case*
by (*simp add*: *RN rule−sub−lem*:*1*:*g*)
next
case (*AOT-subst-o-by-def* Θ Ξ)
AOT-modally-strict {
AOT-have ‹Ξ{ψ} ≡ Ξ{χ}›
using *AOT-subst-o-by-def* **by** *simp*
AOT-thus ‹Θ{ψ} ≡ Θ{χ}›
using ≡*Df*[*OF AOT-subst-o-by-def*(*1*), *of* - ψ]
≡*Df*[*OF AOT-subst-o-by-def*(*1*), *of* - χ]
by (*metis* ≡*E*(*6*) *oth−class−taut*:*3*:*a*)
}
qed

```
qed
end

instantiation fun :: (AOT-Term-id-2, AOT-subst) AOT-subst
begin

definition AOT-subst-cond-fun :: ‹('a ⇒ 'b) ⇒ ('a ⇒ 'b) ⇒ bool› where
  ‹AOT-subst-cond-fun ≡ λ φ ψ . ∀ α . AOT-subst-cond (φ (AOT-term-of-var α))
                                               (ψ (AOT-term-of-var α))›

inductive AOT-subst-fun :: ‹(('a ⇒ 'b) ⇒ o) ⇒ bool› where
  AOT-subst-fun-const[AOT-substI]:
    ‹AOT-subst-fun (λφ. ψ)›
  | AOT-subst-fun-id[AOT-substI]:
    ‹AOT-subst Ψ ⟹ AOT-subst-fun (λφ. Ψ (φ (AOT-term-of-var α)))›
  | AOT-subst-fun-all[AOT-substI]:
    ‹AOT-subst Ψ ⟹ (⋀ α . AOT-subst-fun (Θ (AOT-term-of-var α))) ⟹
     AOT-subst-fun (λφ :: 'a ⇒ 'b. Ψ «∀ α «Θ (α::'a) φ»»)›
  | AOT-subst-fun-not[AOT-substI]:
    ‹AOT-subst Ψ ⟹ AOT-subst-fun (λφ. «¬«Ψ φ»»)›
  | AOT-subst-fun-imp[AOT-substI]:
    ‹AOT-subst Ψ ⟹ AOT-subst Θ ⟹ AOT-subst-fun (λφ. ««Ψ φ» → «Θ φ»»)›
  | AOT-subst-fun-lambda0[AOT-substI]:
    ‹AOT-subst Θ ⟹ AOT-subst-fun (λ φ. (AOT-lambda0 (Θ φ)))›
  | AOT-subst-fun-act[AOT-substI]:
    ‹AOT-subst Θ ⟹ AOT-subst-fun (λ φ. «𝓐«Θ φ»»)›
  | AOT-subst-fun-box[AOT-substI]:
    ‹AOT-subst Θ ⟹ AOT-subst-fun (λ φ. «□«Θ φ»»)›
  | AOT-subst-fun-def[AOT-substI]:
    ‹(⋀ φ . AOT-model-equiv-def (Θ φ) (Ψ φ)) ⟹
     AOT-subst-fun Ψ ⟹ AOT-subst-fun Θ›

instance proof
  fix ψ χ :: ‹'a ⇒ 'b› and φ :: ‹('a ⇒ 'b) ⇒ o›
  assume ‹AOT-subst φ›
  moreover assume cond: ‹AOT-subst-cond ψ χ›
  ultimately AOT-show ‹⊢□ «φ ψ» ≡ «φ χ»›
  proof(induct)
    case (AOT-subst-fun-const ψ)
    then show ?case by (simp add: oth−class−taut:3:a)
  next
  case (AOT-subst-fun-id Ψ x)
  then show ?case by (simp add: AOT-subst AOT-subst-cond-fun-def)
  next
  next
  case (AOT-subst-fun-all Ψ Θ)
  AOT-have ‹⊢□ □(Θ{α, «ψ»} ≡ Θ{α, «χ»})› for α
    using AOT-subst-fun-all.hyps(3) AOT-subst-fun-all.prems RN by presburger
  thus ?case using AOT-subst[OF AOT-subst-fun-all(1)]
    by (simp add: RN rule−sub−lem:1:d
              AOT-subst-cond-fun-def AOT-subst-cond-o-def)
  next
  case (AOT-subst-fun-not Ψ)
  then show ?case by (simp add: RN rule−sub−lem:1:a)
  next
  case (AOT-subst-fun-imp Ψ Θ)
  then show ?case
    unfolding AOT-subst-cond-fun-def AOT-subst-cond-o-def
    by (meson ≡E(5) oth−class−taut:4:c oth−class−taut:4:d →E)
  next
  case (AOT-subst-fun-lambda0 Θ)
  then show ?case by (simp add: RN rule−sub−lem:1:e)
  next
```

```
  case (AOT-subst-fun-act Θ)
  then show ?case by (simp add: RN rule−sub−lem:1:f)
  next
  case (AOT-subst-fun-box Θ)
  then show ?case by (simp add: RN rule−sub−lem:1:g)
  next
  case (AOT-subst-fun-def Θ Ψ)
  then show ?case
    by (meson df−rules−formulas[3] df−rules−formulas[4] ≡I ≡E(5))
  qed
qed
end

ML‹
fun prove-AOT-subst-tac ctxt = REPEAT (SUBGOAL (fn (trm,-) => let
  fun findHeadConst (Const x) = SOME x
    | findHeadConst (A $ -) = findHeadConst A
    | findHeadConst - = NONE
  fun findDef (Const (const-name‹AOT-model-equiv-def›, -) $ lhs $ -)
      = findHeadConst lhs
    | findDef (A $ B) = (case findDef A of SOME x => SOME x | - => findDef B)
    | findDef (Abs (-,-,c)) = findDef c
    | findDef - = NONE
  val const-opt = (findDef trm)
  val defs = case const-opt of SOME const => List.filter (fn thm => let
      val concl = Thm.concl-of thm
      val thmconst = (findDef concl)
      in case thmconst of SOME (c,-) => fst const = c | - => false end)
      (AOT-Definitions.get ctxt)
      | - => []
  val tac = case defs of
            [] => safe-step-tac (ctxt addSIs @{thms AOT-substI}) 1
            | - => resolve-tac ctxt defs 1
  in tac end) 1)
fun getSubstThm ctxt reversed phi p q = let
val p-ty = Term.type-of p
val abs = HOLogic.mk-Trueprop (@{const AOT-subst(-)} $ phi)
val abs = Syntax.check-term ctxt abs
val substThm = Goal.prove ctxt [] [] abs
  (fn {context=ctxt, prems=-} => prove-AOT-subst-tac ctxt)
val substThm = substThm RS @{thm AOT-subst}
in if reversed then let
  val substThm = Drule.instantiate-normalize
          (TVars.empty, Vars.make [(((χ, 0), p-ty), Thm.cterm-of ctxt p),
          (((ψ, 0), p-ty), Thm.cterm-of ctxt q)]) substThm
  val substThm = substThm RS @{thm ≡E(1)}
  in substThm end
else
  let
  val substThm = Drule.instantiate-normalize
          (TVars.empty, Vars.make [(((ψ, 0), p-ty), Thm.cterm-of ctxt p),
          (((χ, 0), p-ty), Thm.cterm-of ctxt q)]) substThm
  val substThm = substThm RS @{thm ≡E(2)}
  in substThm end end
›

method-setup AOT-subst = ‹
Scan.option (Scan.lift (Args.parens (Args.$$$ reverse))) −−
Scan.lift (Parse.embedded-inner-syntax −− Parse.embedded-inner-syntax) −−
Scan.option (Scan.lift (Args.$$$ for −− Args.colon) |−−
Scan.repeat1 (Scan.lift (Parse.embedded-inner-syntax) −−
Scan.option (Scan.lift (Args.$$$ :: |−− Parse.embedded-inner-syntax))))
>> (fn ((reversed,(raw-p,raw-q)),raw-bounds) => (fn ctxt =>
```

```
(Method.SIMPLE-METHOD (Subgoal.FOCUS (fn {context = ctxt, params = -,
  prems = prems, asms = asms, concl = concl, schematics = -} =>
let
val thms = prems
val ctxt' = ctxt
val ctxt = Context-Position.set-visible false ctxt
val raw-bounds = case raw-bounds of SOME bounds => bounds | - => []

val ctxt = (fold (fn (bound, ty) => fn ctxt =>
  let
    val bound = AOT-read-term @{nonterminal τ'} ctxt bound
    val ty = Option.map (Syntax.read-typ ctxt) ty
    val ctxt = case ty of SOME ty => let
        val bound = Const (-type-constraint-, Type (fun, [ty,ty])) $ bound
        val bound = Syntax.check-term ctxt bound
      in Variable.declare-term bound ctxt end | - => ctxt
  in ctxt end)) raw-bounds ctxt

val p = AOT-read-term @{nonterminal φ'} ctxt raw-p
val p = Syntax.check-term ctxt p
val ctxt = Variable.declare-term p ctxt
val q = AOT-read-term @{nonterminal φ'} ctxt raw-q
val q = Syntax.check-term ctxt q
val ctxt = Variable.declare-term q ctxt

val bounds = (map (fn (bound, -) =>
  Syntax.check-term ctxt (AOT-read-term @{nonterminal τ'} ctxt bound)
)) raw-bounds
val p = fold (fn bound => fn p =>
  Term.abs (α, Term.type-of bound) (Term.abstract-over (bound,p)))
  bounds p
val p = Syntax.check-term ctxt p
val p-ty = Term.type-of p

val pat = @{const Trueprop} $
  (@{const AOT-model-valid-in} $ Var ((w,0), @{typ w}) $
   (Var ((φ,0), Type (type-name ‹fun›, [p-ty, @{typ o}])) $ p))
val univ = Unify.matchers (Context.Proof ctxt) [(pat, Thm.term-of concl)]
val univ = hd (Seq.list-of univ) (* TODO: consider all matches *)
val phi = the (Envir.lookup univ
  ((φ,0), Type (type-name ‹fun›, [p-ty, @{typ o}])))

val q = fold (fn bound => fn q =>
  Term.abs (α, Term.type-of bound) (Term.abstract-over (bound,q))) bounds q
val q = Syntax.check-term ctxt q

(* Reparse to report bounds as fixes. *)
val ctxt = Context-Position.restore-visible ctxt' ctxt
val ctxt' = ctxt
fun unsource str = fst (Input.source-content (Syntax.read-input str))
val (-,ctxt') = Proof-Context.add-fixes (map (fn (str,-) =>
  (Binding.make (unsource str, Position.none), NONE, Mixfix.NoSyn)) raw-bounds)
  ctxt'
val - = (map (fn (x,-) =>
  Syntax.check-term ctxt (AOT-read-term @{nonterminal τ'} ctxt' x)))
  raw-bounds
val - = AOT-read-term @{nonterminal φ'} ctxt' raw-p
val - = AOT-read-term @{nonterminal φ'} ctxt' raw-q
val reversed = case reversed of SOME - => true | - => false
val simpThms = [@{thm AOT-subst-cond-o-def}, @{thm AOT-subst-cond-fun-def}]
in
resolve-tac ctxt [getSubstThm ctxt reversed phi p q] 1
THEN simp-tac (ctxt addsimps simpThms) 1
```

```
THEN (REPEAT (resolve-tac ctxt [@{thm allI}] 1))
THEN (TRY (resolve-tac ctxt thms 1))
end
) ctxt 1))))
›
```

method-setup *AOT-subst-def* = ‹

```
Scan.option (Scan.lift (Args.parens (Args.$$$ reverse))) --
Attrib.thm
>> (fn (reversed,fact) => (fn ctxt =>
(Method.SIMPLE-METHOD (Subgoal.FOCUS (fn {context = ctxt, params = -,
  prems = prems, asms = asms, concl = concl, schematics = -} =>
let
val c = Thm.concl-of fact
val (lhs, rhs) = case c of (const‹Trueprop› $
    (const‹AOT-model-equiv-def› $ lhs $ rhs)) => (lhs, rhs)
  | - => raise Fail Definition expected.
val substCond = HOLogic.mk-Trueprop
  (Const (const-name‹AOT-subst-cond›, dummyT) $ lhs $ rhs)
val substCond = Syntax.check-term
  (Proof-Context.set-mode Proof-Context.mode-schematic ctxt)
  substCond
val simpThms = [@{thm AOT-subst-cond-o-def},
  @{thm AOT-subst-cond-fun-def},
  fact RS @{thm ≡Df}]
val substCondThm = Goal.prove ctxt [] [] substCond
  (fn {context=ctxt, prems=prems} =>
      (SUBGOAL (fn (trm,int) =>
        auto-tac (ctxt addsimps simpThms)) 1))
val substThm = substCondThm RSN (2,@{thm AOT-subst})
in
resolve-tac ctxt [substThm RS
  (case reversed of NONE => @{thm ≡E(2)} | - => @{thm ≡E(1)})] 1
THEN prove-AOT-subst-tac ctxt
THEN (TRY (resolve-tac ctxt prems 1))
end
) ctxt 1))))
›
```

method-setup *AOT-subst-thm* = ‹

```
Scan.option (Scan.lift (Args.parens (Args.$$$ reverse))) --
Attrib.thm
>> (fn (reversed,fact) => (fn ctxt =>
(Method.SIMPLE-METHOD (Subgoal.FOCUS (fn {context = ctxt, params = -,
  prems = prems, asms = asms, concl = concl, schematics = -} =>
let
val c = Thm.concl-of fact
val (lhs, rhs) = case c of
  (const‹Trueprop› $
   (const‹AOT-model-valid-in› $ - $
    (const‹AOT-equiv› $ lhs $ rhs))) => (lhs, rhs)
  | - => raise Fail Equivalence expected.

val substCond = HOLogic.mk-Trueprop
  (Const (const-name‹AOT-subst-cond›, dummyT) $ lhs $ rhs)
val substCond = Syntax.check-term
  (Proof-Context.set-mode Proof-Context.mode-schematic ctxt)
  substCond
val simpThms = [@{thm AOT-subst-cond-o-def},
  @{thm AOT-subst-cond-fun-def},
  fact]
val substCondThm = Goal.prove ctxt [] [] substCond
  (fn {context=ctxt, prems=prems} =>
```

```
      (SUBGOAL (fn (trm,int) => auto-tac (ctxt addsimps simpThms)) 1))
val substThm = substCondThm RSN (2,@{thm AOT-subst})
in
resolve-tac ctxt [substThm RS
  (case reversed of NONE => @{thm ≡E(2)} | - => @{thm ≡E(1)})] 1
THEN prove-AOT-subst-tac ctxt
THEN (TRY (resolve-tac ctxt prems 1))
end
) ctxt 1))))
›

AOT-theorem rule−sub−remark:1[1]:
  assumes ‹⊢□ A!x ≡ ¬◇E!x› and ‹¬A!x›
  shows ‹¬¬◇E!x›
  by (AOT-subst (reverse) ‹¬◇E!x› ‹A!x›)
     (auto simp: assms)

AOT-theorem rule−sub−remark:1[2]:
  assumes ‹⊢□ A!x ≡ ¬◇E!x› and  ‹¬¬◇E!x›
  shows ‹¬A!x›
  by (AOT-subst ‹A!x› ‹¬◇E!x›)
     (auto simp: assms)

AOT-theorem rule−sub−remark:2[1]:
  assumes ‹⊢□ [R]xy ≡ ([R]xy & ([Q]a ∨ ¬[Q]a))›
      and ‹p → [R]xy›
  shows ‹p → [R]xy & ([Q]a ∨ ¬[Q]a)›
  by (AOT-subst-thm (reverse) assms(1)) (simp add: assms(2))

AOT-theorem rule−sub−remark:2[2]:
  assumes ‹⊢□ [R]xy ≡ ([R]xy & ([Q]a ∨ ¬[Q]a))›
      and ‹p → [R]xy & ([Q]a ∨ ¬[Q]a)›
  shows ‹p → [R]xy›
  by (AOT-subst-thm assms(1)) (simp add: assms(2))

AOT-theorem rule−sub−remark:3[1]:
  assumes ‹for arbitrary x: ⊢□ A!x ≡ ¬◇E!x›
      and ‹∃x A!x›
    shows ‹∃x ¬◇E!x›
  by (AOT-subst (reverse) ‹¬◇E!x› ‹A!x› for: x)
     (auto simp: assms)

AOT-theorem rule−sub−remark:3[2]:
  assumes ‹for arbitrary x: ⊢□ A!x ≡ ¬◇E!x›
      and ‹∃x ¬◇E!x›
    shows ‹∃x A!x›
  by (AOT-subst ‹A!x› ‹¬◇E!x› for: x)
     (auto simp: assms)

AOT-theorem rule−sub−remark:4[1]:
  assumes ‹⊢□ ¬¬[P]x ≡ [P]x› and ‹𝒜¬¬[P]x›
  shows ‹𝒜[P]x›
  by (AOT-subst-thm (reverse) assms(1)) (simp add: assms(2))

AOT-theorem rule−sub−remark:4[2]:
  assumes ‹⊢□ ¬¬[P]x ≡ [P]x› and ‹𝒜[P]x›
  shows ‹𝒜¬¬[P]x›
  by (AOT-subst-thm assms(1)) (simp add: assms(2))

AOT-theorem rule−sub−remark:5[1]:
  assumes ‹⊢□ (φ → ψ) ≡ (¬ψ → ¬φ)› and ‹□(φ → ψ)›
  shows ‹□(¬ψ → ¬φ)›
  by (AOT-subst-thm (reverse) assms(1)) (simp add: assms(2))
```

AOT-theorem *rule−sub−remark*:*5*[*2*]:
assumes ‹$\vdash_\Box$ (φ → ψ) ≡ (¬ψ → ¬φ)› **and** ‹□(¬ψ → ¬φ)›
shows ‹□(φ → ψ)›
by (*AOT-subst-thm assms*(*1*)) (*simp add*: *assms*(*2*))

AOT-theorem *rule−sub−remark*:*6*[*1*]:
assumes ‹$\vdash_\Box$ ψ ≡ χ› **and** ‹□(φ → ψ)›
shows ‹□(φ → χ)›
by (*AOT-subst-thm* (*reverse*) *assms*(*1*)) (*simp add*: *assms*(*2*))

AOT-theorem *rule−sub−remark*:*6*[*2*]:
assumes ‹$\vdash_\Box$ ψ ≡ χ› **and** ‹□(φ → χ)›
shows ‹□(φ → ψ)›
by (*AOT-subst-thm assms*(*1*)) (*simp add*: *assms*(*2*))

AOT-theorem *rule−sub−remark*:*7*[*1*]:
assumes ‹$\vdash_\Box$ φ ≡ ¬¬φ› **and** ‹□(φ → φ)›
shows ‹□(¬¬φ → φ)›
by (*AOT-subst-thm* (*reverse*) *assms*(*1*)) (*simp add*: *assms*(*2*))

AOT-theorem *rule−sub−remark*:*7*[*2*]:
assumes ‹$\vdash_\Box$ φ ≡ ¬¬φ› **and** ‹□(¬¬φ → φ)›
shows ‹□(φ → φ)›
by (*AOT-subst-thm assms*(*1*)) (*simp add*: *assms*(*2*))

AOT-theorem *KBasic2*:*1*: ‹□¬φ ≡ ¬◇φ›
by (*meson conventions*:*5 contraposition*:*2*
Hypothetical Syllogism df−rules−formulas[*3*]
df−rules−formulas[*4*] ≡*I useful−tautologies*:*1*)

AOT-theorem *KBasic2*:*2*: ‹◇(φ ∨ ψ) ≡ (◇φ ∨ ◇ψ)›
proof −
AOT-have ‹◇(φ ∨ ψ) ≡ ◇¬(¬φ & ¬ψ)›
by (*simp add*: *RE*◇ *oth−class−taut*:*5*:*b*)
also AOT-have ‹. . . ≡ ¬□(¬φ & ¬ψ)›
using *KBasic*:*11* ≡*E*(*6*) *oth−class−taut*:*3*:*a* **by** *blast*
also AOT-have ‹. . . ≡ ¬(□¬φ & □¬ψ)›
using *KBasic*:*3* ≡*E*(*1*) *oth−class−taut*:*4*:*b* **by** *blast*
also AOT-have ‹. . . ≡ ¬(¬◇φ & ¬◇ψ)›
using *KBasic2*:*1*
by (*AOT-subst* ‹□¬φ› ‹¬◇φ›; *AOT-subst* ‹□¬ψ› ‹¬◇ψ›;
auto simp: *oth−class−taut*:*3*:*a*)
also AOT-have ‹. . . ≡ ¬¬(◇φ ∨ ◇ψ)›
using ≡*E*(*6*) *oth−class−taut*:*3*:*b oth−class−taut*:*5*:*b* **by** *blast*
also AOT-have ‹. . . ≡ ◇φ ∨ ◇ψ›
by (*simp add*: ≡*I useful−tautologies*:*1 useful−tautologies*:*2*)
finally show *?thesis* **.**
qed

AOT-theorem *KBasic2*:*3*: ‹◇(φ & ψ) → (◇φ & ◇ψ)›
by (*metis RM*◇ &*I Conjunction Simplification*(*1*,*2*)
→*I modus−tollens*:*1 reductio−aa*:*1*)

AOT-theorem *KBasic2*:*4*: ‹◇(φ → ψ) ≡ (□φ → ◇ψ)›
proof −
AOT-have ‹◇(φ → ψ) ≡ ◇(¬φ ∨ ψ)›
by (*AOT-subst* ‹φ → ψ› ‹¬φ ∨ ψ›)
(*auto simp*: *oth−class−taut*:*1*:*c oth−class−taut*:*3*:*a*)
also AOT-have ‹... ≡ ◇¬φ ∨ ◇ψ›
by (*simp add*: *KBasic2*:*2*)
also AOT-have ‹... ≡ ¬□φ ∨ ◇ψ›
by (*AOT-subst* ‹¬□φ› ‹◇¬φ›)

```
      (auto simp: KBasic:11 oth−class−taut:3:a)
  also AOT-have ‹... ≡ □φ → ◇ψ›
    using ≡E(6) oth−class−taut:1:c oth−class−taut:3:a by blast
  finally show ?thesis .
qed

AOT-theorem KBasic2:5: ‹◇◇φ ≡ ¬□□¬φ›
  using conventions:5[THEN ≡Df]
  by (AOT-subst ‹◇φ› ‹¬□¬φ›;
      AOT-subst ‹◇¬□¬φ› ‹¬□¬¬□¬φ›;
      AOT-subst (reverse) ‹¬¬□¬φ› ‹□¬φ›)
     (auto simp: oth−class−taut:3:b oth−class−taut:3:a)


AOT-theorem KBasic2:6: ‹□(φ ∨ ψ) → (□φ ∨ ◇ψ)›
proof(rule →I; rule raa−cor:1)
  AOT-assume ‹□(φ ∨ ψ)›
  AOT-hence ‹□(¬φ → ψ)›
    using conventions:2[THEN ≡Df]
    by (AOT-subst (reverse) ‹¬φ → ψ› ‹φ ∨ ψ›) simp
  AOT-hence 1: ‹◇¬φ → ◇ψ›
    using KBasic:13 vdash−properties:10 by blast
  AOT-assume ‹¬(□φ ∨ ◇ψ)›
  AOT-hence ‹¬□φ› and ‹¬◇ψ›
    using &E ≡E(1) oth−class−taut:5:d by blast+
  AOT-thus ‹◇ψ & ¬◇ψ›
    using &I(1) 1[THEN →E] KBasic:11 ≡E(4) raa−cor:3 by blast
qed

AOT-theorem KBasic2:7: ‹(□(φ ∨ ψ) & ◇¬φ) → ◇ψ›
proof(rule →I; frule &E(1); drule &E(2))
  AOT-assume ‹□(φ ∨ ψ)›
  AOT-hence 1: ‹□φ ∨ ◇ψ›
    using KBasic2:6 ∨I(2) ∨E(1) by blast
  AOT-assume ‹◇¬φ›
  AOT-hence ‹¬□φ› using KBasic:11 ≡E(2) by blast
  AOT-thus ‹◇ψ› using 1 ∨E(2) by blast
qed

AOT-theorem T−S5−fund:1: ‹φ → ◇φ›
  by (meson ≡dfI conventions:5 contraposition:2
          Hypothetical Syllogism →I qml:2[axiom-inst])
lemmas T◇ = T−S5−fund:1

AOT-theorem T−S5−fund:2: ‹◇□φ → □φ›
proof(rule →I)
  AOT-assume ‹◇□φ›
  AOT-hence ‹¬□◇¬φ›
    using KBasic:14 ≡E(4) raa−cor:3 by blast
  moreover AOT-have ‹◇¬φ → □◇¬φ›
    by (fact qml:3[axiom-inst])
  ultimately AOT-have ‹¬◇¬φ›
    using modus−tollens:1 by blast
  AOT-thus ‹□φ› using KBasic:12 ≡E(2) by blast
qed
lemmas 5◇ = T−S5−fund:2

AOT-theorem Act−Sub:1: ‹𝓐φ ≡ ¬𝓐¬φ›
  by (AOT-subst ‹𝓐¬φ› ‹¬𝓐φ›)
     (auto simp: logic−actual−nec:1[axiom-inst] oth−class−taut:3:b)

AOT-theorem Act−Sub:2: ‹◇φ ≡ 𝓐◇φ›
  using conventions:5[THEN ≡Df]
```

by (*AOT-subst* ‹◇φ› ‹¬□¬φ›)
(*metis deduction−theorem* ≡*I* ≡*E*(*1*) ≡*E*(*2*) ≡*E*(*3*)
logic−actual−nec:*1*[*axiom-inst*] *qml−act*:*2*[*axiom-inst*])

AOT-theorem *Act−Sub*:*3*: ‹𝓐φ → ◇φ›
using *conventions*:*5*[*THEN* ≡*Df*]
by (*AOT-subst* ‹◇φ› ‹¬□¬φ›)
(*metis Act−Sub*:*1* →*I* ≡*E*(*4*) *nec−imp−act reductio−aa*:*2* →*E*)

AOT-theorem *Act−Sub*:*4*: ‹𝓐φ ≡ ◇𝓐φ›
proof (*rule* ≡*I*; *rule* →*I*)
AOT-assume ‹𝓐φ›
AOT-thus ‹◇𝓐φ› **using** *T*◇ *vdash−properties*:*10* **by** *blast*
next
AOT-assume ‹◇𝓐φ›
AOT-hence ‹¬□¬𝓐φ›
using $\equiv_{df}$*E* *conventions*:*5* **by** *blast*
AOT-hence ‹¬□𝓐¬φ›
by (*AOT-subst* ‹𝓐¬φ› ‹¬𝓐φ›)
(*simp add*: *logic−actual−nec*:*1*[*axiom-inst*])
AOT-thus ‹𝓐φ›
using *Act−Basic*:*1* *Act−Basic*:*6* ∨*E*(*3*) ≡*E*(*4*)
reductio−aa:*1* **by** *blast*
qed

AOT-theorem *Act−Sub*:*5*: ‹◇𝓐φ → 𝓐◇φ›
by (*metis Act−Sub*:*2* *Act−Sub*:*3* *Act−Sub*:*4* →*I* ≡*E*(*1*) ≡*E*(*2*) →*E*)

AOT-theorem *S5Basic*:*1*: ‹◇φ ≡ □◇φ›
by (*simp add*: ≡*I* *qml*:*2*[*axiom-inst*] *qml*:*3*[*axiom-inst*])

AOT-theorem *S5Basic*:*2*: ‹□φ ≡ ◇□φ›
by (*simp add*: *T*◇ *5*◇ ≡*I*)

AOT-theorem *S5Basic*:*3*: ‹φ → □◇φ›
using *T*◇ *Hypothetical Syllogism qml*:*3*[*axiom-inst*] **by** *blast*
lemmas *B* = *S5Basic*:*3*

AOT-theorem *S5Basic*:*4*: ‹◇□φ → φ›
using *5*◇ *Hypothetical Syllogism qml*:*2*[*axiom-inst*] **by** *blast*
lemmas *B*◇ = *S5Basic*:*4*

AOT-theorem *S5Basic*:*5*: ‹□φ → □□φ›
using *RM*:*1* *B* *5*◇ *Hypothetical Syllogism* **by** *blast*
lemmas *4* = *S5Basic*:*5*

AOT-theorem *S5Basic*:*6*: ‹□φ ≡ □□φ›
by (*simp add*: *4* ≡*I* *qml*:*2*[*axiom-inst*])

AOT-theorem *S5Basic*:*7*: ‹◇◇φ → ◇φ›
using *conventions*:*5*[*THEN* ≡*Df*] *oth−class−taut*:*3*:*b*
by (*AOT-subst* ‹◇◇φ› ‹¬□¬◇φ›;
AOT-subst ‹◇φ› ‹¬□¬φ›;
AOT-subst (*reverse*) ‹¬¬□¬φ› ‹□¬φ›;
AOT-subst (*reverse*) ‹□□¬φ› ‹□¬φ›)
(*auto simp*: *S5Basic*:*6* *if−p−then−p*)

lemmas *4*◇ = *S5Basic*:*7*

AOT-theorem *S5Basic*:*8*: ‹◇◇φ ≡ ◇φ›
by (*simp add*: *4*◇ *T*◇ ≡*I*)

AOT-theorem *S5Basic*:*9*: ‹□(φ ∨ □ψ) ≡ (□φ ∨ □ψ)›

```
  apply (rule ≡I; rule →I)
  using KBasic2:6 5◇ ∨I(3) if−p−then−p vdash−properties:10
   apply blast
  by (meson KBasic:15 4 ∨I(3) ∨E(1) Disjunction Addition(1)
            con−dis−taut:7 intro−elim:1 Commutativity of ∨)

AOT-theorem S5Basic:10: ‹□(φ ∨ ◇ψ) ≡ (□φ ∨ ◇ψ)›
proof(rule ≡I; rule →I)
  AOT-assume ‹□(φ ∨ ◇ψ)›
  AOT-hence ‹□φ ∨ ◇◇ψ›
    by (meson KBasic2:6 ∨I(2) ∨E(1))
  AOT-thus ‹□φ ∨ ◇ψ›
    by (meson B◇ 4 4◇ T◇ ∨I(3))
next
  AOT-assume ‹□φ ∨ ◇ψ›
  AOT-hence ‹□φ ∨ □◇ψ›
    by (meson S5Basic:1 B◇ S5Basic:6 T◇ 5◇ ∨I(3) intro−elim:1)
  AOT-thus ‹□(φ ∨ ◇ψ)›
    by (meson KBasic:15 ∨I(3) ∨E(1) Disjunction Addition(1,2))
qed

AOT-theorem S5Basic:11: ‹◇(φ & ◇ψ) ≡ (◇φ & ◇ψ)›
proof −
  AOT-have ‹◇(φ & ◇ψ) ≡ ◇¬(¬φ ∨ ¬◇ψ)›
    by (AOT-subst ‹φ & ◇ψ› ‹¬(¬φ ∨ ¬◇ψ)›)
       (auto simp: oth−class−taut:5:a oth−class−taut:3:a)
  also AOT-have ‹... ≡ ◇¬(¬φ ∨ □¬ψ)›
    by (AOT-subst ‹□¬ψ› ‹¬◇ψ›)
       (auto simp: KBasic2:1 oth−class−taut:3:a)
  also AOT-have ‹... ≡ ¬□(¬φ ∨ □¬ψ)›
    using KBasic:11 ≡E(6) oth−class−taut:3:a by blast
  also AOT-have ‹... ≡ ¬(□¬φ ∨ □¬ψ)›
    using S5Basic:9 ≡E(1) oth−class−taut:4:b by blast
  also AOT-have ‹... ≡ ¬(¬◇φ ∨ ¬◇ψ)›
    using KBasic2:1
    by (AOT-subst ‹□¬φ› ‹¬◇φ›; AOT-subst ‹□¬ψ› ‹¬◇ψ›)
       (auto simp:  oth−class−taut:3:a)
  also AOT-have ‹... ≡ ◇φ & ◇ψ›
    using ≡E(6) oth−class−taut:3:a oth−class−taut:5:a by blast
  finally show ?thesis .
qed

AOT-theorem S5Basic:12: ‹◇(φ & □ψ) ≡ (◇φ & □ψ)›
proof (rule ≡I; rule →I)
  AOT-assume ‹◇(φ & □ψ)›
  AOT-hence ‹◇φ & ◇□ψ›
    using KBasic2:3 vdash−properties:6 by blast
  AOT-thus ‹◇φ & □ψ›
    using 5◇ &I &E(1) &E(2) vdash−properties:6 by blast
next
  AOT-assume ‹◇φ & □ψ›
  moreover AOT-have ‹(□□ψ & ◇φ) → ◇(φ & □ψ)›
    by (AOT-subst ‹φ & □ψ› ‹□ψ & φ›)
       (auto simp: Commutativity of & KBasic:16)
  ultimately AOT-show ‹◇(φ & □ψ)›
    by (metis 4 &I Conjunction Simplification(1,2) →E)
qed

AOT-theorem S5Basic:13: ‹□(φ → □ψ) ≡ □(◇φ → ψ)›
proof (rule ≡I)
  AOT-modally-strict {
    AOT-have ‹□(φ → □ψ) → (◇φ → ψ)›
      by (meson KBasic:13 B◇ Hypothetical Syllogism →I)
```

  }
  AOT-hence ‹$\Box\Box(\varphi \rightarrow \Box\psi) \rightarrow \Box(\Diamond\varphi \rightarrow \psi)$›
    by (*rule RM*)
  AOT-thus ‹$\Box(\varphi \rightarrow \Box\psi) \rightarrow \Box(\Diamond\varphi \rightarrow \psi)$›
    using *4 Hypothetical Syllogism* **by** *blast*
next
  AOT-modally-strict {
    AOT-have ‹$\Box(\Diamond\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \Box\psi)$›
      by (*meson B Hypothetical Syllogism* →*I qml*:*1*[*axiom-inst*])
  }
  AOT-hence ‹$\Box\Box(\Diamond\varphi \rightarrow \psi) \rightarrow \Box(\varphi \rightarrow \Box\psi)$›
    by (*rule RM*)
  AOT-thus ‹$\Box(\Diamond\varphi \rightarrow \psi) \rightarrow \Box(\varphi \rightarrow \Box\psi)$›
    using *4 Hypothetical Syllogism* **by** *blast*
qed

AOT-theorem *derived−S5−rules*:*1*:
  assumes ‹$\Gamma \vdash_\Box \Diamond\varphi \rightarrow \psi$›
  shows ‹$\Box\Gamma \vdash_\Box \varphi \rightarrow \Box\psi$›
proof −
  AOT-have ‹$\Box\Gamma \vdash_\Box \Box\Diamond\varphi \rightarrow \Box\psi$›
    using *assms* **by** (*rule RM*:*1*[*prem*])
  AOT-thus ‹$\Box\Gamma \vdash_\Box \varphi \rightarrow \Box\psi$›
    using *B Hypothetical Syllogism* **by** *blast*
qed

AOT-theorem *derived−S5−rules*:*2*:
  assumes ‹$\Gamma \vdash_\Box \varphi \rightarrow \Box\psi$›
  shows ‹$\Box\Gamma \vdash_\Box \Diamond\varphi \rightarrow \psi$›
proof −
  AOT-have ‹$\Box\Gamma \vdash_\Box \Diamond\varphi \rightarrow \Diamond\Box\psi$›
    using *assms* **by** (*rule RM*:*2*[*prem*])
  AOT-thus ‹$\Box\Gamma \vdash_\Box \Diamond\varphi \rightarrow \psi$›
    using *B*◇ *Hypothetical Syllogism* **by** *blast*
qed

AOT-theorem *BFs*:*1*: ‹$\forall\alpha\ \Box\varphi\{\alpha\} \rightarrow \Box\forall\alpha\ \varphi\{\alpha\}$›
proof −
  AOT-modally-strict {
    AOT-have ‹$\Diamond\forall\alpha\ \Box\varphi\{\alpha\} \rightarrow \Diamond\Box\varphi\{\alpha\}$› **for** α
      using *cqt−orig*:*3* **by** (*rule RM*◇)
    AOT-hence ‹$\Diamond\forall\alpha\ \Box\varphi\{\alpha\} \rightarrow \forall\alpha\ \varphi\{\alpha\}$›
      using *B*◇ ∀*I* →*E* →*I* **by** *metis*
  }
  thus *?thesis*
    using *derived−S5−rules*:*1* **by** *blast*
qed
lemmas *BF* = *BFs*:*1*

AOT-theorem *BFs*:*2*: ‹$\Box\forall\alpha\ \varphi\{\alpha\} \rightarrow \forall\alpha\ \Box\varphi\{\alpha\}$›
proof −
  AOT-have ‹$\Box\forall\alpha\ \varphi\{\alpha\} \rightarrow \Box\varphi\{\alpha\}$› **for** α
    using *RM cqt−orig*:*3* **by** *metis*
  thus *?thesis*
    using *cqt−orig*:*2*[*THEN* →*E*] ∀*I* **by** *metis*
qed
lemmas *CBF* = *BFs*:*2*

AOT-theorem *BFs*:*3*: ‹$\Diamond\exists\alpha\ \varphi\{\alpha\} \rightarrow \exists\alpha\ \Diamond\varphi\{\alpha\}$›
proof(*rule* →*I*)
  AOT-modally-strict {
    AOT-have ‹$\Box\forall\alpha\ \neg\varphi\{\alpha\} \equiv \forall\alpha\ \Box\neg\varphi\{\alpha\}$›
      using *BF CBF* ≡*I* **by** *blast*

} **note** ϑ = *this*

AOT-assume ‹◇∃ α φ{α}›
AOT-hence ‹¬□¬(∃ α φ{α})›
using $\equiv_{df}E$ *conventions*:*5* **by** *blast*
AOT-hence ‹¬□∀ α ¬φ{α}›
apply (*AOT-subst* ‹∀ α ¬φ{α}› ‹¬(∃ α φ{α})›)
using $\equiv_{df}I$ *conventions*:*3* *conventions*:*4* &*I*
contraposition:*2* *cqt−further*:*4*
df−rules−formulas[*3*] **by** *blast*
AOT-hence ‹¬∀ α □¬φ{α}›
apply (*AOT-subst* (*reverse*) ‹∀ α □¬φ{α}› ‹□∀ α ¬φ{α}›)
using ϑ **by** *blast*
AOT-hence ‹¬∀ α ¬¬□¬φ{α}›
by (*AOT-subst* (*reverse*) ‹¬¬□¬φ{α}› ‹□¬φ{α}› **for**: α)
(*simp add*: *oth−class−taut*:*3*:*b*)
AOT-hence ‹∃ α ¬□¬φ{α}›
by (*rule conventions*:*4*[*THEN* $\equiv_{df}I$])
AOT-thus ‹∃ α ◇φ{α}›
using *conventions*:*5*[*THEN* ≡*Df*]
by (*AOT-subst* ‹◇φ{α}› ‹¬□¬φ{α}› **for**: α)
qed
lemmas *BF*◇ = *BFs*:*3*

AOT-theorem *BFs*:*4*: ‹∃ α ◇φ{α} → ◇∃ α φ{α}›
proof(*rule* →*I*)
AOT-assume ‹∃ α ◇φ{α}›
AOT-hence ‹¬∀ α ¬◇φ{α}›
using *conventions*:*4*[*THEN* $\equiv_{df}E$] **by** *blast*
AOT-hence ‹¬∀ α □¬φ{α}›
using *KBasic2*:*1*
by (*AOT-subst* ‹□¬φ{α}› ‹¬◇φ{α}› **for**: α)
moreover AOT-have ‹∀ α □¬φ{α} ≡ □∀ α ¬φ{α}›
using ≡*I BF CBF* **by** *metis*
ultimately AOT-have *1*: ‹¬□∀ α ¬φ{α}›
using ≡*E*(*3*) **by** *blast*
AOT-show ‹◇∃ α φ{α}›
apply (*rule conventions*:*5*[*THEN* $\equiv_{df}I$])
apply (*AOT-subst* ‹∃ α φ{α}› ‹¬∀ α ¬φ{α}›)
apply (*simp add*: *conventions*:*4* ≡*Df*)
apply (*AOT-subst* ‹¬¬∀ α ¬φ{α}› ‹∀ α ¬φ{α}›)
by (*auto simp*: *1* ≡*I useful−tautologies*:*1 useful−tautologies*:*2*)
qed
lemmas *CBF*◇ = *BFs*:*4*

AOT-theorem *sign−S5−thm*:*1*: ‹∃ α □φ{α} → □∃ α φ{α}›
proof(*rule* →*I*)
AOT-assume ‹∃ α □φ{α}›
then AOT-obtain α **where** ‹□φ{α}› **using** ∃ *E* **by** *metis*
moreover AOT-have ‹□α↓›
by (*simp add*: *ex*:*1*:*a rule−ui*:*2*[*const-var*] *RN*)
moreover AOT-have ‹□φ{τ}, □τ↓ $\vdash_\Box$ □∃ α φ{α}› **for** τ
proof −
AOT-have ‹φ{τ}, τ↓ $\vdash_\Box$ ∃ α φ{α}› **using** *existential*:*1* **by** *blast*
AOT-thus ‹□φ{τ}, □τ↓ $\vdash_\Box$ □∃ α φ{α}›
using *RN*[*prem*][**where** Γ={φ τ, «τ↓»}, *simplified*] **by** *blast*
qed
ultimately AOT-show ‹□∃ α φ{α}› **by** *blast*
qed
lemmas *Buridan* = *sign−S5−thm*:*1*

AOT-theorem *sign−S5−thm*:*2*: ‹◇∀ α φ{α} → ∀ α ◇φ{α}›
proof −

```
  AOT-have ‹∀α (◊∀α φ{α} → ◊φ{α})›
    by (simp add: RM◊ cqt−orig:3 ∀I)
  AOT-thus ‹◊∀α φ{α} → ∀α ◊φ{α}›
    using ∀E(4) ∀I →E →I by metis
qed
lemmas Buridan◊ = sign−S5−thm:2

AOT-theorem sign−S5−thm:3:
  ‹◊∃α (φ{α} & ψ{α}) → ◊(∃α φ{α} & ∃α ψ{α})›
  apply (rule RM:2)
  by (metis (no-types, lifting) ∃E &I &E(1) &E(2) →I ∃I(2))

AOT-theorem sign−S5−thm:4: ‹◊∃α (φ{α} & ψ{α}) → ◊∃α φ{α}›
  apply (rule RM:2)
  by (meson instantiation &E(1) →I ∃I(2))

AOT-theorem sign−S5−thm:5:
  ‹(□∀α (φ{α} → ψ{α}) & □∀α (ψ{α} → χ{α})) → □∀α (φ{α} → χ{α})›
proof −
  {
    fix φ' ψ' χ'
    AOT-assume ‹⊢□ φ' & ψ' → χ'›
    AOT-hence ‹□φ' & □ψ' → □χ'›
      using RN[prem][where Γ={φ', ψ'}] apply simp
      using &E &I →E →I by metis
  } note R = this
  show ?thesis by (rule R; fact AOT)
qed

AOT-theorem sign−S5−thm:6:
  ‹(□∀α (φ{α} ≡ ψ{α}) & □∀α(ψ{α} ≡ χ{α})) → □∀α(φ{α} ≡ χ{α})›
proof −
  {
    fix φ' ψ' χ'
    AOT-assume ‹⊢□ φ' & ψ' → χ'›
    AOT-hence ‹□φ' & □ψ' → □χ'›
      using RN[prem][where Γ={φ', ψ'}] apply simp
      using &E &I →E →I by metis
  } note R = this
  show ?thesis by (rule R; fact AOT)
qed

AOT-theorem exist−nec2:1: ‹◊τ↓ → τ↓›
  using B◊ RM◊ Hypothetical Syllogism exist−nec by blast

AOT-theorem exists−nec2:2: ‹◊τ↓ ≡ □τ↓›
  by (meson Act−Sub:3 Hypothetical Syllogism exist−nec
          exist−nec2:1 ≡I nec−imp−act)

AOT-theorem exists−nec2:3: ‹¬τ↓ → □¬τ↓›
  using KBasic2:1 →I exist−nec2:1 ≡E(2) modus−tollens:1 by blast

AOT-theorem exists−nec2:4: ‹◊¬τ↓ ≡ □¬τ↓›
  by (metis Act−Sub:3 KBasic:12 →I exist−nec exists−nec2:3
          ≡I ≡E(4) nec−imp−act reductio−aa:1)

AOT-theorem id−nec2:1: ‹◊α = β → α = β›
  using B◊ RM◊ Hypothetical Syllogism id−nec:1 by blast

AOT-theorem id−nec2:2: ‹α ≠ β → □α ≠ β›
  apply (AOT-subst ‹α ≠ β› ‹¬(α = β)›)
  using =−infix[THEN ≡Df] apply blast
  using KBasic2:1 →I id−nec2:1 ≡E(2) modus−tollens:1 by blast
```

```
AOT-theorem id−nec2:3: ‹◇α ≠ β → α ≠ β›
  apply (AOT-subst ‹α ≠ β› ‹¬(α = β)›)
  using =−infix[THEN ≡Df] apply blast
  by (metis KBasic:11 →I id−nec:2 ≡E(3) reductio−aa:2 →E)

AOT-theorem id−nec2:4: ‹◇α = β → □α = β›
  using Hypothetical Syllogism id−nec2:1 id−nec:1 by blast

AOT-theorem id−nec2:5: ‹◇α ≠ β → □α ≠ β›
  using id−nec2:3 id−nec2:2 →I →E by metis

AOT-theorem sc−eq−box−box:1: ‹□(φ → □φ) ≡ (◇φ → □φ)›
  apply (rule ≡I; rule →I)
  using KBasic:13 5◇ Hypothetical Syllogism →E apply blast
  by (metis KBasic2:1 KBasic:1 KBasic:2 S5Basic:13 ≡E(2)
          raa−cor:5 →E)

AOT-theorem sc−eq−box−box:2: ‹(□(φ → □φ) ∨ (◇φ → □φ)) → (◇φ ≡ □φ)›
  by (metis Act−Sub:3 KBasic:13 5◇ ∨E(2) →I ≡I
          nec−imp−act raa−cor:2 →E)

AOT-theorem sc−eq−box−box:3: ‹□(φ → □φ) → (¬□φ ≡ □¬φ)›
proof (rule →I; rule ≡I; rule →I)
  AOT-assume ‹□(φ → □φ)›
  AOT-hence ‹◇φ → □φ› using sc−eq−box−box:1 ≡E by blast
  moreover AOT-assume ‹¬□φ›
  ultimately AOT-have ‹¬◇φ›
    using modus−tollens:1 by blast
  AOT-thus ‹□¬φ›
    using KBasic2:1 ≡E(2) by blast
next
  AOT-assume ‹□(φ → □φ)›
  moreover AOT-assume ‹□¬φ›
  ultimately AOT-show ‹¬□φ›
    using modus−tollens:1 qml:2[axiom-inst] →E by blast
qed

AOT-theorem sc−eq−box−box:4:
  ‹(□(φ → □φ) & □(ψ → □ψ)) → ((□φ ≡ □ψ) → □(φ ≡ ψ))›
proof(rule →I; rule →I)
  AOT-assume ϑ: ‹□(φ → □φ) & □(ψ → □ψ)›
  AOT-assume ξ: ‹□φ ≡ □ψ›
  AOT-hence ‹(□φ & □ψ) ∨ (¬□φ & ¬□ψ)›
    using ≡E(4) oth−class−taut:4:g raa−cor:3 by blast
  moreover {
    AOT-assume ‹□φ & □ψ›
    AOT-hence ‹□(φ ≡ ψ)›
      using KBasic:3 KBasic:8 ≡E(2) vdash−properties:10 by blast
  }
  moreover {
    AOT-assume ‹¬□φ & ¬□ψ›
    moreover AOT-have ‹¬□φ ≡ □¬φ› and ‹¬□ψ ≡ □¬ψ›
      using ϑ Conjunction Simplification(1,2)
          sc−eq−box−box:3 →E by metis+
    ultimately AOT-have ‹□¬φ & □¬ψ›
      by (metis &I Conjunction Simplification(1,2)
              ≡E(4) modus−tollens:1 raa−cor:3)
    AOT-hence ‹□(φ ≡ ψ)›
      using KBasic:3 KBasic:9 ≡E(2) →E by blast
  }
  ultimately AOT-show ‹□(φ ≡ ψ)›
    using ∨E(2) reductio−aa:1 by blast
```

qed

AOT-theorem *sc−eq−box−box*:*5*:
‹$(\Box(\varphi \to \Box\varphi)$ & $\Box(\psi \to \Box\psi)) \to \Box((\varphi \equiv \psi) \to \Box(\varphi \equiv \psi))$›
proof (*rule* →*I*)
  AOT-assume ‹$(\Box(\varphi \to \Box\varphi)$ & $\Box(\psi \to \Box\psi))$›
  AOT-hence ‹$\Box(\Box(\varphi \to \Box\varphi)$ & $\Box(\psi \to \Box\psi))$›
    using *4*[*THEN* →*E*] &*E* &*I* *KBasic*:*3* ≡*E*(*2*) **by** *metis*
  moreover AOT-have ‹$\Box(\Box(\varphi \to \Box\varphi)$ & $\Box(\psi \to \Box\psi)) \to \Box((\varphi \equiv \psi) \to \Box(\varphi \equiv \psi))$›
  proof (*rule RM*; *rule* →*I*; *rule* →*I*)
    AOT-modally-strict {
      AOT-assume *A*: ‹$(\Box(\varphi \to \Box\varphi)$ & $\Box(\psi \to \Box\psi))$›
      AOT-hence ‹$\varphi \to \Box\varphi$› **and** ‹$\psi \to \Box\psi$›
        using &*E* *qml*:*2*[*axiom-inst*] →*E* **by** *blast*+
      moreover AOT-assume ‹$\varphi \equiv \psi$›
      ultimately AOT-have ‹$\Box\varphi \equiv \Box\psi$›
        using →*E* *qml*:*2*[*axiom-inst*] ≡*E* ≡*I* **by** *meson*
      moreover AOT-have ‹$(\Box\varphi \equiv \Box\psi) \to \Box(\varphi \equiv \psi)$›
        using *A* *sc−eq−box−box*:*4* →*E* **by** *blast*
      ultimately AOT-show ‹$\Box(\varphi \equiv \psi)$› **using** →*E* **by** *blast*
    }
  qed
  ultimately AOT-show ‹$\Box((\varphi \equiv \psi) \to \Box(\varphi \equiv \psi))$› **using** →*E* **by** *blast*
qed

AOT-theorem *sc−eq−box−box*:*6*: ‹$\Box(\varphi \to \Box\varphi) \to ((\varphi \to \Box\psi) \to \Box(\varphi \to \psi))$›
proof (*rule* →*I*; *rule* →*I*; *rule raa−cor*:*1*)
  AOT-assume ‹$\neg\Box(\varphi \to \psi)$›
  AOT-hence ‹$\Diamond\neg(\varphi \to \psi)$›
    by (*metis KBasic*:*11* ≡*E*(*1*))
  AOT-hence ‹$\Diamond(\varphi$ & $\neg\psi)$›
    by (*AOT-subst* ‹φ & $\neg\psi$› ‹$\neg(\varphi \to \psi)$›)
      (*meson Commutativity of* ≡ ≡*E*(*1*) *oth−class−taut*:*1*:*b*)
  AOT-hence ‹$\Diamond\varphi$› **and** *2*: ‹$\Diamond\neg\psi$›
    using *KBasic2*:*3*[*THEN* →*E*] &*E* **by** *blast*+
  moreover AOT-assume ‹$\Box(\varphi \to \Box\varphi)$›
  ultimately AOT-have ‹$\Box\varphi$›
    by (*metis* ≡*E*(*1*) *sc−eq−box−box*:*1* →*E*)
  AOT-hence φ
    using *qml*:*2*[*axiom-inst*, *THEN* →*E*] **by** *blast*
  moreover AOT-assume ‹$\varphi \to \Box\psi$›
  ultimately AOT-have ‹$\Box\psi$›
    using →*E* **by** *blast*
  moreover AOT-have ‹$\neg\Box\psi$›
    using *2* *KBasic*:*12* ¬¬*I* *intro−elim*:*3*:*d* **by** *blast*
  ultimately AOT-show ‹$\Box\psi$ & $\neg\Box\psi$›
    using &*I* **by** *blast*
qed

AOT-theorem *sc−eq−box−box*:*7*: ‹$\Box(\varphi \to \Box\varphi) \to ((\varphi \to \boldsymbol{\mathcal{A}}\psi) \to \boldsymbol{\mathcal{A}}(\varphi \to \psi))$›
proof (*rule* →*I*; *rule* →*I*; *rule raa−cor*:*1*)
  AOT-assume ‹$\neg\boldsymbol{\mathcal{A}}(\varphi \to \psi)$›
  AOT-hence ‹$\boldsymbol{\mathcal{A}}\neg(\varphi \to \psi)$›
    by (*metis Act−Basic*:*1* ∨*E*(*2*))
  AOT-hence ‹$\boldsymbol{\mathcal{A}}(\varphi$ & $\neg\psi)$›
    by (*AOT-subst* ‹φ & $\neg\psi$› ‹$\neg(\varphi \to \psi)$›)
      (*meson Commutativity of* ≡ ≡*E*(*1*) *oth−class−taut*:*1*:*b*)
  AOT-hence ‹$\boldsymbol{\mathcal{A}}\varphi$› **and** *2*: ‹$\boldsymbol{\mathcal{A}}\neg\psi$›
    using *Act−Basic*:*2*[*THEN* ≡*E*(*1*)] &*E* **by** *blast*+
  AOT-hence ‹$\Diamond\varphi$›
    by (*metis Act−Sub*:*3* →*E*)
  moreover AOT-assume ‹$\Box(\varphi \to \Box\varphi)$›
  ultimately AOT-have ‹$\Box\varphi$›

by (*metis* ≡*E*(*1*) *sc−eq−box−box*:*1* →*E*)
AOT-hence φ
using *qml*:*2*[*axiom-inst*, *THEN* →*E*] **by** *blast*
moreover AOT-assume ‹$\varphi \rightarrow \mathcal{A}\psi$›
ultimately AOT-have ‹$\mathcal{A}\psi$›
using →*E* **by** *blast*
moreover AOT-have ‹$\neg\mathcal{A}\psi$›
using *2* **by** (*meson Act−Sub*:*1* ≡*E*(*4*) *raa−cor*:*3*)
ultimately AOT-show ‹$\mathcal{A}\psi$ & $\neg\mathcal{A}\psi$›
using &*I* **by** *blast*
qed

AOT-theorem *sc−eq−fur*:*1*: ‹$\Diamond\mathcal{A}\varphi \equiv \Box\mathcal{A}\varphi$›
using *Act−Basic*:*6 Act−Sub*:*4* ≡*E*(*6*) **by** *blast*

AOT-theorem *sc−eq−fur*:*2*: ‹$\Box(\varphi \rightarrow \Box\varphi) \rightarrow (\mathcal{A}\varphi \equiv \varphi)$›
by (*metis B◇ Act−Sub*:*3 KBasic*:*13 T◇ Hypothetical Syllogism* →*I* ≡*I nec−imp−act*)

AOT-theorem *sc−eq−fur*:*3*:
‹$\Box\forall x\ (\varphi\{x\} \rightarrow \Box\varphi\{x\}) \rightarrow (\exists!x\ \varphi\{x\} \rightarrow \iota x\ \varphi\{x\}\downarrow)$›
proof (*rule* →*I*; *rule* →*I*)
AOT-assume ‹$\Box\forall x\ (\varphi\{x\} \rightarrow \Box\varphi\{x\})$›
AOT-hence *A*: ‹$\forall x\ \Box(\varphi\{x\} \rightarrow \Box\varphi\{x\})$›
using *CBF* →*E* **by** *blast*
AOT-assume ‹$\exists!x\ \varphi\{x\}$›
then AOT-obtain *a* **where** *a-def*: ‹$\varphi\{a\}$ & $\forall y\ (\varphi\{y\} \rightarrow y = a)$›
using ∃*E*[*rotated 1*, *OF uniqueness*:*1*[*THEN* $\equiv_{df}$*E*]] **by** *blast*
moreover AOT-have ‹$\Box\varphi\{a\}$›
using *calculation A* ∀*E*(*2*) *qml*:*2*[*axiom-inst*] →*E* &*E*(*1*) **by** *blast*
AOT-hence ‹$\mathcal{A}\varphi\{a\}$›
using *nec−imp−act* →*E* **by** *blast*
moreover AOT-have ‹$\forall y\ (\mathcal{A}\varphi\{y\} \rightarrow y = a)$›
proof (*rule* ∀*I*; *rule* →*I*)
fix *b*
AOT-assume ‹$\mathcal{A}\varphi\{b\}$›
AOT-hence ‹$\Diamond\varphi\{b\}$›
using *Act−Sub*:*3* →*E* **by** *blast*
moreover {
AOT-have ‹$\Box(\varphi\{b\} \rightarrow \Box\varphi\{b\})$›
using *A* ∀*E*(*2*) **by** *blast*
AOT-hence ‹$\Diamond\varphi\{b\} \rightarrow \Box\varphi\{b\}$›
using *KBasic*:*13 5◇ Hypothetical Syllogism* →*E* **by** *blast*
}
ultimately AOT-have ‹$\Box\varphi\{b\}$›
using →*E* **by** *blast*
AOT-hence ‹$\varphi\{b\}$›
using *qml*:*2*[*axiom-inst*] →*E* **by** *blast*
AOT-thus ‹$b = a$›
using *a-def*[*THEN* &*E*(*2*)] ∀*E*(*2*) →*E* **by** *blast*
qed
ultimately AOT-have ‹$\mathcal{A}\varphi\{a\}$ & $\forall y\ (\mathcal{A}\varphi\{y\} \rightarrow y = a)$›
using &*I* **by** *blast*
AOT-hence ‹$\exists x\ (\mathcal{A}\varphi\{x\}$ & $\forall y\ (\mathcal{A}\varphi\{y\} \rightarrow y = x))$›
using ∃*I* **by** *fast*
AOT-hence ‹$\exists!x\ \mathcal{A}\varphi\{x\}$›
using *uniqueness*:*1*[*THEN* $\equiv_{df}$*I*] **by** *fast*
AOT-thus ‹$\iota x\ \varphi\{x\}\downarrow$›
using *actual−desc*:*1*[*THEN* ≡*E*(*2*)] **by** *blast*
qed

AOT-theorem *sc−eq−fur*:*4*:
‹$\Box\forall x\ (\varphi\{x\} \rightarrow \Box\varphi\{x\}) \rightarrow (x = \iota x\ \varphi\{x\} \equiv (\varphi\{x\}$ & $\forall z\ (\varphi\{z\} \rightarrow z = x)))$›

```
proof (rule →I)
  AOT-assume ‹□∀ x (φ{x} → □φ{x})›
  AOT-hence ‹∀ x □(φ{x} → □φ{x})›
    using CBF →E by blast
  AOT-hence A: ‹𝓐φ{α} ≡ φ{α}› for α
    using sc−eq−fur:2 ∀ E →E by fast
  AOT-show ‹x = ιx φ{x} ≡ (φ{x} & ∀ z (φ{z} → z = x))›
  proof (rule ≡I; rule →I)
    AOT-assume ‹x = ιx φ{x}›
    AOT-hence B: ‹𝓐φ{x} & ∀ z (𝓐φ{z} → z = x)›
      using nec−hintikka−scheme[THEN ≡E(1)] by blast
    AOT-show ‹φ{x} & ∀ z (φ{z} → z = x)›
    proof (rule &I; (rule ∀ I; rule →I)?)
      AOT-show ‹φ{x}›
        using A B[THEN &E(1)] ≡E(1) by blast
    next
      AOT-show ‹z = x› if ‹φ{z}› for z
        using that B[THEN &E(2)] ∀ E(2) →E A[THEN ≡E(2)] by blast
    qed
  next
    AOT-assume B: ‹φ{x} & ∀ z (φ{z} → z = x)›
    AOT-have ‹𝓐φ{x} & ∀ z (𝓐φ{z} → z = x)›
    proof(rule &I; (rule ∀ I; rule →I)?)
      AOT-show ‹𝓐φ{x}›
        using B[THEN &E(1)] A[THEN ≡E(2)] by blast
    next
      AOT-show ‹b = x› if ‹𝓐φ{b}› for b
        using A[THEN ≡E(1)] that
              B[THEN &E(2), THEN ∀ E(2), THEN →E] by blast
    qed
    AOT-thus ‹x = ιx φ{x}›
      using nec−hintikka−scheme[THEN ≡E(2)] by blast
  qed
qed

AOT-theorem id−act:1: ‹α = β ≡ 𝓐α = β›
  by (meson Act−Sub:3 Hypothetical Syllogism
          id−nec2:1 id−nec:2 ≡I nec−imp−act)

AOT-theorem id−act:2: ‹α ≠ β ≡ 𝓐α ≠ β›
proof (AOT-subst ‹α ≠ β› ‹¬(α = β)›)
  AOT-modally-strict {
    AOT-show ‹α ≠ β ≡ ¬(α = β)›
      by (simp add: =−infix ≡Df)
  }
next
  AOT-show ‹¬(α = β) ≡ 𝓐¬(α = β)›
  proof (safe intro!: ≡I →I)
    AOT-assume ‹¬α = β›
    AOT-hence ‹¬𝓐α = β› using id−act:1 ≡E(3) by blast
    AOT-thus ‹𝓐¬α = β›
      using ¬¬E Act−Sub:1 ≡E(3) by blast
  next
    AOT-assume ‹𝓐¬α = β›
    AOT-hence ‹¬𝓐α = β›
      using ¬¬I Act−Sub:1 ≡E(4) by blast
    AOT-thus ‹¬α = β›
      using id−act:1 ≡E(4) by blast
  qed
qed

AOT-theorem A−Exists:1: ‹𝓐∃ !α φ{α} ≡ ∃ !α 𝓐φ{α}›
proof −
```

```
  AOT-have ‹𝓐∃!α φ{α} ≡ 𝓐∃α∀β (φ{β} ≡ β = α)›
    by (AOT-subst ‹∃!α φ{α}› ‹∃α∀β (φ{β} ≡ β = α)›)
       (auto simp add: oth−class−taut:3:a uniqueness:2)
  also AOT-have ‹... ≡ ∃α 𝓐∀β (φ{β} ≡ β = α)›
    by (simp add: Act−Basic:10)
  also AOT-have ‹... ≡ ∃α∀β 𝓐(φ{β} ≡ β = α)›
    by (AOT-subst ‹𝓐∀β (φ{β} ≡ β = α)› ‹∀β 𝓐(φ{β} ≡ β = α)› for: α)
       (auto simp: logic−actual−nec:3[axiom-inst] oth−class−taut:3:a)
  also AOT-have ‹... ≡ ∃α∀β (𝓐φ{β} ≡ 𝓐β = α)›
    by (AOT-subst (reverse) ‹𝓐φ{β} ≡ 𝓐β = α›
                             ‹𝓐(φ{β} ≡ β = α)› for: α β :: 'a)
       (auto simp: Act−Basic:5 cqt−further:7)
  also AOT-have ‹... ≡ ∃α∀β (𝓐φ{β} ≡ β = α)›
    by (AOT-subst (reverse) ‹𝓐β = α› ‹β = α› for: α β :: 'a)
       (auto simp: id−act:1 cqt−further:7)
  also AOT-have ‹... ≡ ∃!α 𝓐φ{α}›
    using uniqueness:2 Commutativity of ≡[THEN ≡E(1)] by fast
  finally show ?thesis.
qed

AOT-theorem A−Exists:2: ‹ιx φ{x}↓ ≡ 𝓐∃!x φ{x}›
  by (AOT-subst ‹𝓐∃!x φ{x}› ‹∃!x 𝓐φ{x}›)
     (auto simp: actual−desc:1 A−Exists:1)

AOT-theorem id−act−desc:1: ‹ιx (x = y)↓›
proof(rule existence:1[THEN ≡df I]; rule ∃I)
  AOT-show ‹[λx E!x → E!x]ιx (x = y)›
  proof (rule russell−axiom[exe,1].nec−russell−axiom[THEN ≡E(2)];
         rule ∃I; (rule &I)+)
    AOT-show ‹𝓐y = y› by (simp add: RA[2] id−eq:1)
  next
    AOT-show ‹∀z (𝓐z = y → z = y)›
      apply (rule ∀I)
      using id−act:1[THEN ≡E(2)] →I by blast
  next
    AOT-show ‹[λx E!x → E!x]y›
    proof (rule lambda−predicates:2[axiom-inst, THEN →E, THEN ≡E(2)])
      AOT-show ‹[λx E!x → E!x]↓›
        by cqt:2[lambda]
    next
      AOT-show ‹E!y → E!y›
        by (simp add: if−p−then−p)
    qed
  qed
next
  AOT-show ‹[λx E!x → E!x]↓›
    by cqt:2[lambda]
qed

AOT-theorem id−act−desc:2: ‹y = ιx (x = y)›
  by (rule descriptions[axiom-inst, THEN ≡E(2)];
      rule ∀I; rule id−act:1[symmetric])

AOT-theorem pre−en−eq:1[1]: ‹x1[F] → □x1[F]›
  by (simp add: encoding vdash−properties:1[2])

AOT-theorem pre−en−eq:1[2]: ‹x1x2[F] → □x1x2[F]›
proof (rule →I)
  AOT-assume ‹x1x2[F]›
  AOT-hence ‹x1[λy [F]yx2]› and ‹x2[λy [F]x1y]›
    using nary−encoding[2][axiom-inst, THEN ≡E(1)] &E by blast+
  moreover AOT-have ‹[λy [F]yx2]↓› by cqt:2
  moreover AOT-have ‹[λy [F]x1y]↓› by cqt:2
```

```
  ultimately AOT-have ‹□x₁[λy [F]yx₂]› and ‹□x₂[λy [F]x₁y]›
    using encoding[axiom-inst, unvarify F] →E &I by blast+
  note A = this
  AOT-hence ‹□(x₁[λy [F]yx₂] & x₂[λy [F]x₁y])›
    using KBasic:3[THEN ≡E(2)] &I by blast
  AOT-thus ‹□x₁x₂[F]›
    by (rule nary−encoding[2][axiom-inst, THEN RN,
                              THEN KBasic:6[THEN →E],
                              THEN ≡E(2)])
qed

AOT-theorem pre−en−eq:1[3]: ‹x₁x₂x₃[F] → □x₁x₂x₃[F]›
proof (rule →I)
  AOT-assume ‹x₁x₂x₃[F]›
  AOT-hence ‹x₁[λy [F]yx₂x₃]›
        and ‹x₂[λy [F]x₁yx₃]›
        and ‹x₃[λy [F]x₁x₂y]›
    using nary−encoding[3][axiom-inst, THEN ≡E(1)] &E by blast+
  moreover AOT-have ‹[λy [F]yx₂x₃]↓› by cqt:2
  moreover AOT-have ‹[λy [F]x₁yx₃]↓› by cqt:2
  moreover AOT-have ‹[λy [F]x₁x₂y]↓› by cqt:2
  ultimately AOT-have ‹□x₁[λy [F]yx₂x₃]›
                  and ‹□x₂[λy [F]x₁yx₃]›
                  and ‹□x₃[λy [F]x₁x₂y]›
    using encoding[axiom-inst, unvarify F] →E by blast+
  note A = this
  AOT-have B: ‹□(x₁[λy [F]yx₂x₃] & x₂[λy [F]x₁yx₃] & x₃[λy [F]x₁x₂y])›
    by (rule KBasic:3[THEN ≡E(2)] &I A)+
  AOT-thus ‹□x₁x₂x₃[F]›
    by (rule nary−encoding[3][axiom-inst, THEN RN,
                THEN KBasic:6[THEN →E], THEN ≡E(2)])
qed

AOT-theorem pre−en−eq:1[4]: ‹x₁x₂x₃x₄[F] → □x₁x₂x₃x₄[F]›
proof (rule →I)
  AOT-assume ‹x₁x₂x₃x₄[F]›
  AOT-hence ‹x₁[λy [F]yx₂x₃x₄]›
        and ‹x₂[λy [F]x₁yx₃x₄]›
        and ‹x₃[λy [F]x₁x₂yx₄]›
        and ‹x₄[λy [F]x₁x₂x₃y]›
    using nary−encoding[4][axiom-inst, THEN ≡E(1)] &E by metis+
  moreover AOT-have ‹[λy [F]yx₂x₃x₄]↓› by cqt:2
  moreover AOT-have ‹[λy [F]x₁yx₃x₄]↓› by cqt:2
  moreover AOT-have ‹[λy [F]x₁x₂yx₄]↓› by cqt:2
  moreover AOT-have ‹[λy [F]x₁x₂x₃y]↓› by cqt:2
  ultimately AOT-have ‹□x₁[λy [F]yx₂x₃x₄]›
                  and ‹□x₂[λy [F]x₁yx₃x₄]›
                  and ‹□x₃[λy [F]x₁x₂yx₄]›
                  and ‹□x₄[λy [F]x₁x₂x₃y]›
    using →E encoding[axiom-inst, unvarify F] by blast+
  note A = this
  AOT-have B: ‹□(x₁[λy [F]yx₂x₃x₄] &
                x₂[λy [F]x₁yx₃x₄] &
                x₃[λy [F]x₁x₂yx₄] &
                x₄[λy [F]x₁x₂x₃y])›
    by (rule KBasic:3[THEN ≡E(2)] &I A)+
  AOT-thus ‹□x₁x₂x₃x₄[F]›
    by (rule nary−encoding[4][axiom-inst, THEN RN,
              THEN KBasic:6[THEN →E], THEN ≡E(2)])
qed

AOT-theorem pre−en−eq:2[1]: ‹¬x₁[F] → □¬x₁[F]›
proof (rule →I; rule raa−cor:1)
```

AOT-assume ‹$\neg\Box\neg x_1[F]$›
AOT-hence ‹$\Diamond x_1[F]$›
by (*rule conventions*:*5*[*THEN* $\equiv_{df}I$])
AOT-hence ‹$x_1[F]$›
by(*rule S5Basic*:*13*[*THEN* $\equiv E(1)$, *OF pre−en−eq*:*1*[*1*][*THEN RN*],
THEN qml:*2*[*axiom-inst*, *THEN* $\rightarrow E$], *THEN* $\rightarrow E$])
moreover AOT-assume ‹$\neg x_1[F]$›
ultimately AOT-show ‹$x_1[F]$ & $\neg x_1[F]$› **by** (*rule* &*I*)
qed
AOT-theorem *pre−en−eq*:*2*[*2*]: ‹$\neg x_1x_2[F] \rightarrow \Box\neg x_1x_2[F]$›
proof (*rule* $\rightarrow I$; *rule raa−cor*:*1*)
AOT-assume ‹$\neg\Box\neg x_1x_2[F]$›
AOT-hence ‹$\Diamond x_1x_2[F]$›
by (*rule conventions*:*5*[*THEN* $\equiv_{df}I$])
AOT-hence ‹$x_1x_2[F]$›
by(*rule S5Basic*:*13*[*THEN* $\equiv E(1)$, *OF pre−en−eq*:*1*[*2*][*THEN RN*],
THEN qml:*2*[*axiom-inst*, *THEN* $\rightarrow E$], *THEN* $\rightarrow E$])
moreover AOT-assume ‹$\neg x_1x_2[F]$›
ultimately AOT-show ‹$x_1x_2[F]$ & $\neg x_1x_2[F]$› **by** (*rule* &*I*)
qed

AOT-theorem *pre−en−eq*:*2*[*3*]: ‹$\neg x_1x_2x_3[F] \rightarrow \Box\neg x_1x_2x_3[F]$›
proof (*rule* $\rightarrow I$; *rule raa−cor*:*1*)
AOT-assume ‹$\neg\Box\neg x_1x_2x_3[F]$›
AOT-hence ‹$\Diamond x_1x_2x_3[F]$›
by (*rule conventions*:*5*[*THEN* $\equiv_{df}I$])
AOT-hence ‹$x_1x_2x_3[F]$›
by(*rule S5Basic*:*13*[*THEN* $\equiv E(1)$, *OF pre−en−eq*:*1*[*3*][*THEN RN*],
THEN qml:*2*[*axiom-inst*, *THEN* $\rightarrow E$], *THEN* $\rightarrow E$])
moreover AOT-assume ‹$\neg x_1x_2x_3[F]$›
ultimately AOT-show ‹$x_1x_2x_3[F]$ & $\neg x_1x_2x_3[F]$› **by** (*rule* &*I*)
qed

AOT-theorem *pre−en−eq*:*2*[*4*]: ‹$\neg x_1x_2x_3x_4[F] \rightarrow \Box\neg x_1x_2x_3x_4[F]$›
proof (*rule* $\rightarrow I$; *rule raa−cor*:*1*)
AOT-assume ‹$\neg\Box\neg x_1x_2x_3x_4[F]$›
AOT-hence ‹$\Diamond x_1x_2x_3x_4[F]$›
by (*rule conventions*:*5*[*THEN* $\equiv_{df}I$])
AOT-hence ‹$x_1x_2x_3x_4[F]$›
by(*rule S5Basic*:*13*[*THEN* $\equiv E(1)$, *OF pre−en−eq*:*1*[*4*][*THEN RN*],
THEN qml:*2*[*axiom-inst*, *THEN* $\rightarrow E$], *THEN* $\rightarrow E$])
moreover AOT-assume ‹$\neg x_1x_2x_3x_4[F]$›
ultimately AOT-show ‹$x_1x_2x_3x_4[F]$ & $\neg x_1x_2x_3x_4[F]$› **by** (*rule* &*I*)
qed

AOT-theorem *en−eq*:*1*[*1*]: ‹$\Diamond x_1[F] \equiv \Box x_1[F]$›
using *pre−en−eq*:*1*[*1*][*THEN RN*] *sc−eq−box−box*:*2* $\vee I$ $\rightarrow E$ **by** *metis*
AOT-theorem *en−eq*:*1*[*2*]: ‹$\Diamond x_1x_2[F] \equiv \Box x_1x_2[F]$›
using *pre−en−eq*:*1*[*2*][*THEN RN*] *sc−eq−box−box*:*2* $\vee I$ $\rightarrow E$ **by** *metis*
AOT-theorem *en−eq*:*1*[*3*]: ‹$\Diamond x_1x_2x_3[F] \equiv \Box x_1x_2x_3[F]$›
using *pre−en−eq*:*1*[*3*][*THEN RN*] *sc−eq−box−box*:*2* $\vee I$ $\rightarrow E$ **by** *fast*
AOT-theorem *en−eq*:*1*[*4*]: ‹$\Diamond x_1x_2x_3x_4[F] \equiv \Box x_1x_2x_3x_4[F]$›
using *pre−en−eq*:*1*[*4*][*THEN RN*] *sc−eq−box−box*:*2* $\vee I$ $\rightarrow E$ **by** *fast*

AOT-theorem *en−eq*:*2*[*1*]: ‹$x_1[F] \equiv \Box x_1[F]$›
by (*simp add*: $\equiv I$ *pre−en−eq*:*1*[*1*] *qml*:*2*[*axiom-inst*])
AOT-theorem *en−eq*:*2*[*2*]: ‹$x_1x_2[F] \equiv \Box x_1x_2[F]$›
by (*simp add*: $\equiv I$ *pre−en−eq*:*1*[*2*] *qml*:*2*[*axiom-inst*])
AOT-theorem *en−eq*:*2*[*3*]: ‹$x_1x_2x_3[F] \equiv \Box x_1x_2x_3[F]$›
by (*simp add*: $\equiv I$ *pre−en−eq*:*1*[*3*] *qml*:*2*[*axiom-inst*])
AOT-theorem *en−eq*:*2*[*4*]: ‹$x_1x_2x_3x_4[F] \equiv \Box x_1x_2x_3x_4[F]$›
by (*simp add*: $\equiv I$ *pre−en−eq*:*1*[*4*] *qml*:*2*[*axiom-inst*])

AOT-theorem *en−eq:3[1]*: ‹$\Diamond x_1[F] \equiv x_1[F]$›
using $T\Diamond$ *derived−S5−rules:2[OF pre−en−eq:1[1]] ≡I* **by** *blast*
AOT-theorem *en−eq:3[2]*: ‹$\Diamond x_1 x_2[F] \equiv x_1 x_2[F]$›
using $T\Diamond$ *derived−S5−rules:2[OF pre−en−eq:1[2]] ≡I* **by** *blast*
AOT-theorem *en−eq:3[3]*: ‹$\Diamond x_1 x_2 x_3[F] \equiv x_1 x_2 x_3[F]$›
using $T\Diamond$ *derived−S5−rules:2[OF pre−en−eq:1[3]] ≡I* **by** *blast*
AOT-theorem *en−eq:3[4]*: ‹$\Diamond x_1 x_2 x_3 x_4[F] \equiv x_1 x_2 x_3 x_4[F]$›
using $T\Diamond$ *derived−S5−rules:2[OF pre−en−eq:1[4]] ≡I* **by** *blast*

AOT-theorem *en−eq:4[1]*:
‹$(x_1[F] \equiv y_1[G]) \equiv (\Box x_1[F] \equiv \Box y_1[G])$›
apply *(rule ≡I; rule →I; rule ≡I; rule →I)*
using *qml:2[axiom-inst, THEN →E] ≡E(1,2) en−eq:2[1]* **by** *blast+*
AOT-theorem *en−eq:4[2]*:
‹$(x_1 x_2[F] \equiv y_1 y_2[G]) \equiv (\Box x_1 x_2[F] \equiv \Box y_1 y_2[G])$›
apply *(rule ≡I; rule →I; rule ≡I; rule →I)*
using *qml:2[axiom-inst, THEN →E] ≡E(1,2) en−eq:2[2]* **by** *blast+*
AOT-theorem *en−eq:4[3]*:
‹$(x_1 x_2 x_3[F] \equiv y_1 y_2 y_3[G]) \equiv (\Box x_1 x_2 x_3[F] \equiv \Box y_1 y_2 y_3[G])$›
apply *(rule ≡I; rule →I; rule ≡I; rule →I)*
using *qml:2[axiom-inst, THEN →E] ≡E(1,2) en−eq:2[3]* **by** *blast+*
AOT-theorem *en−eq:4[4]*:
‹$(x_1 x_2 x_3 x_4[F] \equiv y_1 y_2 y_3 y_4[G]) \equiv (\Box x_1 x_2 x_3 x_4[F] \equiv \Box y_1 y_2 y_3 y_4[G])$›
apply *(rule ≡I; rule →I; rule ≡I; rule →I)*
using *qml:2[axiom-inst, THEN →E] ≡E(1,2) en−eq:2[4]* **by** *blast+*

AOT-theorem *en−eq:5[1]*:
‹$\Box(x_1[F] \equiv y_1[G]) \equiv (\Box x_1[F] \equiv \Box y_1[G])$›
apply *(rule ≡I; rule →I)*
using *en−eq:4[1][THEN ≡E(1)] qml:2[axiom-inst, THEN →E]*
apply *blast*
using *sc−eq−box−box:4[THEN →E, THEN →E]*
&I[OF pre−en−eq:1[1][THEN RN], OF pre−en−eq:1[1][THEN RN]]
by *blast*
AOT-theorem *en−eq:5[2]*:
‹$\Box(x_1 x_2[F] \equiv y_1 y_2[G]) \equiv (\Box x_1 x_2[F] \equiv \Box y_1 y_2[G])$›
apply *(rule ≡I; rule →I)*
using *en−eq:4[2][THEN ≡E(1)] qml:2[axiom-inst, THEN →E]*
apply *blast*
using *sc−eq−box−box:4[THEN →E, THEN →E]*
&I[OF pre−en−eq:1[2][THEN RN], OF pre−en−eq:1[2][THEN RN]]
by *blast*
AOT-theorem *en−eq:5[3]*:
‹$\Box(x_1 x_2 x_3[F] \equiv y_1 y_2 y_3[G]) \equiv (\Box x_1 x_2 x_3[F] \equiv \Box y_1 y_2 y_3[G])$›
apply *(rule ≡I; rule →I)*
using *en−eq:4[3][THEN ≡E(1)] qml:2[axiom-inst, THEN →E]*
apply *blast*
using *sc−eq−box−box:4[THEN →E, THEN →E]*
&I[OF pre−en−eq:1[3][THEN RN], OF pre−en−eq:1[3][THEN RN]]
by *blast*
AOT-theorem *en−eq:5[4]*:
‹$\Box(x_1 x_2 x_3 x_4[F] \equiv y_1 y_2 y_3 y_4[G]) \equiv (\Box x_1 x_2 x_3 x_4[F] \equiv \Box y_1 y_2 y_3 y_4[G])$›
apply *(rule ≡I; rule →I)*
using *en−eq:4[4][THEN ≡E(1)] qml:2[axiom-inst, THEN →E]*
apply *blast*
using *sc−eq−box−box:4[THEN →E, THEN →E]*
&I[OF pre−en−eq:1[4][THEN RN], OF pre−en−eq:1[4][THEN RN]]
by *blast*

AOT-theorem *en−eq:6[1]*:
‹$(x_1[F] \equiv y_1[G]) \equiv \Box(x_1[F] \equiv y_1[G])$›
using *en−eq:5[1][symmetric] en−eq:4[1] ≡E(5)* **by** *fast*
AOT-theorem *en−eq:6[2]*:

‹$(x_1x_2[F] \equiv y_1y_2[G]) \equiv \Box(x_1x_2[F] \equiv y_1y_2[G])$›
using *en−eq:5[2][symmetric] en−eq:4[2]* $\equiv E(5)$ **by** *fast*
AOT-theorem *en−eq:6[3]*:
‹$(x_1x_2x_3[F] \equiv y_1y_2y_3[G]) \equiv \Box(x_1x_2x_3[F] \equiv y_1y_2y_3[G])$›
using *en−eq:5[3][symmetric] en−eq:4[3]* $\equiv E(5)$ **by** *fast*
AOT-theorem *en−eq:6[4]*:
‹$(x_1x_2x_3x_4[F] \equiv y_1y_2y_3y_4[G]) \equiv \Box(x_1x_2x_3x_4[F] \equiv y_1y_2y_3y_4[G])$›
using *en−eq:5[4][symmetric] en−eq:4[4]* $\equiv E(5)$ **by** *fast*

AOT-theorem *en−eq:7[1]*: ‹$\neg x_1[F] \equiv \Box\neg x_1[F]$›
using *pre−en−eq:2[1] qml:2[axiom-inst]* $\equiv I$ **by** *blast*
AOT-theorem *en−eq:7[2]*: ‹$\neg x_1x_2[F] \equiv \Box\neg x_1x_2[F]$›
using *pre−en−eq:2[2] qml:2[axiom-inst]* $\equiv I$ **by** *blast*
AOT-theorem *en−eq:7[3]*: ‹$\neg x_1x_2x_3[F] \equiv \Box\neg x_1x_2x_3[F]$›
using *pre−en−eq:2[3] qml:2[axiom-inst]* $\equiv I$ **by** *blast*
AOT-theorem *en−eq:7[4]*: ‹$\neg x_1x_2x_3x_4[F] \equiv \Box\neg x_1x_2x_3x_4[F]$›
using *pre−en−eq:2[4] qml:2[axiom-inst]* $\equiv I$ **by** *blast*

AOT-theorem *en−eq:8[1]*: ‹$\Diamond\neg x_1[F] \equiv \neg x_1[F]$›
using *en−eq:2[1][THEN oth−class−taut:4:b[THEN* $\equiv E(1)$*]]*
KBasic:11 $\equiv E(5)$*[symmetric]* **by** *blast*
AOT-theorem *en−eq:8[2]*: ‹$\Diamond\neg x_1x_2[F] \equiv \neg x_1x_2[F]$›
using *en−eq:2[2][THEN oth−class−taut:4:b[THEN* $\equiv E(1)$*]]*
KBasic:11 $\equiv E(5)$*[symmetric]* **by** *blast*
AOT-theorem *en−eq:8[3]*: ‹$\Diamond\neg x_1x_2x_3[F] \equiv \neg x_1x_2x_3[F]$›
using *en−eq:2[3][THEN oth−class−taut:4:b[THEN* $\equiv E(1)$*]]*
KBasic:11 $\equiv E(5)$*[symmetric]* **by** *blast*
AOT-theorem *en−eq:8[4]*: ‹$\Diamond\neg x_1x_2x_3x_4[F] \equiv \neg x_1x_2x_3x_4[F]$›
using *en−eq:2[4][THEN oth−class−taut:4:b[THEN* $\equiv E(1)$*]]*
KBasic:11 $\equiv E(5)$*[symmetric]* **by** *blast*

AOT-theorem *en−eq:9[1]*: ‹$\Diamond\neg x_1[F] \equiv \Box\neg x_1[F]$›
using *en−eq:7[1] en−eq:8[1]* $\equiv E(5)$ **by** *blast*
AOT-theorem *en−eq:9[2]*: ‹$\Diamond\neg x_1x_2[F] \equiv \Box\neg x_1x_2[F]$›
using *en−eq:7[2] en−eq:8[2]* $\equiv E(5)$ **by** *blast*
AOT-theorem *en−eq:9[3]*: ‹$\Diamond\neg x_1x_2x_3[F] \equiv \Box\neg x_1x_2x_3[F]$›
using *en−eq:7[3] en−eq:8[3]* $\equiv E(5)$ **by** *blast*
AOT-theorem *en−eq:9[4]*: ‹$\Diamond\neg x_1x_2x_3x_4[F] \equiv \Box\neg x_1x_2x_3x_4[F]$›
using *en−eq:7[4] en−eq:8[4]* $\equiv E(5)$ **by** *blast*

AOT-theorem *en−eq:10[1]*: ‹$\mathcal{A}x_1[F] \equiv x_1[F]$›
by (*metis Act−Sub:3 deduction−theorem* $\equiv I$ $\equiv E(1)$
nec−imp−act en−eq:3[1] pre−en−eq:1[1])
AOT-theorem *en−eq:10[2]*: ‹$\mathcal{A}x_1x_2[F] \equiv x_1x_2[F]$›
by (*metis Act−Sub:3 deduction−theorem* $\equiv I$ $\equiv E(1)$
nec−imp−act en−eq:3[2] pre−en−eq:1[2])
AOT-theorem *en−eq:10[3]*: ‹$\mathcal{A}x_1x_2x_3[F] \equiv x_1x_2x_3[F]$›
by (*metis Act−Sub:3 deduction−theorem* $\equiv I$ $\equiv E(1)$
nec−imp−act en−eq:3[3] pre−en−eq:1[3])
AOT-theorem *en−eq:10[4]*: ‹$\mathcal{A}x_1x_2x_3x_4[F] \equiv x_1x_2x_3x_4[F]$›
by (*metis Act−Sub:3 deduction−theorem* $\equiv I$ $\equiv E(1)$
nec−imp−act en−eq:3[4] pre−en−eq:1[4])

AOT-theorem *oa−facts:1*: ‹$O!x \to \Box O!x$›
proof(*rule* $\to I$)
AOT-modally-strict {
AOT-have ‹$[\lambda x\ \Diamond E!x]x \equiv \Diamond E!x$›
by (*rule lambda−predicates:2[axiom-inst, THEN* $\to E$*]*) *cqt:2*
} note ϑ = *this*
AOT-assume ‹$O!x$›
AOT-hence ‹$[\lambda x\ \Diamond E!x]x$›
by (*rule* $=_{df}E(2)$*[OF AOT-ordinary, rotated 1]*) *cqt:2*
AOT-hence ‹$\Diamond E!x$› **using** ϑ*[THEN* $\equiv E(1)$*]* **by** *blast*

AOT-hence ‹$\Box\Diamond E!x$› **using** *qml:3*[*axiom-inst*, *THEN* $\rightarrow$*E*] **by** *blast*
AOT-hence ‹$\Box[\lambda x\ \Diamond E!x]x$›
by (*AOT-subst* ‹$[\lambda x\ \Diamond E!x]x$› ‹$\Diamond E!x$›)
(*auto simp*: ϑ)
AOT-thus ‹$\Box O!x$›
by (*rule* $=_{df}I(2)$[*OF AOT-ordinary*, *rotated 1*]) *cqt:2*
qed

AOT-theorem *oa−facts:2*: ‹$A!x \rightarrow \Box A!x$›
proof(*rule* $\rightarrow$*I*)
AOT-modally-strict {
AOT-have ‹$[\lambda x\ \neg\Diamond E!x]x \equiv \neg\Diamond E!x$›
by (*rule lambda−predicates:2*[*axiom-inst*, *THEN* $\rightarrow$*E*]) *cqt:2*
} note ϑ = *this*
AOT-assume ‹$A!x$›
AOT-hence ‹$[\lambda x\ \neg\Diamond E!x]x$›
by (*rule* $=_{df}E(2)$[*OF AOT-abstract*, *rotated 1*]) *cqt:2*
AOT-hence ‹$\neg\Diamond E!x$› **using** ϑ[*THEN* $\equiv$*E*(*1*)] **by** *blast*
AOT-hence ‹$\Box\neg E!x$› **using** *KBasic2:1*[*THEN* $\equiv$*E*(*2*)] **by** *blast*
AOT-hence ‹$\Box\Box\neg E!x$› **using** *4*[*THEN* $\rightarrow$*E*] **by** *blast*
AOT-hence ‹$\Box\neg\Diamond E!x$›
using *KBasic2:1*
by (*AOT-subst* (*reverse*) ‹$\neg\Diamond E!x$› ‹$\Box\neg E!x$›) *blast*
AOT-hence ‹$\Box[\lambda x\ \neg\Diamond E!x]x$›
by (*AOT-subst* ‹$[\lambda x\ \neg\Diamond E!x]x$› ‹$\neg\Diamond E!x$›)
(*auto simp*: ϑ)
AOT-thus ‹$\Box A!x$›
by (*rule* $=_{df}I(2)$[*OF AOT-abstract*, *rotated 1*]) *cqt:2*[*lambda*]
qed

AOT-theorem *oa−facts:3*: ‹$\Diamond O!x \rightarrow O!x$›
using *oa−facts:1 B$\Diamond$ RM$\Diamond$ Hypothetical Syllogism* **by** *blast*
AOT-theorem *oa−facts:4*: ‹$\Diamond A!x \rightarrow A!x$›
using *oa−facts:2 B$\Diamond$ RM$\Diamond$ Hypothetical Syllogism* **by** *blast*

AOT-theorem *oa−facts:5*: ‹$\Diamond O!x \equiv \Box O!x$›
by (*meson Act−Sub:3 Hypothetical Syllogism* $\equiv$*I nec−imp−act*
oa−facts:1 oa−facts:3)

AOT-theorem *oa−facts:6*: ‹$\Diamond A!x \equiv \Box A!x$›
by (*meson Act−Sub:3 Hypothetical Syllogism* $\equiv$*I nec−imp−act*
oa−facts:2 oa−facts:4)

AOT-theorem *oa−facts:7*: ‹$O!x \equiv \mathcal{A}O!x$›
by (*meson Act−Sub:3 Hypothetical Syllogism* $\equiv$*I nec−imp−act*
oa−facts:1 oa−facts:3)

AOT-theorem *oa−facts:8*: ‹$A!x \equiv \mathcal{A}A!x$›
by (*meson Act−Sub:3 Hypothetical Syllogism* $\equiv$*I nec−imp−act*
oa−facts:2 oa−facts:4)

8.10 The Theory of Relations

AOT-theorem *beta−C−meta*:
‹$[\lambda\mu_1...\mu_n\ \varphi\{\mu_1...\mu_n,\ \nu_1...\nu_n\}]\downarrow \rightarrow$
$([\lambda\mu_1...\mu_n\ \varphi\{\mu_1...\mu_n,\ \nu_1...\nu_n\}]\nu_1...\nu_n \equiv \varphi\{\nu_1...\nu_n,\ \nu_1...\nu_n\})$›
using *lambda−predicates:2*[*axiom-inst*] **by** *blast*

AOT-theorem *beta−C−cor:1*:
‹$(\forall\nu_1...\forall\nu_n([\lambda\mu_1...\mu_n\ \varphi\{\mu_1...\mu_n,\ \nu_1...\nu_n\}]\downarrow)) \rightarrow$
$\forall\nu_1...\forall\nu_n\ ([\lambda\mu_1...\mu_n\ \varphi\{\mu_1...\mu_n,\ \nu_1...\nu_n\}]\nu_1...\nu_n \equiv \varphi\{\nu_1...\nu_n,\ \nu_1...\nu_n\})$›
apply (*rule cqt−basic:14*[**where** *'a='a*, *THEN* $\rightarrow$*E*])
using *beta−C−meta* $\forall$*I* **by** *fast*

```
AOT-theorem beta−C−cor:2:
  ‹[λμ1...μn φ{μ1...μn}]↓ →
   ∀ν1...∀νn ([λμ1...μn φ{μ1...μn}]ν1...νn ≡ φ{ν1...νn})›
  apply (rule →I; rule ∀I)
  using beta−C−meta[THEN →E] by fast


theorem beta−C−cor:3:
  assumes ‹⋀ν1νn. AOT-instance-of-cqt-2 (φ (AOT-term-of-var ν1νn))›
  shows ‹[v ⊨ ∀ν1...∀νn ([λμ1...μn φ{ν1...νn, μ1...μn}]ν1...νn ≡
                         φ{ν1...νn, ν1...νn})]›
  using cqt:2[lambda][axiom-inst, OF assms]
        beta−C−cor:1[THEN →E] ∀I by fast

AOT-theorem betaC:1:a: ‹[λμ1...μn φ{μ1...μn}]κ1...κn ⊢□ φ{κ1...κn}›
proof −
  AOT-modally-strict {
    AOT-assume ‹[λμ1...μn φ{μ1...μn}]κ1...κn›
    moreover AOT-have ‹[λμ1...μn φ{μ1...μn}]↓› and ‹κ1...κn↓›
      using calculation cqt:5:a[axiom-inst, THEN →E] &E by blast+
    ultimately AOT-show ‹φ{κ1...κn}›
      using beta−C−cor:2[THEN →E, THEN ∀E(1), THEN ≡E(1)] by blast
  }
qed

AOT-theorem betaC:1:b: ‹¬φ{κ1...κn} ⊢□ ¬[λμ1...μn φ{μ1...μn}]κ1...κn›
  using betaC:1:a raa−cor:3 by blast

lemmas β→C = betaC:1:a betaC:1:b

AOT-theorem betaC:2:a:
  ‹[λμ1...μn φ{μ1...μn}]↓, κ1...κn↓, φ{κ1...κn} ⊢□
   [λμ1...μn φ{μ1...μn}]κ1...κn›
proof −
  AOT-modally-strict {
    AOT-assume 1: ‹[λμ1...μn φ{μ1...μn}]↓›
           and 2: ‹κ1...κn↓›
           and 3: ‹φ{κ1...κn}›
    AOT-hence ‹[λμ1...μn φ{μ1...μn}]κ1...κn›
      using beta−C−cor:2[THEN →E, OF 1, THEN ∀E(1), THEN ≡E(2)]
      by blast
  }
  AOT-thus ‹[λμ1...μn φ{μ1...μn}]↓, κ1...κn↓, φ{κ1...κn} ⊢□
            [λμ1...μn φ{μ1...μn}]κ1...κn›
    by blast
qed

AOT-theorem betaC:2:b:
  ‹[λμ1...μn φ{μ1...μn}]↓, κ1...κn↓, ¬[λμ1...μn φ{μ1...μn}]κ1...κn ⊢□
   ¬φ{κ1...κn}›
  using betaC:2:a raa−cor:3 by blast

lemmas β←C = betaC:2:a betaC:2:b

AOT-theorem eta−conversion−lemma1:1: ‹Π↓ → [λx1...xn [Π]x1...xn] = Π›
  using lambda−predicates:3[axiom-inst] ∀I ∀E(1) →I by fast


AOT-theorem eta−conversion−lemma1:2: ‹Π↓ → [λν1...νn [Π]ν1...νn] = Π›
  using eta−conversion−lemma1:1.
```

Note: not explicitly part of PLM.

```
AOT-theorem id-sym:
  assumes ‹τ = τ'›
  shows ‹τ' = τ›
  using rule=E[where φ=λ τ' . «τ' = τ», rotated 1, OF assms]
        =I(1)[OF t=t-proper:1[THEN →E, OF assms]] by auto
declare id-sym[sym]
```

Note: not explicitly part of PLM.

```
AOT-theorem id-trans:
  assumes ‹τ = τ'› and ‹τ' = τ''›
  shows ‹τ = τ''›
  using rule=E assms by blast
declare id-trans[trans]

method ηC for Π :: ‹<'a::{AOT-Term-id-2,AOT-κs}>› =
  (match conclusion in [v ⊨ τ{Π} = τ'{Π}] for v τ τ' ⇒ ‹
   rule rule=E[rotated 1, OF eta-conversion-lemma1:2
    [THEN →E, of v «[Π]», symmetric]]›)

AOT-theorem sub-des-lam:1:
  ‹[λz1...zn  χ{z1...zn, ιx φ{x}}]↓ & ιx φ{x} = ιx ψ{x} →
   [λz1...zn χ{z1...zn, ιx φ{x}}] = [λz1...zn χ{z1...zn, ιx ψ{x}}]›
proof(rule →I)
  AOT-assume A: ‹[λz1...zn  χ{z1...zn, ιx φ{x}}]↓ & ιx φ{x} = ιx ψ{x}›
  AOT-show ‹[λz1...zn χ{z1...zn, ιx φ{x}}] = [λz1...zn χ{z1...zn, ιx ψ{x}}]›
    using rule=E[where φ=λ τ . «[λz1...zn χ{z1...zn, ιx φ{x}}] =
                                [λz1...zn χ{z1...zn, τ}]»,
               OF =I(1)[OF A[THEN &E(1)]], OF A[THEN &E(2)]]
    by blast
qed

AOT-theorem sub-des-lam:2:
  ‹ιx φ{x} = ιx ψ{x} → χ{ιx φ{x}} = χ{ιx ψ{x}}› for χ :: ‹κ ⇒ o›
  using rule=E[where φ=λ τ . «χ{ιx φ{x}} = χ{τ}»,
                 OF =I(1)[OF log-prop-prop:2]] →I by blast

AOT-theorem prop-equiv: ‹F = G ≡ ∀ x (x[F] ≡ x[G])›
proof(rule ≡I; rule →I)
  AOT-assume ‹F = G›
  AOT-thus ‹∀ x (x[F] ≡ x[G])›
    by (rule rule=E[rotated]) (fact oth-class-taut:3:a[THEN GEN])
next
  AOT-assume ‹∀ x (x[F] ≡ x[G])›
  AOT-hence ‹x[F] ≡ x[G]› for x
    using ∀ E by blast
  AOT-hence ‹□(x[F] ≡ x[G])› for x
    using en-eq:6[1][THEN ≡E(1)] by blast
  AOT-hence ‹∀ x □(x[F] ≡ x[G])›
    by (rule GEN)
  AOT-hence ‹□∀ x (x[F] ≡ x[G])›
    using BF[THEN →E] by fast
  AOT-thus F = G
    using p-identity-thm2:1[THEN ≡E(2)] by blast
qed

AOT-theorem relations:1:
  assumes ‹INSTANCE-OF-CQT-2(φ)›
  shows ‹∃ F □∀ x1...∀ xn ([F]x1...xn ≡ φ{x1...xn})›
  apply (rule ∃ I(1)[where τ=«[λx1...xn φ{x1...xn}]»])
  using cqt:2[lambda][OF assms, axiom-inst]
        beta-C-cor:2[THEN →E, THEN RN] by blast+

AOT-theorem relations:2:
```

```
  assumes ‹INSTANCE-OF-CQT-2(φ)›
  shows ‹∃ F □∀ x ([F]x ≡ φ{x})›
  using relations:1 assms by blast

AOT-theorem block−paradox:1: ‹¬[λx ∃ G (x[G] & ¬[G]x)]↓›
proof(rule raa−cor:2)
  let ?K=«[λx ∃ G (x[G] & ¬[G]x)]»
  AOT-assume A: ‹«?K»↓›
  AOT-have ‹∃ x (A!x & ∀ F (x[F] ≡ F = «?K»))›
    using A−objects[axiom-inst] by fast
  then AOT-obtain a where ξ: ‹A!a & ∀ F (a[F] ≡ F = «?K»)›
    using ∃ E[rotated] by blast
  AOT-show ‹p & ¬p› for p
  proof (rule ∨E(1)[OF exc−mid]; rule →I)
    AOT-assume B: ‹[«?K»]a›
    AOT-hence ‹∃ G (a[G] & ¬[G]a)›
      using β→C A by blast
    then AOT-obtain P where ‹a[P] & ¬[P]a›
      using ∃ E[rotated] by blast
    moreover AOT-have ‹P = [«?K»]›
      using ξ[THEN &E(2), THEN ∀ E(2), THEN ≡E(1)]
            calculation[THEN &E(1)] by blast
    ultimately AOT-have ‹¬[«?K»]a›
      using rule=E &E(2) by fast
    AOT-thus ‹p & ¬p›
      using B RAA by blast
  next
    AOT-assume B: ‹¬[«?K»]a›
    AOT-hence ‹¬∃ G (a[G] & ¬[G]a)›
      using β←C cqt:2[const-var][of a, axiom-inst] A by blast
    AOT-hence C: ‹∀ G ¬(a[G] & ¬[G]a)›
      using cqt−further:4[THEN →E] by blast
    AOT-have ‹∀ G (a[G] → [G]a)›
      by (AOT-subst ‹a[G] → [G]a› ‹¬(a[G] & ¬[G]a)› for: G)
         (auto simp: oth−class−taut:1:a C)
    AOT-hence ‹a[«?K»] → [«?K»]a›
      using ∀ E A by blast
    moreover AOT-have ‹a[«?K»]›
      using ξ[THEN &E(2), THEN ∀ E(1), OF A, THEN ≡E(2)]
      using =I(1)[OF A] by blast
    ultimately AOT-show ‹p & ¬p›
      using B →E RAA by blast
  qed
qed

AOT-theorem block−paradox:2: ‹¬∃ F ∀ x([F]x ≡ ∃ G(x[G] & ¬[G]x))›
proof(rule RAA(2))
  AOT-assume ‹∃ F ∀ x ([F]x ≡ ∃ G (x[G] & ¬[G]x))›
  then AOT-obtain F where F-prop: ‹∀ x ([F]x ≡ ∃ G (x[G] & ¬[G]x))›
    using ∃ E[rotated] by blast
  AOT-have ‹∃ x (A!x & ∀ G (x[G] ≡ G = F))›
    using A−objects[axiom-inst] by fast
  then AOT-obtain a where ξ: ‹A!a & ∀ G (a[G] ≡ G = F)›
    using ∃ E[rotated] by blast
  AOT-show ‹¬∃ F ∀ x([F]x ≡ ∃ G(x[G] & ¬[G]x))›
  proof (rule ∨E(1)[OF exc−mid]; rule →I)
    AOT-assume B: ‹[F]a›
    AOT-hence ‹∃ G (a[G] & ¬[G]a)›
      using F-prop[THEN ∀ E(2), THEN ≡E(1)] by blast
    then AOT-obtain P where ‹a[P] & ¬[P]a›
      using ∃ E[rotated] by blast
    moreover AOT-have ‹P = F›
      using ξ[THEN &E(2), THEN ∀ E(2), THEN ≡E(1)]
```

```
        calculation[THEN &E(1)] by blast
    ultimately AOT-have ‹¬[F]a›
      using rule=E &E(2) by fast
    AOT-thus ‹¬∃ F ∀ x([F]x ≡ ∃ G(x[G] & ¬[G]x))›
      using B RAA by blast
  next
    AOT-assume B: ‹¬[F]a›
    AOT-hence ‹¬∃ G (a[G] & ¬[G]a)›
      using oth−class−taut:4:b[THEN ≡E(1),
            OF F-prop[THEN ∀ E(2)[of - - a]], THEN ≡E(1)]
      by simp
    AOT-hence C: ‹∀ G ¬(a[G] & ¬[G]a)›
      using cqt−further:4[THEN →E] by blast
    AOT-have ‹∀ G (a[G] → [G]a)›
      by (AOT-subst ‹a[G] → [G]a› ‹¬(a[G] & ¬[G]a)› for: G)
        (auto simp: oth−class−taut:1:a C)
    AOT-hence ‹a[F] → [F]a›
      using ∀ E by blast
    moreover AOT-have ‹a[F]›
      using ξ[THEN &E(2), THEN ∀ E(2), of F, THEN ≡E(2)]
      using =I(2) by blast
    ultimately AOT-show ‹¬∃ F ∀ x([F]x ≡ ∃ G(x[G] & ¬[G]x))›
      using B →E RAA by blast
  qed
qed(simp)

AOT-theorem block−paradox:3: ‹¬∀ y [λz z = y]↓›
proof(rule RAA(2))
  AOT-assume ϑ: ‹∀ y [λz z = y]↓›
  AOT-have ‹∃ x (A!x & ∀ F (x[F] ≡ ∃ y(F = [λz z = y] & ¬y[F])))›
    using A−objects[axiom-inst] by force
  then AOT-obtain a where
    a-prop: ‹A!a & ∀ F (a[F] ≡ ∃ y (F = [λz z = y] & ¬y[F]))›
    using ∃ E[rotated] by blast
  AOT-have ζ: ‹a[λz z = a] ≡ ∃ y ([λz z = a] = [λz z = y] & ¬y[λz z = a])›
    using ϑ[THEN ∀ E(2)] a-prop[THEN &E(2), THEN ∀ E(1)] by blast
  AOT-show ‹¬∀ y [λz z = y]↓›
  proof (rule ∨E(1)[OF exc−mid]; rule →I)
    AOT-assume A: ‹a[λz z = a]›
    AOT-hence ‹∃ y ([λz z = a] = [λz z = y] & ¬y[λz z = a])›
      using ζ[THEN ≡E(1)] by blast
    then AOT-obtain b where b-prop: ‹[λz z = a] = [λz z = b] & ¬b[λz z = a]›
      using ∃ E[rotated] by blast
    moreover AOT-have ‹a = a› by (rule =I)
    moreover AOT-have ‹[λz z = a]↓› using ϑ ∀ E by blast
    moreover AOT-have ‹a↓› using cqt:2[const-var][axiom-inst] .
    ultimately AOT-have ‹[λz z = a]a› using β←C by blast
    AOT-hence ‹[λz z = b]a› using rule=E b-prop[THEN &E(1)] by fast
    AOT-hence ‹a = b› using β→C by blast
    AOT-hence ‹b[λz z = a]› using A rule=E by fast
    AOT-thus ‹¬∀ y [λz z = y]↓› using b-prop[THEN &E(2)] RAA by blast
  next
    AOT-assume A: ‹¬a[λz z = a]›
    AOT-hence ‹¬∃ y ([λz z = a] = [λz z = y] & ¬y[λz z = a])›
      using ζ oth−class−taut:4:b[THEN ≡E(1), THEN ≡E(1)] by blast
    AOT-hence ‹∀ y ¬([λz z = a] = [λz z = y] & ¬y[λz z = a])›
      using cqt−further:4[THEN →E] by blast
    AOT-hence ‹¬([λz z = a] = [λz z = a] & ¬a[λz z = a])›
      using ∀ E by blast
    AOT-hence ‹[λz z = a] = [λz z = a] → a[λz z = a]›
      by (metis &I deduction−theorem raa−cor:4)
    AOT-hence ‹a[λz z = a]› using =I(1) ϑ[THEN ∀ E(2)] →E by blast
    AOT-thus ‹¬∀ y [λz z = y]↓› using A RAA by blast
```

qed
qed(*simp*)

AOT-theorem *block−paradox:4*: ‹$\neg\forall y \exists F \forall x([F]x \equiv x = y)$›
proof(*rule RAA(2)*)
AOT-assume ϑ: ‹$\forall y \exists F \forall x([F]x \equiv x = y)$›
AOT-have ‹$\exists x\ (A!x\ \&\ \forall F\ (x[F] \equiv \exists z\ (\forall y([F]y \equiv y = z)\ \&\ \neg z[F])))$›
using *A−objects[axiom-inst]* **by** *force*
then AOT-obtain *a* **where**
a-prop: ‹$A!a\ \&\ \forall F\ (a[F] \equiv \exists z\ (\forall y([F]y \equiv y = z)\ \&\ \neg z[F]))$›
using $\exists E$[*rotated*] **by** *blast*
AOT-obtain *F* **where** *F-prop*: ‹$\forall x\ ([F]x \equiv x = a)$›
using ϑ[*THEN* $\forall E(2)$] $\exists E$[*rotated*] **by** *blast*
AOT-have ζ: ‹$a[F] \equiv \exists z\ (\forall y\ ([F]y \equiv y = z)\ \&\ \neg z[F])$›
using *a-prop*[*THEN* $\&E(2)$, *THEN* $\forall E(2)$] **by** *blast*
AOT-show ‹$\neg\forall y \exists F \forall x([F]x \equiv x = y)$›
proof (*rule* $\vee E(1)$[*OF exc−mid*]; *rule* $\rightarrow I$)
AOT-assume *A*: ‹$a[F]$›
AOT-hence ‹$\exists z\ (\forall y\ ([F]y \equiv y = z)\ \&\ \neg z[F])$›
using ζ[*THEN* $\equiv E(1)$] **by** *blast*
then AOT-obtain *b* **where** *b-prop*: ‹$\forall y\ ([F]y \equiv y = b)\ \&\ \neg b[F]$›
using $\exists E$[*rotated*] **by** *blast*
moreover AOT-have ‹$[F]a$›
using *F-prop*[*THEN* $\forall E(2)$, *THEN* $\equiv E(2)$] $=I(2)$ **by** *blast*
ultimately AOT-have ‹$a = b$›
using $\forall E(2)$ $\equiv E(1)$ $\&E$ **by** *fast*
AOT-hence ‹$a = b$›
using $\beta{\rightarrow}C$ **by** *blast*
AOT-hence ‹$b[F]$›
using *A rule=E* **by** *fast*
AOT-thus ‹$\neg\forall y \exists F \forall x([F]x \equiv x = y)$›
using *b-prop*[*THEN* $\&E(2)$] *RAA* **by** *blast*
next
AOT-assume *A*: ‹$\neg a[F]$›
AOT-hence ‹$\neg\exists z\ (\forall y\ ([F]y \equiv y = z)\ \&\ \neg z[F])$›
using ζ *oth−class−taut:4:b*[*THEN* $\equiv E(1)$, *THEN* $\equiv E(1)$] **by** *blast*
AOT-hence ‹$\forall z\ \neg(\forall y\ ([F]y \equiv y = z)\ \&\ \neg z[F])$›
using *cqt−further:4*[*THEN* $\rightarrow E$] **by** *blast*
AOT-hence ‹$\neg(\forall y\ ([F]y \equiv y = a)\ \&\ \neg a[F])$›
using $\forall E$ **by** *blast*
AOT-hence ‹$\forall y\ ([F]y \equiv y = a) \rightarrow a[F]$›
by (*metis* $\&I$ *deduction−theorem raa−cor:4*)
AOT-hence ‹$a[F]$› **using** *F-prop* $\rightarrow E$ **by** *blast*
AOT-thus ‹$\neg\forall y \exists F \forall x([F]x \equiv x = y)$›
using *A RAA* **by** *blast*
qed
qed(*simp*)

AOT-theorem *block−paradox:5*: ‹$\neg\exists F\forall x\forall y([F]xy \equiv y = x)$›
proof(*rule raa−cor:2*)
AOT-assume ‹$\exists F\forall x\forall y([F]xy \equiv y = x)$›
then AOT-obtain *F* **where** *F-prop*: ‹$\forall x\forall y([F]xy \equiv y = x)$›
using $\exists E$[*rotated*] **by** *blast*
{
fix *x*
AOT-have *1*: ‹$\forall y([F]xy \equiv y = x)$›
using *F-prop* $\forall E$ **by** *blast*
AOT-have *2*: ‹$[\lambda z\ [F]xz]\downarrow$› **by** *cqt:2*
moreover AOT-have ‹$\forall y([\lambda z\ [F]xz]y \equiv y = x)$›
proof(*rule* $\forall I$)
fix *y*
AOT-have ‹$[\lambda z\ [F]xz]y \equiv [F]xy$›
using *beta−C−meta*[*THEN* $\rightarrow E$] *2* **by** *fast*

```
    also AOT-have ‹... ≡ y = x›
      using 1 ∀E by fast
    finally AOT-show ‹[λz [F]xz]y ≡ y = x›.
  qed
  ultimately AOT-have ‹∃F∀y([F]y ≡ y = x)›
    using ∃I by fast
}
AOT-hence ‹∀x∃F∀y([F]y ≡ y = x)›
  by (rule GEN)
AOT-thus ‹∀x∃F∀y([F]y ≡ y = x) & ¬∀x∃F∀y([F]y ≡ y = x)›
  using &I block−paradox:4 by blast
qed

AOT-act-theorem block−paradox2:1:
‹∀x [G]x → ¬[λx [G]ιy (y = x & ∃H (x[H] & ¬[H]x))]↓›
proof(rule →I; rule raa−cor:2)
  AOT-assume antecedant: ‹∀x [G]x›
  AOT-have Lemma: ‹∀x ([G]ιy(y = x & ∃H (x[H] & ¬[H]x)) ≡ ∃H (x[H] & ¬[H]x))›
  proof(rule GEN)
    fix x
    AOT-have A: ‹[G]ιy (y = x & ∃H (x[H] & ¬[H]x)) ≡
                ∃!y (y = x & ∃H (x[H] & ¬[H]x))›
    proof(rule ≡I; rule →I)
      AOT-assume ‹[G]ιy (y = x & ∃H (x[H] & ¬[H]x))›
      AOT-hence ‹ιy (y = x & ∃H (x[H] & ¬[H]x))↓›
        using cqt:5:a[axiom-inst, THEN →E, THEN &E(2)] by blast
      AOT-thus ‹∃!y (y = x & ∃H (x[H] & ¬[H]x))›
        using !−exists:1[THEN ≡E(1)] by blast
    next
      AOT-assume A: ‹∃!y (y = x & ∃H (x[H] & ¬[H]x))›
      AOT-obtain a where a-1: ‹a = x & ∃H (x[H] & ¬[H]x)›
                and a-2: ‹∀z (z = x & ∃H (x[H] & ¬[H]x) → z = a)›
        using uniqueness:1[THEN ≡df E, OF A] &E ∃E[rotated] by blast
      AOT-have a-3: ‹[G]a›
        using antecedant ∀E by blast
      AOT-show ‹[G]ιy (y = x & ∃H (x[H] & ¬[H]x))›
        apply (rule russell−axiom[exe,1].russell−axiom[THEN ≡E(2)])
        apply (rule ∃I(2))
        using a-1 a-2 a-3 &I by blast
    qed
    also AOT-have B: ‹... ≡ ∃H (x[H] & ¬[H]x)›
    proof (rule ≡I; rule →I)
      AOT-assume A: ‹∃!y (y = x & ∃H (x[H] & ¬[H]x))›
      AOT-obtain a where ‹a = x & ∃H (x[H] & ¬[H]x)›
        using uniqueness:1[THEN ≡df E, OF A] &E ∃E[rotated] by blast
      AOT-thus ‹∃H (x[H] & ¬[H]x)› using &E by blast
    next
      AOT-assume ‹∃H (x[H] & ¬[H]x)›
      AOT-hence ‹x = x & ∃H (x[H] & ¬[H]x)›
        using id−eq:1 &I by blast
      moreover AOT-have ‹∀z (z = x & ∃H (x[H] & ¬[H]x) → z = x)›
        by (simp add: Conjunction Simplification(1) universal−cor)
      ultimately AOT-show ‹∃!y (y = x & ∃H (x[H] & ¬[H]x))›
        using uniqueness:1[THEN ≡df I] &I ∃I(2) by fast
    qed
    finally AOT-show ‹([G]ιy(y = x & ∃H (x[H] & ¬[H]x)) ≡ ∃H (x[H] & ¬[H]x))› .
  qed

  AOT-assume A: ‹[λx [G]ιy (y = x & ∃H (x[H] & ¬[H]x))]↓›
  AOT-have ϑ: ‹∀x ([λx [G]ιy (y = x & ∃H (x[H] & ¬[H]x))]x ≡
                [G]ιy(y = x & ∃H (x[H] & ¬[H]x)))›
    using beta−C−meta[THEN →E, OF A] ∀I by fast
  AOT-have ‹∀x ([λx [G]ιy (y = x & ∃H (x[H] & ¬[H]x))]x ≡ ∃H (x[H] & ¬[H]x))›
```

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    using ϑ Lemma cqt−basic:10[THEN →E] &I by fast
  AOT-hence ‹∃ F ∀ x ([F]x ≡ ∃ H (x[H] & ¬[H]x))›
    using ∃ I(1) A by fast
  AOT-thus ‹(∃ F ∀ x ([F]x ≡ ∃ H (x[H] & ¬[H]x))) &
           (¬∃ F ∀ x ([F]x ≡ ∃ H (x[H] & ¬[H]x)))›
    using block−paradox:2 &I by blast
qed
```

Note: Strengthens the above to a modally-strict theorem. Not explicitly part of PLM.

```
AOT-theorem block−paradox2:1[strict]:
  ‹∀ x 𝓐[G]x → ¬[λx [G]ιy (y = x & ∃ H (x[H] & ¬[H]x))]↓›
proof(rule →I; rule raa−cor:2)
  AOT-assume antecedant: ‹∀ x 𝓐[G]x›
  AOT-have Lemma: ‹𝓐∀ x ([G]ιy(y = x & ∃ H (x[H] & ¬[H]x)) ≡ ∃ H (x[H] & ¬[H]x))›
  proof(safe intro!: GEN Act−Basic:5[THEN ≡E(2)]
                    logic−actual−nec:3[axiom-inst, THEN ≡E(2)])
    fix x
    AOT-have A: ‹𝓐[G]ιy (y = x & ∃ H (x[H] & ¬[H]x)) ≡
                ∃ !y 𝓐(y = x & ∃ H (x[H] & ¬[H]x))›
    proof(rule ≡I; rule →I)
      AOT-assume ‹𝓐[G]ιy (y = x & ∃ H (x[H] & ¬[H]x))›
      moreover AOT-have ‹□([G]ιy (y = x & ∃ H (x[H] & ¬[H]x)) →
                           □ιy (y = x & ∃ H (x[H] & ¬[H]x))↓)›
      proof(rule RN; rule →I)
        AOT-modally-strict {
          AOT-assume ‹[G]ιy (y = x & ∃ H (x[H] & ¬[H]x))›
          AOT-hence ‹ιy (y = x & ∃ H (x[H] & ¬[H]x))↓›
            using cqt:5:a[axiom-inst, THEN →E, THEN &E(2)] by blast
          AOT-thus ‹□ιy (y = x & ∃ H (x[H] & ¬[H]x))↓›
            using exist−nec[THEN →E] by blast
        }
      qed
      ultimately AOT-have ‹𝓐□ιy (y = x & ∃ H (x[H] & ¬[H]x))↓›
        using act−cond[THEN →E, THEN →E] nec−imp−act[THEN →E] by blast
      AOT-hence ‹ιy (y = x & ∃ H (x[H] & ¬[H]x))↓›
        using Act−Sub:3 B◇ vdash−properties:10 by blast
      AOT-thus ‹∃ !y 𝓐(y = x & ∃ H (x[H] & ¬[H]x))›
        using actual−desc:1[THEN ≡E(1)] by blast
    next
      AOT-assume A: ‹∃ !y 𝓐(y = x & ∃ H (x[H] & ¬[H]x))›
      AOT-obtain a where a-1: ‹𝓐(a = x & ∃ H (x[H] & ¬[H]x))›
                 and a-2: ‹∀ z (𝓐(z = x & ∃ H (x[H] & ¬[H]x)) → z = a)›
        using uniqueness:1[THEN ≡df E, OF A] &E ∃ E[rotated] by blast
      AOT-have a-3: ‹𝓐[G]a›
        using antecedant ∀ E by blast
      moreover AOT-have ‹a = ιy(y = x & ∃ H (x[H] & ¬[H]x))›
        using nec−hintikka−scheme[THEN ≡E(2), OF &I] a-1 a-2 by auto
      ultimately AOT-show ‹𝓐[G]ιy (y = x & ∃ H (x[H] & ¬[H]x))›
        using rule=E by fast
    qed
    also AOT-have B: ‹... ≡ 𝓐∃ H (x[H] & ¬[H]x)›
    proof (rule ≡I; rule →I)
      AOT-assume A: ‹∃ !y 𝓐(y = x & ∃ H (x[H] & ¬[H]x))›
      AOT-obtain a where ‹𝓐(a = x & ∃ H (x[H] & ¬[H]x))›
        using uniqueness:1[THEN ≡df E, OF A] &E ∃ E[rotated] by blast
      AOT-thus ‹𝓐∃ H (x[H] & ¬[H]x)›
        using Act−Basic:2[THEN ≡E(1), THEN &E(2)] by blast
    next
      AOT-assume ‹𝓐∃ H (x[H] & ¬[H]x)›
      AOT-hence ‹𝓐x = x & 𝓐∃ H (x[H] & ¬[H]x)›
        using id−eq:1 &I RA[2] by blast
      AOT-hence ‹𝓐(x = x & ∃ H (x[H] & ¬[H]x))›
        using act−conj−act:3 Act−Basic:2 ≡E by blast
```

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      moreover AOT-have ‹∀ z (𝓐(z = x & ∃ H (x[H] & ¬[H]x)) → z = x)›
      proof(safe intro!: GEN →I)
        fix z
        AOT-assume ‹𝓐(z = x & ∃ H (x[H] & ¬[H]x))›
        AOT-hence ‹𝓐(z = x)›
          using Act−Basic:2[THEN ≡E(1), THEN &E(1)] by blast
        AOT-thus ‹z = x›
           by (metis id−act:1 intro−elim:3:b)
      qed
      ultimately AOT-show ‹∃!y 𝓐(y = x & ∃ H (x[H] & ¬[H]x))›
        using uniqueness:1[THEN ≡df I] &I ∃ I(2) by fast
    qed
    finally AOT-show ‹(𝓐[G]ιy(y = x & ∃ H (x[H] & ¬[H]x)) ≡ 𝓐∃ H (x[H] & ¬[H]x))›.
  qed

  AOT-assume A: ‹[λx [G]ιy (y = x & ∃ H (x[H] & ¬[H]x))]↓›
  AOT-hence ‹𝓐[λx [G]ιy (y = x & ∃ H (x[H] & ¬[H]x))]↓›
    using exist−nec →E nec−imp−act[THEN →E] by blast
  AOT-hence ‹𝓐([λx [G]ιy (y = x & ∃ H (x[H] & ¬[H]x))]↓ &
               ∀ x ([G]ιy(y = x & ∃ H (x[H] & ¬[H]x)) ≡ ∃ H (x[H] & ¬[H]x)))›
    using Lemma Act−Basic:2[THEN ≡E(2)] &I by blast
  moreover AOT-have ‹𝓐([λx [G]ιy (y = x & ∃ H (x[H] & ¬[H]x))]↓ &
               ∀ x ([G]ιy(y = x & ∃ H (x[H] & ¬[H]x)) ≡ ∃ H (x[H] & ¬[H]x)))
      → 𝓐∃ p (p & ¬p)›
  proof (rule logic−actual−nec:2[axiom-inst, THEN ≡E(1)];
         rule RA[2]; rule →I)
    AOT-modally-strict {
      AOT-assume 0: ‹[λx [G]ιy (y = x & ∃ H (x[H] & ¬[H]x))]↓ &
               ∀ x ([G]ιy(y = x & ∃ H (x[H] & ¬[H]x)) ≡ ∃ H (x[H] & ¬[H]x))›
      AOT-have ‹∃ F ∀ x ([F]x ≡ ∃ G (x[G] & ¬[G]x))›
      proof(rule ∃ I(1))
        AOT-show ‹∀ x ([λx [G]ιy (y = x & ∃ H (x[H] & ¬[H]x))]x ≡ ∃ H (x[H] & ¬[H]x))›
        proof(safe intro!: GEN ≡I →I β←C dest!: β→C)
          fix x
          AOT-assume ‹[G]ιy(y = x & ∃ H (x[H] & ¬[H]x))›
          AOT-thus ‹∃ H (x[H] & ¬[H]x)›
            using 0 &E ∀ E(2) ≡E(1) by blast
        next
          fix x
          AOT-assume ‹∃ H (x[H] & ¬[H]x)›
          AOT-thus ‹[G]ιy(y = x & ∃ H (x[H] & ¬[H]x))›
            using 0 &E ∀ E(2) ≡E(2) by blast
        qed(auto intro!: 0[THEN &E(1)] cqt:2)
      next
        AOT-show ‹[λx [G]ιy (y = x & ∃ H (x[H] & ¬[H]x))]↓›
          using 0 &E(1) by blast
      qed
      AOT-thus ‹∃ p (p & ¬p)›
        using block−paradox:2 reductio−aa:1 by blast
    }
  qed
  ultimately AOT-have ‹𝓐∃ p (p & ¬p)›
    using →E by blast
  AOT-hence ‹∃ p 𝓐(p & ¬p)›
    by (metis Act−Basic:10 intro−elim:3:a)
  then AOT-obtain p where ‹𝓐(p & ¬p)›
    using ∃ E[rotated] by blast
  moreover AOT-have ‹¬𝓐(p & ¬p)›
    using non−contradiction[THEN RA[2]]
    by (meson Act−Sub:1 ¬¬I intro−elim:3:d)
  ultimately AOT-show ‹p & ¬p› for p
    by (metis raa−cor:3)
qed
```

AOT-act-theorem *block−paradox2*:*2*:
‹$\exists\, G\ \neg[\lambda x\ [G]\iota y\ (y = x\ \&\ \exists\, H\ (x[H]\ \&\ \neg[H]x))]\downarrow$›
proof(*rule* $\exists\, I(1)$)
AOT-have *0*: ‹$[\lambda x\ \forall\, p\ (p \rightarrow p)]\downarrow$›
by *cqt*:*2*[*lambda*]
moreover AOT-have ‹$\forall\, x\ [\lambda x\ \forall\, p\ (p \rightarrow p)]x$›
apply (*rule GEN*)
apply (*rule beta−C−cor*:*2*[*THEN* →*E*, *OF 0*, *THEN* ∀*E*(*2*), *THEN* ≡*E*(*2*)])
using *if−p−then−p GEN* **by** *fast*
moreover AOT-have ‹$\forall\, G\ (\forall\, x\ [G]x \rightarrow \neg[\lambda x\ [G]\iota y\ (y = x\ \&\ \exists\, H\ (x[H]\ \&\ \neg[H]x))]\downarrow)$›
using *block−paradox2*:*1* ∀ *I* **by** *fast*
ultimately AOT-show ‹$\neg[\lambda x\ [\lambda x\ \forall\, p\ (p \rightarrow p)]\iota y\ (y = x\ \&\ \exists\, H\ (x[H]\ \&\ \neg[H]x))]\downarrow$›
using ∀ *E*(*1*) →*E* **by** *blast*
qed(*cqt*:*2*[*lambda*])

AOT-theorem *propositions*: ‹$\exists\, p\ \Box(p \equiv \varphi)$›
proof(*rule* $\exists\, I(1)$)
AOT-show ‹$\Box(\varphi \equiv \varphi)$›
by (*simp add*: *RN oth−class−taut*:*3*:*a*)
next
AOT-show ‹$\varphi\downarrow$›
by (*simp add*: *log−prop−prop*:*2*)
qed

AOT-theorem *pos−not−equiv−ne*:*1*:
‹$(\Diamond\neg\forall\, x_1 ...\forall\, x_n\ ([F]x_1...x_n \equiv [G]x_1...x_n)) \rightarrow F \neq G$›
proof (*rule* →*I*)
AOT-assume ‹$\Diamond\neg\forall\, x_1...\forall\, x_n\ ([F]x_1...x_n \equiv [G]x_1...x_n)$›
AOT-hence ‹$\neg\Box\forall\, x_1...\forall\, x_n\ ([F]x_1...x_n \equiv [G]x_1...x_n)$›
using *KBasic*:*11*[*THEN* ≡*E*(*2*)] **by** *blast*
AOT-hence ‹$\neg(F = G)$›
using *id−rel−nec−equiv*:*1 modus−tollens*:*1* **by** *blast*
AOT-thus ‹$F \neq G$›
using *=−infix*[*THEN* $\equiv_{df}$*I*] **by** *blast*
qed

AOT-theorem *pos−not−equiv−ne*:*2*: ‹$(\Diamond\neg(\varphi\{F\} \equiv \varphi\{G\})) \rightarrow F \neq G$›
proof (*rule* →*I*)
AOT-modally-strict {
AOT-have ‹$\neg(\varphi\{F\} \equiv \varphi\{G\}) \rightarrow \neg(F = G)$›
proof (*rule* →*I*; *rule raa−cor*:*2*)
AOT-assume *1*: ‹$F = G$›
AOT-hence ‹$\varphi\{F\} \rightarrow \varphi\{G\}$›
using *l−identity*[*axiom-inst*, *THEN* →*E*] **by** *blast*
moreover {
AOT-have ‹$G = F$›
using *1 id-sym* **by** *blast*
AOT-hence ‹$\varphi\{G\} \rightarrow \varphi\{F\}$›
using *l−identity*[*axiom-inst*, *THEN* →*E*] **by** *blast*
}
ultimately AOT-have ‹$\varphi\{F\} \equiv \varphi\{G\}$›
using ≡*I* **by** *blast*
moreover AOT-assume ‹$\neg(\varphi\{F\} \equiv \varphi\{G\})$›
ultimately AOT-show ‹$(\varphi\{F\} \equiv \varphi\{G\})\ \&\ \neg(\varphi\{F\} \equiv \varphi\{G\})$›
using &*I* **by** *blast*
qed
}
AOT-hence ‹$\Diamond\neg(\varphi\{F\} \equiv \varphi\{G\}) \rightarrow \Diamond\neg(F = G)$›
using *RM*:*2*[*prem*] **by** *blast*
moreover AOT-assume ‹$\Diamond\neg(\varphi\{F\} \equiv \varphi\{G\})$›
ultimately AOT-have *0*: ‹$\Diamond\neg(F = G)$› **using** →*E* **by** *blast*
AOT-have ‹$\Diamond(F \neq G)$›

by (*AOT-subst* ‹$F \neq G$› ‹$\neg(F = G)$›)
(*auto simp*: *=−infix* *≡Df 0*)
AOT-thus ‹$F \neq G$›
using *id−nec2*:*3*[*THEN* →*E*] **by** *blast*
qed

AOT-theorem *pos−not−equiv−ne*:*2*[*zero*]: ‹$(\Diamond\neg(\varphi\{p\} \equiv \varphi\{q\})) \rightarrow p \neq q$›
proof (*rule* →*I*)
AOT-modally-strict {
AOT-have ‹$\neg(\varphi\{p\} \equiv \varphi\{q\}) \rightarrow \neg(p = q)$›
proof (*rule* →*I*; *rule raa−cor*:*2*)
AOT-assume *1*: ‹$p = q$›
AOT-hence ‹$\varphi\{p\} \rightarrow \varphi\{q\}$›
using *l−identity*[*axiom-inst*, *THEN* →*E*] **by** *blast*
moreover {
AOT-have ‹$q = p$›
using *1 id-sym* **by** *blast*
AOT-hence ‹$\varphi\{q\} \rightarrow \varphi\{p\}$›
using *l−identity*[*axiom-inst*, *THEN* →*E*] **by** *blast*
}
ultimately AOT-have ‹$\varphi\{p\} \equiv \varphi\{q\}$›
using ≡*I* **by** *blast*
moreover AOT-assume ‹$\neg(\varphi\{p\} \equiv \varphi\{q\})$›
ultimately AOT-show ‹$(\varphi\{p\} \equiv \varphi\{q\})\ \&\ \neg(\varphi\{p\} \equiv \varphi\{q\})$›
using &*I* **by** *blast*
qed
}
AOT-hence ‹$\Diamond\neg(\varphi\{p\} \equiv \varphi\{q\}) \rightarrow \Diamond\neg(p = q)$›
using *RM*:*2*[*prem*] **by** *blast*
moreover AOT-assume ‹$\Diamond\neg(\varphi\{p\} \equiv \varphi\{q\})$›
ultimately AOT-have *0*: ‹$\Diamond\neg(p = q)$› **using** →*E* **by** *blast*
AOT-have ‹$\Diamond(p \neq q)$›
by (*AOT-subst* ‹$p \neq q$› ‹$\neg(p = q)$›)
(*auto simp*: *0* *=−infix* *≡Df*)
AOT-thus ‹$p \neq q$›
using *id−nec2*:*3*[*THEN* →*E*] **by** *blast*
qed

AOT-theorem *pos−not−equiv−ne*:*3*:
‹$(\neg\forall x_1...\forall x_n\ ([F]x_1...x_n \equiv [G]x_1...x_n)) \rightarrow F \neq G$›
using →*I pos−not−equiv−ne*:*1*[*THEN* →*E*] *T*◇[*THEN* →*E*] **by** *blast*

AOT-theorem *pos−not−equiv−ne*:*4*: ‹$(\neg(\varphi\{F\} \equiv \varphi\{G\})) \rightarrow F \neq G$›
using →*I pos−not−equiv−ne*:*2*[*THEN* →*E*] *T*◇[*THEN* →*E*] **by** *blast*

AOT-theorem *pos−not−equiv−ne*:*4*[*zero*]: ‹$(\neg(\varphi\{p\} \equiv \varphi\{q\})) \rightarrow p \neq q$›
using →*I pos−not−equiv−ne*:*2*[*zero*][*THEN* →*E*]
T◇[*THEN* →*E*] **by** *blast*

AOT-define *relation-negation* :: Π ⇒ Π (‹-⁻›)
df−relation−negation: $[F]^- =_{df} [\lambda x_1...x_n\ \neg[F]x_1...x_n]$

nonterminal *φneg*
syntax :: *φneg* ⇒ τ (‹-›)
syntax :: *φneg* ⇒ φ (‹′(-′)›)

AOT-define *relation-negation-0* :: ‹φ ⇒ *φneg*› (‹′(-′)⁻›)
df−relation−negation[*zero*]: $(p)^- =_{df} [\lambda\ \neg p]$

AOT-theorem *rel−neg−T*:*1*: ‹$[\lambda x_1...x_n\ \neg[\Pi]x_1...x_n]\downarrow$›
by *cqt*:*2*[*lambda*]

AOT-theorem *rel−neg−T*:*1*[*zero*]: ‹$[\lambda\ \neg\varphi]\downarrow$›

using *cqt:2[lambda0][axiom-inst]* **by** *blast*

AOT-theorem *rel−neg−T:2*: ‹$[\Pi]^- = [\lambda x_1...x_n\ \neg[\Pi]x_1...x_n]$›
using *=I(1)[OF rel−neg−T:1]*
by *(rule* $=_{df}$*I(1)[OF df−relation−negation, OF rel−neg−T:1])*

AOT-theorem *rel−neg−T:2[zero]*: ‹$(\varphi)^- = [\lambda\ \neg\varphi]$›
using *=I(1)[OF rel−neg−T:1[zero]]*
by *(rule* $=_{df}$*I(1)[OF df−relation−negation[zero], OF rel−neg−T:1[zero]])*

AOT-theorem *rel−neg−T:3*: ‹$[\Pi]^-\downarrow$›
using $=_{df}$*I(1)[OF df−relation−negation, OF rel−neg−T:1]*
rel−neg−T:1 **by** *blast*

AOT-theorem *rel−neg−T:3[zero]*: ‹$(\varphi)^-\downarrow$›
using *log−prop−prop:2* **by** *blast*

AOT-theorem *thm−relation−negation:1*: ‹$[F]^-x_1...x_n \equiv \neg[F]x_1...x_n$›
proof −
AOT-have ‹$[F]^-x_1...x_n \equiv [\lambda x_1...x_n\ \neg[F]x_1...x_n]x_1...x_n$›
using *rule=E[rotated, OF rel−neg−T:2]*
rule=E[rotated, OF rel−neg−T:2[THEN id-sym]]
$\rightarrow$*I* $\equiv$*I* **by** *fast*
also AOT-have ‹$... \equiv \neg[F]x_1...x_n$›
using *beta−C−meta[THEN* $\rightarrow$*E, OF rel−neg−T:1]* **by** *fast*
finally show *?thesis*.
qed

AOT-theorem *thm−relation−negation:2*: ‹$\neg[F]^-x_1...x_n \equiv [F]x_1...x_n$›
apply *(AOT-subst* ‹$[F]x_1...x_n$› ‹$\neg\neg[F]x_1...x_n$›*)*
apply *(simp add: oth−class−taut:3:b)*
apply *(rule oth−class−taut:4:b[THEN* $\equiv$*E(1)])*
using *thm−relation−negation:1*.

AOT-theorem *thm−relation−negation:3*: ‹$((p)^-) \equiv \neg p$›
proof −
AOT-have ‹$(p)^- = [\lambda\ \neg p]$› **using** *rel−neg−T:2[zero]* **by** *blast*
AOT-hence ‹$((p)^-) \equiv [\lambda\ \neg p]$›
using *df−relation−negation[zero] log−prop−prop:2*
oth−class−taut:3:a rule−id−df:2:a **by** *blast*
also AOT-have ‹$[\lambda\ \neg p] \equiv \neg p$›
by *(simp add: propositions−lemma:2)*
finally show *?thesis*.
qed

AOT-theorem *thm−relation−negation:4*: ‹$(\neg((p)^-)) \equiv p$›
using *thm−relation−negation:3[THEN* $\equiv$*E(1)]*
thm−relation−negation:3[THEN $\equiv$*E(2)]*
$\equiv$*I* $\rightarrow$*I RAA* **by** *metis*

AOT-theorem *thm−relation−negation:5*: ‹$[F] \neq [F]^-$›
proof −
AOT-have ‹$\neg([F] = [F]^-)$›
proof *(rule RAA(2))*
AOT-show ‹$[F]x_1...x_n \rightarrow [F]x_1...x_n$› **for** x_1x_n
using *if−p−then−p*.
next
AOT-assume ‹$[F] = [F]^-$›
AOT-hence ‹$[F]^- = [F]$› **using** *id-sym* **by** *blast*
AOT-hence ‹$[F]x_1...x_n \equiv \neg[F]x_1...x_n$› **for** x_1x_n
using *rule=E thm−relation−negation:1* **by** *fast*
AOT-thus ‹$\neg([F]x_1...x_n \rightarrow [F]x_1...x_n)$› **for** x_1x_n
using $\equiv$*E RAA* **by** *metis*

  qed
  thus *?thesis*
    using $\equiv_{df}I$ *=−infix* **by** *blast*
qed

AOT-theorem *thm−relation−negation:6*: ‹$p \neq (p)^-$›
proof −
  AOT-have ‹$\neg(p = (p)^-)$›
  proof (*rule RAA(2)*)
    AOT-show ‹$p \rightarrow p$›
      using *if−p−then−p*.
  next
    AOT-assume ‹$p = (p)^-$›
    AOT-hence ‹$(p)^- = p$› **using** *id-sym* **by** *blast*
    AOT-hence ‹$p \equiv \neg p$›
      using *rule=E thm−relation−negation:3* **by** *fast*
    AOT-thus ‹$\neg(p \rightarrow p)$›
      using $\equiv E$ *RAA* **by** *metis*
  qed
  thus *?thesis*
    using $\equiv_{df}I$ *=−infix* **by** *blast*
qed

AOT-theorem *thm−relation−negation:7*: ‹$(p)^- = (\neg p)$›
  apply (*rule df−relation−negation*[*zero*][*THEN* $=_{df}E(1)$])
  using *cqt:2*[*lambda0*][*axiom-inst*] *rel−neg−T:2*[*zero*]
      propositions−lemma:1 id-trans **by** *blast+*

AOT-theorem *thm−relation−negation:8*: ‹$p = q \rightarrow (\neg p) = (\neg q)$›
proof(*rule* $\rightarrow I$)
  AOT-assume ‹$p = q$›
  moreover AOT-have ‹$(\neg p)\downarrow$› **using** *log−prop−prop:2*.
  moreover AOT-have ‹$(\neg p) = (\neg p)$› **using** *calculation(2)* $=I$ **by** *blast*
  ultimately AOT-show ‹$(\neg p) = (\neg q)$›
    using *rule=E* **by** *fast*
qed

AOT-theorem *thm−relation−negation:9*: ‹$p = q \rightarrow (p)^- = (q)^-$›
proof(*rule* $\rightarrow I$)
  AOT-assume ‹$p = q$›
  AOT-hence ‹$(\neg p) = (\neg q)$› **using** *thm−relation−negation:8* $\rightarrow E$ **by** *blast*
  AOT-thus ‹$(p)^- = (q)^-$›
    using *thm−relation−negation:7 id-sym id-trans* **by** *metis*
qed

AOT-define *Necessary* :: ‹$\Pi \Rightarrow \varphi$› (‹*Necessary'*(-')›)
  contingent−properties:1:
  ‹*Necessary*([*F*]) $\equiv_{df} \Box\forall x_1...\forall x_n\ [F]x_1...x_n$›

AOT-define *Necessary0* :: ‹$\varphi \Rightarrow \varphi$› (‹*Necessary0'*(-')›)
  contingent−properties:1[*zero*]:
  ‹*Necessary0*(*p*) $\equiv_{df} \Box p$›

AOT-define *Impossible* :: ‹$\Pi \Rightarrow \varphi$› (‹*Impossible'*(-')›)
  contingent−properties:2:
  ‹*Impossible*([*F*]) $\equiv_{df} F\downarrow\ \&\ \Box\forall x_1...\forall x_n\ \neg[F]x_1...x_n$›

AOT-define *Impossible0* :: ‹$\varphi \Rightarrow \varphi$› (‹*Impossible0'*(-')›)
  contingent−properties:2[*zero*]:
  ‹*Impossible0*(*p*) $\equiv_{df} \Box\neg p$›

AOT-define *NonContingent* :: ‹$\Pi \Rightarrow \varphi$› (‹*NonContingent'*(-')›)
  contingent−properties:3:

‹*NonContingent*([*F*]) $\equiv_{df}$ *Necessary*([*F*]) ∨ *Impossible*([*F*])›

AOT-define *NonContingent0* :: ‹φ ⇒ φ› (‹*NonContingent0*′(-′)›)
contingent−properties:*3*[*zero*]:
‹*NonContingent0*(*p*) $\equiv_{df}$ *Necessary0*(*p*) ∨ *Impossible0*(*p*)›

AOT-define *Contingent* :: ‹Π ⇒ φ› (‹*Contingent*′(-′)›)
contingent−properties:*4*:
‹*Contingent*([*F*]) $\equiv_{df}$ *F*↓ & ¬(*Necessary*([*F*]) ∨ *Impossible*([*F*]))›

AOT-define *Contingent0* :: ‹φ ⇒ φ› (‹*Contingent0*′(-′)›)
contingent−properties:*4*[*zero*]:
‹*Contingent0*(*p*) $\equiv_{df}$ ¬(*Necessary0*(*p*) ∨ *Impossible0*(*p*))›

AOT-theorem *thm−cont−prop*:*1*: ‹*NonContingent*([*F*]) ≡ *NonContingent*([*F*]$^-$)›
proof (*rule* ≡*I*; *rule* →*I*)
  AOT-assume ‹*NonContingent*([*F*])›
  AOT-hence ‹*Necessary*([*F*]) ∨ *Impossible*([*F*])›
    using $\equiv_{df}$*E*[*OF contingent−properties*:*3*] **by** *blast*
  moreover {
    AOT-assume ‹*Necessary*([*F*])›
    AOT-hence ‹□(∀ x_1...∀ x_n [*F*]x_1...x_n)›
      using $\equiv_{df}$*E*[*OF contingent−properties*:*1*] **by** *blast*
    moreover AOT-modally-strict {
      AOT-assume ‹∀ x_1...∀ x_n [*F*]x_1...x_n›
      AOT-hence ‹[*F*]x_1...x_n› **for** $x_1 x_n$ **using** ∀ *E* **by** *blast*
      AOT-hence ‹¬[*F*]$^-$$x_1$...$x_n$› **for** $x_1 x_n$
        by (*meson* ≡*E*(*6*) *oth−class−taut*:*3*:*a*
              thm−relation−negation:*2* ≡*E*(*1*))
      AOT-hence ‹∀ x_1...∀ x_n ¬[*F*]$^-$$x_1$...$x_n$› **using** ∀ *I* **by** *fast*
    }
    ultimately AOT-have ‹□(∀ x_1...∀ x_n ¬[*F*]$^-$$x_1$...$x_n$)›
      using *RN*[*prem*][**where** Γ={«∀ x_1...∀ x_n [*F*]x_1...x_n»}, *simplified*] **by** *blast*
    AOT-hence ‹*Impossible*([*F*]$^-$)›
      using ≡*Df*[*OF contingent−properties*:*2*, *THEN* ≡*S*(*1*),
              OF rel−neg−T:*3*, *THEN* ≡*E*(*2*)]
      by *blast*
  }
  moreover {
    AOT-assume ‹*Impossible*([*F*])›
    AOT-hence ‹□(∀ x_1...∀ x_n ¬[*F*]x_1...x_n)›
      using ≡*Df*[*OF contingent−properties*:*2*, *THEN* ≡*S*(*1*),
              OF cqt:*2*[*const-var*][*axiom-inst*], *THEN* ≡*E*(*1*)]
      by *blast*
    moreover AOT-modally-strict {
      AOT-assume ‹∀ x_1...∀ x_n ¬[*F*]x_1...x_n›
      AOT-hence ‹¬[*F*]x_1...x_n› **for** $x_1 x_n$ **using** ∀ *E* **by** *blast*
      AOT-hence ‹[*F*]$^-$$x_1$...$x_n$› **for** $x_1 x_n$
        by (*meson* ≡*E*(*6*) *oth−class−taut*:*3*:*a*
              thm−relation−negation:*1* ≡*E*(*1*))
      AOT-hence ‹∀ x_1...∀ x_n [*F*]$^-$$x_1$...$x_n$› **using** ∀ *I* **by** *fast*
    }
    ultimately AOT-have ‹□(∀ x_1...∀ x_n [*F*]$^-$$x_1$...$x_n$)›
      using *RN*[*prem*][**where** Γ={«∀ x_1...∀ x_n ¬[*F*]x_1...x_n»}] **by** *blast*
    AOT-hence ‹*Necessary*([*F*]$^-$)›
      using $\equiv_{df}$*I*[*OF contingent−properties*:*1*] **by** *blast*
  }
  ultimately AOT-have ‹*Necessary*([*F*]$^-$) ∨ *Impossible*([*F*]$^-$)›
    using ∨*E*(*1*) ∨*I* →*I* **by** *metis*
  AOT-thus ‹*NonContingent*([*F*]$^-$)›
    using $\equiv_{df}$*I*[*OF contingent−properties*:*3*] **by** *blast*
next

```
  AOT-assume ‹NonContingent([F]⁻)›
  AOT-hence ‹Necessary([F]⁻) ∨ Impossible([F]⁻)›
    using ≡df E[OF contingent−properties:3] by blast
  moreover {
    AOT-assume ‹Necessary([F]⁻)›
    AOT-hence ‹□(∀ x1...∀ xn [F]⁻x1...xn)›
      using ≡df E[OF contingent−properties:1] by blast
    moreover AOT-modally-strict {
      AOT-assume ‹∀ x1...∀ xn [F]⁻x1...xn›
      AOT-hence ‹[F]⁻x1...xn› for x1xn using ∀ E by blast
      AOT-hence ‹¬[F]x1...xn› for x1xn
        by (meson ≡E(6) oth−class−taut:3:a
                  thm−relation−negation:1 ≡E(2))
      AOT-hence ‹∀ x1...∀ xn ¬[F]x1...xn› using ∀ I by fast
    }
    ultimately AOT-have ‹□∀ x1...∀ xn ¬[F]x1...xn›
      using RN[prem][where Γ={«∀ x1...∀ xn [F]⁻x1...xn»}] by blast
    AOT-hence ‹Impossible([F])›
      using ≡Df[OF contingent−properties:2, THEN ≡S(1),
                OF cqt:2[const-var][axiom-inst], THEN ≡E(2)]
      by blast
  }
  moreover {
    AOT-assume ‹Impossible([F]⁻)›
    AOT-hence ‹□(∀ x1...∀ xn ¬[F]⁻x1...xn)›
      using ≡Df[OF contingent−properties:2, THEN ≡S(1),
                OF rel−neg−T:3, THEN ≡E(1)]
      by blast
    moreover AOT-modally-strict {
      AOT-assume ‹∀ x1...∀ xn ¬[F]⁻x1...xn›
      AOT-hence ‹¬[F]⁻x1...xn› for x1xn using ∀ E by blast
      AOT-hence ‹[F]x1...xn› for x1xn
        using thm−relation−negation:1[THEN
              oth−class−taut:4:b[THEN ≡E(1)], THEN ≡E(1)]
            useful−tautologies:1[THEN →E] by blast
      AOT-hence ‹∀ x1...∀ xn [F]x1...xn› using ∀ I by fast
    }
    ultimately AOT-have ‹□(∀ x1...∀ xn [F]x1...xn)›
      using RN[prem][where Γ={«∀ x1...∀ xn ¬[F]⁻x1...xn»}] by blast
    AOT-hence ‹Necessary([F])›
      using ≡df I[OF contingent−properties:1] by blast
  }
  ultimately AOT-have ‹Necessary([F]) ∨ Impossible([F])›
    using ∨E(1) ∨I →I by metis
  AOT-thus ‹NonContingent([F])›
    using ≡df I[OF contingent−properties:3] by blast
qed

AOT-theorem thm−cont−prop:2: ‹Contingent([F]) ≡ ◇∃ x [F]x & ◇∃ x ¬[F]x›
proof −
  AOT-have ‹Contingent([F]) ≡ ¬(Necessary([F]) ∨ Impossible([F]))›
    using contingent−properties:4[THEN ≡Df, THEN ≡S(1),
                                  OF cqt:2[const-var][axiom-inst]]
    by blast
  also AOT-have ‹... ≡ ¬Necessary([F]) & ¬Impossible([F])›
    using oth−class−taut:5:d by fastforce
  also AOT-have ‹... ≡ ¬Impossible([F]) & ¬Necessary([F])›
    by (simp add: Commutativity of &)
  also AOT-have ‹... ≡ ◇∃ x [F]x & ¬Necessary([F])›
  proof (rule oth−class−taut:4:e[THEN →E])
    AOT-have ‹¬Impossible([F]) ≡ ¬□¬ ∃ x [F]x›
      apply (rule oth−class−taut:4:b[THEN ≡E(1)])
      apply (AOT-subst ‹∃ x [F]x› ‹¬ ∀ x ¬[F]x›)
```

apply (*simp add*: *conventions*:*4* ≡*Df*)
apply (*AOT-subst* (*reverse*) ‹¬¬∀ *x* ¬[*F*]*x*› ‹∀ *x* ¬[*F*]*x*›)
apply (*simp add*: *oth−class−taut*:*3*:*b*)
using *contingent−properties*:*2*[*THEN* ≡*Df*, *THEN* ≡*S*(*1*),
OF cqt:*2*[*const-var*][*axiom-inst*]]
by *blast*
also AOT-have ‹... ≡ ◊∃ *x* [*F*]*x*›
using *conventions*:*5*[*THEN* ≡*Df*, *symmetric*] **by** *blast*
finally AOT-show ‹¬*Impossible*([*F*]) ≡ ◊∃ *x* [*F*]*x*› **.**
qed
also AOT-have ‹... ≡ ◊∃ *x* [*F*]*x* & ◊∃ *x* ¬[*F*]*x*›
proof (*rule oth−class−taut*:*4*:*f*[*THEN* →*E*])
AOT-have ‹¬*Necessary*([*F*]) ≡ ¬□¬∃ *x* ¬[*F*]*x*›
apply (*rule oth−class−taut*:*4*:*b*[*THEN* ≡*E*(*1*)])
apply (*AOT-subst* ‹∃ *x* ¬[*F*]*x*› ‹¬ ∀ *x* ¬¬[*F*]*x*›)
apply (*simp add*: *conventions*:*4* ≡*Df*)
apply (*AOT-subst* (*reverse*) ‹¬¬[*F*]*x*› ‹[*F*]*x*› **for**: *x*)
apply (*simp add*: *oth−class−taut*:*3*:*b*)
apply (*AOT-subst* (*reverse*) ‹¬¬∀ *x* [*F*]*x*› ‹∀ *x* [*F*]*x*›)
by (*auto simp*: *oth−class−taut*:*3*:*b contingent−properties*:*1* ≡*Df*)
also AOT-have ‹... ≡ ◊∃ *x* ¬[*F*]*x*›
using *conventions*:*5*[*THEN* ≡*Df*, *symmetric*] **by** *blast*
finally AOT-show ‹¬*Necessary*([*F*]) ≡ ◊∃ *x* ¬[*F*]*x*›**.**
qed
finally show *?thesis***.**
qed

AOT-theorem *thm−cont−prop*:*3*:
‹*Contingent*([*F*]) ≡ *Contingent*([*F*]⁻)› **for** *F*::‹<κ> *AOT-var*›
proof −
{
fix Π :: ‹<κ>›
AOT-assume ‹Π↓›
moreover AOT-have ‹∀ *F* (*Contingent*([*F*]) ≡ ◊∃ *x* [*F*]*x* & ◊∃ *x* ¬[*F*]*x*)›
using *thm−cont−prop*:*2 GEN* **by** *fast*
ultimately AOT-have ‹*Contingent*([Π]) ≡ ◊∃ *x* [Π]*x* & ◊∃ *x* ¬[Π]*x*›
using *thm−cont−prop*:*2* ∀ *E* **by** *fast*
} note *1* = *this*
AOT-have ‹*Contingent*([*F*]) ≡ ◊∃ *x* [*F*]*x* & ◊∃ *x* ¬[*F*]*x*›
using *thm−cont−prop*:*2* **by** *blast*
also AOT-have ‹... ≡ ◊∃ *x* ¬[*F*]*x* & ◊∃ *x* [*F*]*x*›
by (*simp add*: *Commutativity of* &)
also AOT-have ‹... ≡ ◊∃ *x* [*F*]⁻ *x* & ◊∃ *x* [*F*]*x*›
by (*AOT-subst* ‹[*F*]⁻*x*› ‹¬[*F*]*x*› **for**: *x*)
(*auto simp*: *thm−relation−negation*:*1 oth−class−taut*:*3*:*a*)
also AOT-have ‹... ≡ ◊∃ *x* [*F*]⁻ *x* & ◊∃ *x* ¬[*F*]⁻ *x*›
by (*AOT-subst* (*reverse*) ‹[*F*]*x*› ‹¬[*F*]⁻ *x*› **for**: *x*)
(*auto simp*: *thm−relation−negation*:*2 oth−class−taut*:*3*:*a*)
also AOT-have ‹... ≡ *Contingent*([*F*]⁻)›
using *1*[*OF rel−neg−T*:*3*, *symmetric*] **by** *blast*
finally show *?thesis***.**
qed

AOT-define *concrete-if-concrete* :: ‹Π› (‹*L*›)
L-def: ‹*L* =$_{df}$ [λ*x E*!*x* → *E*!*x*]›

AOT-theorem *thm−noncont−e−e*:*1*: ‹*Necessary*(*L*)›
proof −
AOT-modally-strict {
fix *x*
AOT-have ‹[λ*x E*!*x* → *E*!*x*]↓› **by** *cqt*:*2*[*lambda*]
moreover AOT-have ‹*x*↓› **using** *cqt*:*2*[*const-var*][*axiom-inst*] **by** *blast*
moreover AOT-have ‹*E*!*x* → *E*!*x*› **using** *if−p−then−p* **by** *blast*

ultimately AOT-have ‹[λx E!x → E!x]x›
using *β←C* **by** *blast*
}
AOT-hence *0*: ‹□∀ x [λx E!x → E!x]x›
using *RN GEN* **by** *blast*
show *?thesis*
apply (*rule* $=_{df}$*I*(*2*)[*OF L-def*])
apply *cqt*:*2*[*lambda*]
by (*rule contingent−properties*:*1*[*THEN* $\equiv_{df}$*I*, *OF 0*])
qed

AOT-theorem *thm−noncont−e−e*:*2*: ‹*Impossible*([*L*]⁻)›
proof −
AOT-modally-strict {
fix *x*

AOT-have *0*: ‹∀ *F* (¬[*F*]⁻*x* ≡ [*F*]*x*)›
using *thm−relation−negation*:*2 GEN* **by** *fast*
AOT-have ‹¬[λx E!x → E!x]⁻x ≡ [λx E!x → E!x]x›
by (*rule 0*[*THEN* ∀ *E*(*1*)]) *cqt*:*2*[*lambda*]
moreover {
AOT-have ‹[λx E!x → E!x]↓› **by** *cqt*:*2*[*lambda*]
moreover AOT-have ‹x↓› **using** *cqt*:*2*[*const-var*][*axiom-inst*] **by** *blast*
moreover AOT-have ‹E!x → E!x› **using** *if−p−then−p* **by** *blast*
ultimately AOT-have ‹[λx E!x → E!x]x›
using *β←C* **by** *blast*
}
ultimately AOT-have ‹¬[λx E!x → E!x]⁻x›
using ≡*E* **by** *blast*
}
AOT-hence *0*: ‹□∀ x ¬[λx E!x → E!x]⁻x›
using *RN GEN* **by** *fast*
show *?thesis*
apply (*rule* $=_{df}$*I*(*2*)[*OF L-def*])
apply *cqt*:*2*[*lambda*]
apply (*rule contingent−properties*:*2*[*THEN* $\equiv_{df}$*I*]; *rule* &*I*)
using *rel−neg−T*:*3*
apply *blast*
using *0*
by *blast*
qed

AOT-theorem *thm−noncont−e−e*:*3*: ‹*NonContingent*(*L*)›
using *thm−noncont−e−e*:*1*
by (*rule contingent−properties*:*3*[*THEN* $\equiv_{df}$*I*, *OF* ∨*I*(*1*)])

AOT-theorem *thm−noncont−e−e*:*4*: ‹*NonContingent*([*L*]⁻)›
proof −
AOT-have *0*: ‹∀ *F* (*NonContingent*([*F*]) ≡ *NonContingent*([*F*]⁻))›
using *thm−cont−prop*:*1* ∀ *I* **by** *fast*
moreover AOT-have *1*: ‹L↓›
by (*rule* $=_{df}$*I*(*2*)[*OF L-def*]) *cqt*:*2*[*lambda*]+
AOT-show ‹*NonContingent*([*L*]⁻)›
using ∀ *E*(*1*)[*OF 0*, *OF 1*, *THEN* ≡*E*(*1*), *OF thm−noncont−e−e*:*3*] **by** *blast*
qed

AOT-theorem *thm−noncont−e−e*:*5*:
‹∃ *F* ∃ *G* (*F* ≠ «*G*::<κ>» & *NonContingent*([*F*]) & *NonContingent*([*G*]))›
proof (*rule* ∃ *I*)+
{
AOT-have ‹∀ *F* [*F*] ≠ [*F*]⁻›
using *thm−relation−negation*:*5 GEN* **by** *fast*
moreover AOT-have ‹L↓›

```
      by (rule =dfI(2)[OF L-def]) cqt:2[lambda]+
    ultimately AOT-have ‹L ≠ [L]⁻›
      using ∀E by blast
  }
  AOT-thus ‹L ≠ [L]⁻ & NonContingent(L) & NonContingent([L]⁻)›
    using thm−noncont−e−e:3 thm−noncont−e−e:4 &I by metis
next
  AOT-show ‹[L]⁻↓›
    using rel−neg−T:3 by blast
next
  AOT-show ‹L↓›
      by (rule =dfI(2)[OF L-def]) cqt:2[lambda]+
qed

AOT-theorem lem−cont−e:1: ‹◇∃x ([F]x & ◇¬[F]x) ≡ ◇∃x (¬[F]x & ◇[F]x)›
proof −
  AOT-have ‹◇∃x ([F]x & ◇¬[F]x) ≡ ∃x ◇([F]x & ◇¬[F]x)›
    using BF◇ CBF◇ ≡I by blast
  also AOT-have ‹... ≡ ∃x (◇[F]x &  ◇¬[F]x)›
    by (AOT-subst ‹◇([F]x & ◇¬[F]x)› ‹◇[F]x &  ◇¬[F]x› for: x)
       (auto simp: S5Basic:11 cqt−further:7)
  also AOT-have ‹... ≡ ∃x (◇¬[F]x & ◇[F]x)›
    by (AOT-subst ‹◇¬[F]x & ◇[F]x›  ‹◇[F]x & ◇¬[F]x› for: x)
       (auto simp: Commutativity of & cqt−further:7)
  also AOT-have ‹... ≡ ∃x ◇(¬[F]x & ◇[F]x)›
    by (AOT-subst ‹◇(¬[F]x & ◇[F]x)› ‹◇¬[F]x & ◇[F]x› for: x)
       (auto simp: S5Basic:11 oth−class−taut:3:a)
  also AOT-have ‹... ≡ ◇∃x (¬[F]x & ◇[F]x)›
    using BF◇ CBF◇ ≡I by fast
  finally show ?thesis.
qed

AOT-theorem lem−cont−e:2:
  ‹◇∃x ([F]x & ◇¬[F]x) ≡ ◇∃x ([F]⁻x & ◇¬[F]⁻x)›
proof −
  AOT-have ‹◇∃x ([F]x & ◇¬[F]x) ≡ ◇∃x (¬[F]x & ◇[F]x)›
    using lem−cont−e:1.
  also AOT-have ‹... ≡ ◇∃x ([F]⁻x & ◇¬[F]⁻x)›
    apply (AOT-subst ‹¬[F]⁻x› ‹[F]x› for: x)
     apply (simp add: thm−relation−negation:2)
    apply (AOT-subst ‹[F]⁻x› ‹¬[F]x› for: x)
     apply (simp add: thm−relation−negation:1)
    by (simp add: oth−class−taut:3:a)
  finally show ?thesis.
qed

AOT-theorem thm−cont−e:1: ‹◇∃x (E!x & ◇¬E!x)›
proof (rule CBF◇[THEN →E])
  AOT-have ‹∃x ◇(E!x & ¬𝓐E!x)›
    using qml:4[axiom-inst] BF◇[THEN →E] by blast
  then AOT-obtain a where ‹◇(E!a & ¬𝓐E!a)›
    using ∃E[rotated] by blast
  AOT-hence ϑ: ‹◇E!a & ◇¬𝓐E!a›
    using KBasic2:3[THEN →E] by blast
  AOT-have ξ: ‹◇E!a & ◇𝓐¬E!a›
    by (AOT-subst  ‹𝓐¬E!a› ‹¬𝓐E!a›)
       (auto simp: logic−actual−nec:1[axiom-inst] ϑ)
  AOT-have ζ: ‹◇E!a & 𝓐¬E!a›
    by (AOT-subst ‹𝓐¬E!a› ‹◇𝓐¬E!a›)
       (auto simp add: Act−Sub:4 ξ)
  AOT-hence ‹◇E!a & ◇¬E!a›
    using &E &I Act−Sub:3[THEN →E] by blast
  AOT-hence ‹◇(E!a & ◇¬E!a)›
```

```
    using S5Basic:11[THEN ≡E(2)] by simp
  AOT-thus ‹∃ x ◊(E!x & ◊¬E!x)›
    using ∃ I(2) by fast
qed

AOT-theorem thm−cont−e:2: ‹◊∃ x (¬E!x & ◊E!x)›
proof −
  AOT-have ‹∀ F (◊∃ x ([F]x & ◊¬[F]x) ≡ ◊∃ x (¬[F]x & ◊[F]x))›
    using lem−cont−e:1 GEN by fast
  AOT-hence ‹(◊∃ x (E!x & ◊¬E!x) ≡ ◊∃ x (¬E!x & ◊E!x))›
    using ∀ E(2) by blast
  thus ?thesis using thm−cont−e:1 ≡E by blast
qed

AOT-theorem thm−cont−e:3: ‹◊∃ x E!x›
proof (rule CBF◊[THEN →E])
  AOT-obtain a where ‹◊(E!a & ◊¬E!a)›
    using ∃ E[rotated, OF thm−cont−e:1[THEN BF◊[THEN →E]]] by blast
  AOT-hence ‹◊E!a›
    using KBasic2:3[THEN →E, THEN &E(1)] by blast
  AOT-thus ‹∃ x ◊E!x› using ∃ I by fast
qed

AOT-theorem thm−cont−e:4: ‹◊∃ x ¬E!x›
proof (rule CBF◊[THEN →E])
  AOT-obtain a where ‹◊(E!a & ◊¬E!a)›
    using ∃ E[rotated, OF thm−cont−e:1[THEN BF◊[THEN →E]]] by blast
  AOT-hence ‹◊◊¬E!a›
    using KBasic2:3[THEN →E, THEN &E(2)] by blast
  AOT-hence ‹◊¬E!a›
    using 4◊[THEN →E] by blast
  AOT-thus ‹∃ x ◊¬E!x› using ∃ I by fast
qed

AOT-theorem thm−cont−e:5: ‹Contingent([E!])›
proof −
  AOT-have ‹∀ F (Contingent([F]) ≡ ◊∃ x [F]x & ◊∃ x ¬[F]x)›
    using thm−cont−prop:2 GEN by fast
  AOT-hence ‹Contingent([E!]) ≡ ◊∃ x E!x & ◊∃ x ¬E!x›
    using ∀ E(2) by blast
  thus ?thesis
    using thm−cont−e:3 thm−cont−e:4 ≡E(2) &I by blast
qed

AOT-theorem thm−cont−e:6: ‹Contingent([E!]⁻)›
proof −
  AOT-have ‹∀ F (Contingent([«F::<κ>»]) ≡ Contingent([F]⁻))›
    using thm−cont−prop:3 GEN by fast
  AOT-hence ‹Contingent([E!]) ≡ Contingent([E!]⁻)›
    using ∀ E(2) by fast
  thus ?thesis using thm−cont−e:5 ≡E by blast
qed

AOT-theorem thm−cont−e:7:
  ‹∃ F∃ G (Contingent([«F::<κ>»]) & Contingent([G]) & F ≠ G)›
proof (rule ∃ I)+
  AOT-have ‹∀ F [«F::<κ>»] ≠ [F]⁻›
    using thm−relation−negation:5 GEN by fast
  AOT-hence ‹[E!] ≠ [E!]⁻›
    using ∀ E by fast
  AOT-thus ‹Contingent([E!]) & Contingent([E!]⁻) & [E!] ≠ [E!]⁻›
    using thm−cont−e:5 thm−cont−e:6 &I by metis
next
```

```
  AOT-show ‹E!⁻↓›
    by (fact AOT)
qed(cqt:2)

AOT-theorem property−facts:1:
  ‹NonContingent([F]) → ¬∃ G (Contingent([G]) & G = F)›
proof (rule →I; rule raa−cor:2)
  AOT-assume ‹NonContingent([F])›
  AOT-hence 1: ‹Necessary([F]) ∨ Impossible([F])›
    using contingent−properties:3[THEN ≡df E] by blast
  AOT-assume ‹∃ G (Contingent([G]) & G = F)›
  then AOT-obtain G where ‹Contingent([G]) & G = F›
    using ∃ E[rotated] by blast
  AOT-hence ‹Contingent([F])› using rule=E &E by blast
  AOT-hence ‹¬(Necessary([F]) ∨ Impossible([F]))›
    using contingent−properties:4[THEN ≡Df, THEN ≡S(1),
          OF cqt:2[const-var][axiom-inst], THEN ≡E(1)] by blast
  AOT-thus ‹(Necessary([F]) ∨ Impossible([F])) &
          ¬(Necessary([F]) ∨ Impossible([F]))›
    using 1 &I by blast
qed

AOT-theorem property−facts:2:
  ‹Contingent([F]) → ¬∃ G (NonContingent([G]) & G = F)›
proof (rule →I; rule raa−cor:2)
  AOT-assume ‹Contingent([F])›
  AOT-hence 1: ‹¬(Necessary([F]) ∨ Impossible([F]))›
    using contingent−properties:4[THEN ≡Df, THEN ≡S(1),
          OF cqt:2[const-var][axiom-inst], THEN ≡E(1)] by blast
  AOT-assume ‹∃ G (NonContingent([G]) & G = F)›
  then AOT-obtain G where ‹NonContingent([G]) & G = F›
    using ∃ E[rotated] by blast
  AOT-hence ‹NonContingent([F])›
    using rule=E &E by blast
  AOT-hence ‹Necessary([F]) ∨ Impossible([F])›
    using contingent−properties:3[THEN ≡df E] by blast
  AOT-thus ‹(Necessary([F]) ∨ Impossible([F])) &
          ¬(Necessary([F]) ∨ Impossible([F]))›
    using 1 &I by blast
qed

AOT-theorem property−facts:3:
  ‹L ≠ [L]⁻ & L ≠ E! & L ≠ E!⁻ & [L]⁻ ≠ [E!]⁻ & E! ≠ [E!]⁻›
proof −
  AOT-have noneqI: ‹Π ≠ Π′› if ‹φ{Π}› and ‹¬φ{Π′}› for φ and Π Π′ :: ‹<κ>›
    apply (rule =−infix[THEN ≡df I]; rule raa−cor:2)
    using rule=E[where φ=φ and τ=Π and σ = Π′] that &I by blast
  AOT-have contingent-denotes: ‹Π↓› if ‹Contingent([Π])› for Π :: ‹<κ>›
    using that contingent−properties:4[THEN ≡df E, THEN &E(1)] by blast
  AOT-have not-noncontingent-if-contingent:
    ‹¬NonContingent([Π])› if ‹Contingent([Π])› for Π :: ‹<κ>›
  proof(rule RAA(2))
    AOT-show ‹¬(Necessary([Π]) ∨ Impossible([Π]))›
      using that contingent−properties:4[THEN ≡Df, THEN ≡S(1),
                OF contingent-denotes[OF that], THEN ≡E(1)]
      by blast
  next
    AOT-assume ‹NonContingent([Π])›
    AOT-thus ‹Necessary([Π]) ∨ Impossible([Π])›
      using contingent−properties:3[THEN ≡df E] by blast
  qed

  show ?thesis
```

```
  proof (safe intro!: &I)
    AOT-show ‹L ≠ [L]⁻›
      apply (rule =df I(2)[OF L-def])
       apply cqt:2[lambda]
      apply (rule ∀ E(1)[where φ=λ Π . «Π ≠ [Π]⁻»])
       apply (rule GEN) apply (fact AOT)
      by cqt:2[lambda]
  next
    AOT-show ‹L ≠ E!›
      apply (rule noneqI)
      using thm−noncont−e−e:3
            not-noncontingent-if-contingent[OF thm−cont−e:5]
      by auto
  next
    AOT-show ‹L ≠ E!⁻›
      apply (rule noneqI)
      using thm−noncont−e−e:3 apply fast
      apply (rule not-noncontingent-if-contingent)
      apply (rule ∀ E(1)[
            where φ=λ Π . «Contingent([Π]) ≡ Contingent([Π]⁻)»,
            rotated, OF contingent-denotes, THEN ≡E(1), rotated])
      using thm−cont−prop:3 GEN apply fast
      using thm−cont−e:5 by fast+
  next
    AOT-show ‹[L]⁻ ≠ E!⁻›
      apply (rule noneqI)
      using thm−noncont−e−e:4 apply fast
      apply (rule not-noncontingent-if-contingent)
      apply (rule ∀ E(1)[
            where φ=λ Π . «Contingent([Π]) ≡ Contingent([Π]⁻)»,
            rotated, OF contingent-denotes, THEN ≡E(1), rotated])
      using thm−cont−prop:3 GEN apply fast
      using thm−cont−e:5 by fast+
  next
    AOT-show ‹E! ≠ E!⁻›
      apply (rule =df I(2)[OF L-def])
       apply cqt:2[lambda]
      apply (rule ∀ E(1)[where φ=λ Π . «Π ≠ [Π]⁻»])
       apply (rule GEN) apply (fact AOT)
      by cqt:2
  qed
qed

AOT-theorem thm−cont−propos:1:
  ‹NonContingent0(p) ≡ NonContingent0(((p)⁻))›
proof(rule ≡I; rule →I)
  AOT-assume ‹NonContingent0(p)›
  AOT-hence ‹Necessary0(p) ∨ Impossible0(p)›
    using contingent−properties:3[zero][THEN ≡df E] by blast
  moreover {
    AOT-assume ‹Necessary0(p)›
    AOT-hence 1: ‹□p›
      using contingent−properties:1[zero][THEN ≡df E] by blast
    AOT-have ‹□¬((p)⁻)›
      by (AOT-subst ‹¬((p)⁻)› ‹p›)
         (auto simp add: 1 thm−relation−negation:4)
    AOT-hence ‹Impossible0(((p)⁻))›
      by (rule contingent−properties:2[zero][THEN ≡df I])
  }
  moreover {
    AOT-assume ‹Impossible0(p)›
    AOT-hence 1: ‹□¬p›
      by (rule contingent−properties:2[zero][THEN ≡df E])
```

AOT-have ‹$\Box((p)^-)$›
by (AOT-$subst$ ‹$((p)^-)$› ‹$\neg p$›)
($auto\ simp$: 1 $thm{-}relation{-}negation{:}3$)
AOT-hence ‹$Necessary0(((p)^-))$›
by ($rule\ contingent{-}properties{:}1[zero][THEN \equiv_{df} I]$)
}
ultimately AOT-have ‹$Necessary0(((p)^-)) \vee Impossible0(((p)^-))$›
using $\vee E(1)$ $\vee I$ $\rightarrow I$ **by** *metis*
AOT-thus ‹$NonContingent0(((p)^-))$›
using $contingent{-}properties{:}3[zero][THEN \equiv_{df} I]$ **by** *blast*
next
AOT-assume ‹$NonContingent0(((p)^-))$›
AOT-hence ‹$Necessary0(((p)^-)) \vee Impossible0(((p)^-))$›
using $contingent{-}properties{:}3[zero][THEN \equiv_{df} E]$ **by** *blast*
moreover {
AOT-assume ‹$Impossible0(((p)^-))$›
AOT-hence 1: ‹$\Box\neg((p)^-)$›
by ($rule\ contingent{-}properties{:}2[zero][THEN \equiv_{df} E]$)
AOT-have ‹$\Box p$›
by (AOT-$subst$ ($reverse$) ‹p› ‹$\neg((p)^-)$›)
($auto\ simp$: 1 $thm{-}relation{-}negation{:}4$)
AOT-hence ‹$Necessary0(p)$›
using $contingent{-}properties{:}1[zero][THEN \equiv_{df} I]$ **by** *blast*
}
moreover {
AOT-assume ‹$Necessary0(((p)^-))$›
AOT-hence 1: ‹$\Box((p)^-)$›
by ($rule\ contingent{-}properties{:}1[zero][THEN \equiv_{df} E]$)
AOT-have ‹$\Box\neg p$›
by (AOT-$subst$ ($reverse$) ‹$\neg p$› ‹$((p)^-)$›)
($auto\ simp$: 1 $thm{-}relation{-}negation{:}3$)
AOT-hence ‹$Impossible0(p)$›
by ($rule\ contingent{-}properties{:}2[zero][THEN \equiv_{df} I]$)
}
ultimately AOT-have ‹$Necessary0(p) \vee Impossible0(p)$›
using $\vee E(1)$ $\vee I$ $\rightarrow I$ **by** *metis*
AOT-thus ‹$NonContingent0(p)$›
using $contingent{-}properties{:}3[zero][THEN \equiv_{df} I]$ **by** *blast*
qed

AOT-theorem $thm{-}cont{-}propos{:}2$: ‹$Contingent0(\varphi) \equiv \Diamond\varphi\ \&\ \Diamond\neg\varphi$›
proof −
AOT-have ‹$Contingent0(\varphi) \equiv \neg(Necessary0(\varphi) \vee Impossible0(\varphi))$›
using $contingent{-}properties{:}4[zero][THEN \equiv Df]$ **by** *simp*
also AOT-have ‹$\ldots \equiv \neg Necessary0(\varphi)\ \&\ \neg Impossible0(\varphi)$›
by ($fact\ AOT$)
also AOT-have ‹$\ldots \equiv \neg Impossible0(\varphi)\ \&\ \neg Necessary0(\varphi)$›
by ($fact\ AOT$)
also AOT-have ‹$\ldots \equiv \Diamond\varphi\ \&\ \Diamond\neg\varphi$›
apply (AOT-$subst$ ‹$\Diamond\varphi$› ‹$\neg\Box\neg\varphi$›)
apply ($simp\ add$: $conventions{:}5 \equiv Df$)
apply (AOT-$subst$ ‹$Impossible0(\varphi)$› ‹$\Box\neg\varphi$›)
apply ($simp\ add$: $contingent{-}properties{:}2[zero] \equiv Df$)
apply (AOT-$subst$ ($reverse$) ‹$\Diamond\neg\varphi$› ‹$\neg\Box\varphi$›)
apply ($simp\ add$: $KBasic{:}11$)
apply (AOT-$subst$ ‹$Necessary0(\varphi)$› ‹$\Box\varphi$›)
apply ($simp\ add$: $contingent{-}properties{:}1[zero] \equiv Df$)
by ($simp\ add$: $oth{-}class{-}taut{:}3{:}a$)
finally show *?thesis*.
qed

AOT-theorem $thm{-}cont{-}propos{:}3$: ‹$Contingent0(p) \equiv Contingent0(((p)^-))$›
proof −

```
  AOT-have ‹Contingent0(p) ≡ ◊p & ◊¬p› using thm−cont−propos:2.
  also AOT-have ‹... ≡ ◊¬p & ◊p› by (fact AOT)
  also AOT-have ‹... ≡ ◊((p)⁻) & ◊p›
    by (AOT-subst ‹((p)⁻)› ‹¬p›)
       (auto simp: thm−relation−negation:3 oth−class−taut:3:a)
  also AOT-have ‹... ≡ ◊((p)⁻) & ◊¬((p)⁻)›
    by (AOT-subst ‹¬((p)⁻)› ‹p›)
       (auto simp: thm−relation−negation:4 oth−class−taut:3:a)
  also AOT-have ‹... ≡ Contingent0(((p)⁻))›
    using thm−cont−propos:2[symmetric] by blast
  finally show ?thesis.
qed

AOT-define noncontingent-prop :: ‹φ› (‹p0›)
  p0-def: (p0) =df (∀x (E!x → E!x))

AOT-theorem thm−noncont−propos:1:  ‹Necessary0((p0))›
proof(rule contingent−properties:1[zero][THEN ≡df I])
  AOT-show ‹□(p0)›
    apply (rule =df I(2)[OF p0-def])
    using log−prop−prop:2 apply simp
    using if−p−then−p RN GEN by fast
qed

AOT-theorem thm−noncont−propos:2: ‹Impossible0(((p0)⁻))›
proof(rule contingent−properties:2[zero][THEN ≡df I])
  AOT-show ‹□¬((p0)⁻)›
    apply (AOT-subst ‹((p0)⁻)› ‹¬p0›)
    using thm−relation−negation:3 GEN ∀E(1)[rotated, OF log−prop−prop:2]
     apply fast
    apply (AOT-subst (reverse) ‹¬¬p0› ‹p0›)
     apply (simp add: oth−class−taut:3:b)
    apply (rule =df I(2)[OF p0-def])
    using log−prop−prop:2 apply simp
    using if−p−then−p RN GEN by fast
qed

AOT-theorem thm−noncont−propos:3: ‹NonContingent0((p0))›
  apply(rule contingent−properties:3[zero][THEN ≡df I])
  using thm−noncont−propos:1 ∨I by blast

AOT-theorem thm−noncont−propos:4: ‹NonContingent0(((p0)⁻))›
  apply(rule contingent−properties:3[zero][THEN ≡df I])
  using thm−noncont−propos:2 ∨I by blast

AOT-theorem thm−noncont−propos:5:
  ‹∃p∃q (NonContingent0((p)) & NonContingent0((q)) & p ≠ q)›
proof(rule ∃I)+
  AOT-have 0: ‹φ ≠ (φ)⁻› for φ
    using thm−relation−negation:6 ∀I
          ∀E(1)[rotated, OF log−prop−prop:2] by fast
  AOT-thus ‹NonContingent0((p0)) & NonContingent0(((p0)⁻)) & (p0) ≠ (p0)⁻›
    using thm−noncont−propos:3 thm−noncont−propos:4 &I by auto
qed(auto simp: log−prop−prop:2)

AOT-act-theorem no−cnac: ‹¬∃x(E!x & ¬𝓐E!x)›
proof(rule raa−cor:2)
  AOT-assume ‹∃x(E!x & ¬𝓐E!x)›
  then AOT-obtain a where a: ‹E!a & ¬𝓐E!a›
    using ∃E[rotated] by blast
  AOT-hence ‹𝓐¬E!a›
    using &E logic−actual−nec:1[axiom-inst, THEN ≡E(2)] by blast
  AOT-hence ‹¬E!a›
```

```
    using logic−actual[act-axiom-inst, THEN →E] by blast
  AOT-hence ‹E!a & ¬E!a›
    using a &E &I by blast
  AOT-thus ‹p & ¬p› for p using raa−cor:1 by blast
qed

AOT-theorem pos−not−pna:1: ‹¬𝓐∃ x (E!x & ¬𝓐E!x)›
proof(rule raa−cor:2)
  AOT-assume ‹𝓐∃ x (E!x & ¬𝓐E!x)›
  AOT-hence ‹∃ x 𝓐(E!x & ¬𝓐E!x)›
    using Act−Basic:10[THEN ≡E(1)] by blast
  then AOT-obtain a where ‹𝓐(E!a & ¬𝓐E!a)›
    using ∃ E[rotated] by blast
  AOT-hence 1: ‹𝓐E!a & 𝓐¬𝓐E!a›
    using Act−Basic:2[THEN ≡E(1)] by blast
  AOT-hence ‹¬𝓐𝓐E!a›
    using &E(2) logic−actual−nec:1[axiom-inst, THEN ≡E(1)] by blast
  AOT-hence ‹¬𝓐E!a›
    using logic−actual−nec:4[axiom-inst, THEN ≡E(1)] RAA by blast
  AOT-thus ‹p & ¬p› for p using 1[THEN &E(1)] &I raa−cor:1 by blast
qed

AOT-theorem pos−not−pna:2: ‹◇¬∃ x(E!x & ¬𝓐E!x)›
proof (rule RAA(1))
  AOT-show ‹¬𝓐∃ x (E!x & ¬𝓐E!x)›
    using pos−not−pna:1 by blast
next
  AOT-assume ‹¬◇¬∃ x (E!x & ¬𝓐E!x)›
  AOT-hence ‹□∃ x (E!x & ¬𝓐E!x)›
    using KBasic:12[THEN ≡E(2)] by blast
  AOT-thus ‹𝓐∃ x (E!x & ¬𝓐E!x)›
    using nec−imp−act[THEN →E] by blast
qed

AOT-theorem pos−not−pna:3: ‹∃ x (◇E!x & ¬𝓐E!x)›
proof −
  AOT-obtain a where ‹◇(E!a & ¬𝓐E!a)›
    using qml:4[axiom-inst] BF◇[THEN →E] ∃ E[rotated] by blast
  AOT-hence ϑ: ‹◇E!a› and ξ: ‹◇¬𝓐E!a›
    using KBasic2:3[THEN →E] &E by blast+
  AOT-have ‹¬□𝓐E!a›
    using ξ KBasic:11[THEN ≡E(2)] by blast
  AOT-hence ‹¬𝓐E!a›
    using Act−Basic:6[THEN oth−class−taut:4:b[THEN ≡E(1)],
                      THEN ≡E(2)] by blast
  AOT-hence ‹◇E!a & ¬𝓐E!a› using ϑ &I by blast
  thus ?thesis using ∃ I by fast
qed

AOT-define contingent-prop :: φ (‹q0›)
  q0-def: ‹(q0) =df (∃ x (E!x & ¬𝓐E!x))›

AOT-theorem q0-prop: ‹◇q0 & ◇¬q0›
  apply (rule =df I(2)[OF q0-def])
  apply (fact log−prop−prop:2)
  apply (rule &I)
   apply (fact qml:4[axiom-inst])
  by (fact pos−not−pna:2)

AOT-theorem basic−prop:1: ‹Contingent0((q0))›
proof(rule contingent−properties:4[zero][THEN ≡df I])
  AOT-have ‹¬Necessary0((q0)) & ¬Impossible0((q0))›
  proof (rule &I;
```

```
        rule =df I(2)[OF q0-def];
        (rule log−prop−prop:2 | rule raa−cor:2))
    AOT-assume ‹Necessary0(∃x (E!x & ¬𝒜E!x))›
    AOT-hence ‹□∃x (E!x & ¬𝒜E!x)›
      using contingent−properties:1[zero][THEN ≡df E] by blast
    AOT-hence ‹𝒜∃x (E!x & ¬𝒜E!x)›
      using Act−Basic:8[THEN →E] qml:2[axiom-inst, THEN →E] by blast
    AOT-thus ‹𝒜∃x (E!x & ¬𝒜E!x) & ¬𝒜∃x (E!x & ¬𝒜E!x)›
      using pos−not−pna:1 &I by blast
  next
    AOT-assume ‹Impossible0(∃x (E!x & ¬𝒜E!x))›
    AOT-hence ‹□¬(∃x (E!x & ¬𝒜E!x))›
      using contingent−properties:2[zero][THEN ≡df E] by blast
    AOT-hence ‹¬◇(∃x (E!x & ¬𝒜E!x))›
      using KBasic2:1[THEN ≡E(1)] by blast
    AOT-thus ‹◇(∃x (E!x & ¬𝒜E!x)) & ¬◇(∃x (E!x & ¬𝒜E!x))›
      using qml:4[axiom-inst] &I by blast
  qed
  AOT-thus ‹¬(Necessary0((q0)) ∨ Impossible0((q0)))›
    using oth−class−taut:5:d ≡E(2) by blast
qed

AOT-theorem basic−prop:2: ‹∃p Contingent0((p))›
  using ∃I(1)[rotated, OF log−prop−prop:2] basic−prop:1 by blast

AOT-theorem basic−prop:3: ‹Contingent0(((q0)−))›
  apply (AOT-subst ‹((q0)−)› ‹¬q0›)
   apply (insert thm−relation−negation:3 ∀I
              ∀E(1)[rotated, OF log−prop−prop:2]; fast)
  apply (rule contingent−properties:4[zero][THEN ≡df I])
  apply (rule oth−class−taut:5:d[THEN ≡E(2)])
  apply (rule &I)
   apply (rule contingent−properties:1[zero][THEN df−rules−formulas[3],
              THEN useful−tautologies:5[THEN →E], THEN →E])
   apply (rule conventions:5[THEN ≡df E])
   apply (rule =df E(2)[OF q0-def])
    apply (rule log−prop−prop:2)
   apply (rule q0-prop[THEN &E(1)])
  apply (rule contingent−properties:2[zero][THEN df−rules−formulas[3],
             THEN useful−tautologies:5[THEN →E], THEN →E])
  apply (rule conventions:5[THEN ≡df E])
  by (rule q0-prop[THEN &E(2)])

AOT-theorem basic−prop:4:
  ‹∃p∃q (p ≠ q & Contingent0(p) & Contingent0(q))›
proof(rule ∃I)+
  AOT-have 0: ‹φ ≠ (φ)−› for φ
    using thm−relation−negation:6 ∀I
        ∀E(1)[rotated, OF log−prop−prop:2] by fast
  AOT-show ‹(q0) ≠ (q0)− & Contingent0(q0) & Contingent0(((q0)−))›
    using basic−prop:1 basic−prop:3 &I 0 by presburger
qed(auto simp: log−prop−prop:2)

AOT-theorem proposition−facts:1:
  ‹NonContingent0(p) → ¬∃q (Contingent0(q) & q = p)›
proof(rule →I; rule raa−cor:2)
  AOT-assume ‹NonContingent0(p)›
  AOT-hence 1: ‹Necessary0(p) ∨ Impossible0(p)›
    using contingent−properties:3[zero][THEN ≡df E] by blast
  AOT-assume ‹∃q (Contingent0(q) & q = p)›
  then AOT-obtain q where ‹Contingent0(q) & q = p›
    using ∃E[rotated] by blast
  AOT-hence ‹Contingent0(p)›
```

using *rule*=*E* &*E* **by** *fast*
AOT-thus ‹(*Necessary0*(*p*) ∨ *Impossible0*(*p*)) &
¬(*Necessary0*(*p*) ∨ *Impossible0*(*p*))›
using *contingent−properties*:*4*[*zero*][*THEN* $\equiv_{df}$*E*] *1* &*I* **by** *blast*
qed

AOT-theorem *proposition−facts*:*2*:
‹*Contingent0*(*p*) → ¬∃ *q* (*NonContingent0*(*q*) & *q* = *p*)›
proof(*rule* →*I*; *rule raa*−*cor*:*2*)
AOT-assume ‹*Contingent0*(*p*)›
AOT-hence *1*: ‹¬(*Necessary0*(*p*) ∨ *Impossible0*(*p*))›
using *contingent−properties*:*4*[*zero*][*THEN* $\equiv_{df}$*E*] **by** *blast*
AOT-assume ‹∃ *q* (*NonContingent0*(*q*) & *q* = *p*)›
then AOT-obtain *q* **where** ‹*NonContingent0*(*q*) & *q* = *p*›
using ∃ *E*[*rotated*] **by** *blast*
AOT-hence ‹*NonContingent0*(*p*)›
using *rule*=*E* &*E* **by** *fast*
AOT-thus ‹(*Necessary0*(*p*) ∨ *Impossible0*(*p*)) &
¬(*Necessary0*(*p*) ∨ *Impossible0*(*p*))›
using *contingent−properties*:*3*[*zero*][*THEN* $\equiv_{df}$*E*] *1* &*I* **by** *blast*
qed

AOT-theorem *proposition−facts*:*3*:
‹$(p_0) \neq (p_0)^- \;\&\; (p_0) \neq (q_0) \;\&\; (p_0) \neq (q_0)^- \;\&\; (p_0)^- \neq (q_0)^- \;\&\; (q_0) \neq (q_0)^-$›
proof −
{
fix χ φ ψ
AOT-assume ‹χ{φ}›
moreover AOT-assume ‹¬χ{ψ}›
ultimately AOT-have ‹¬(χ{φ} ≡ χ{ψ})›
using *RAA* ≡*E* **by** *metis*
moreover {
AOT-have ‹∀ *p*∀ *q* ((¬(χ{*p*} ≡ χ{*q*})) → *p* ≠ *q*)›
by (*rule* ∀ *I*; *rule* ∀ *I*; *rule pos*−*not*−*equiv*−*ne*:*4*[*zero*])
AOT-hence ‹((¬(χ{φ} ≡ χ{ψ})) → φ ≠ ψ)›
using ∀ *E log*−*prop*−*prop*:*2* **by** *blast*
}
ultimately AOT-have ‹φ ≠ ψ›
using →*E* **by** *blast*
} note *0* = *this*
AOT-have *contingent-neg*: ‹*Contingent0*(φ) ≡ *Contingent0*(((φ)$^-$))› **for** φ
using *thm*−*cont*−*propos*:*3* ∀ *I*
∀ *E*(*1*)[*rotated*, *OF log*−*prop*−*prop*:*2*] **by** *fast*
AOT-have *not-noncontingent-if-contingent*:
‹¬*NonContingent0*(φ)› **if** ‹*Contingent0*(φ)› **for** φ
apply (*rule contingent*−*properties*:*3*[*zero*][*THEN* ≡*Df*,
THEN oth−*class*−*taut*:*4*:*b*[*THEN* ≡*E*(*1*)], *THEN* ≡*E*(*2*)])
using *that contingent*−*properties*:*4*[*zero*][*THEN* $\equiv_{df}$*E*] **by** *blast*
show *?thesis*
apply (*rule* &*I*)+
using *thm*−*relation*−*negation*:*6* ∀ *I*
∀ *E*(*1*)[*rotated*, *OF log*−*prop*−*prop*:*2*]
apply *fast*
apply (*rule 0*)
using *thm*−*noncont*−*propos*:*3* **apply** *fast*
apply (*rule not-noncontingent-if-contingent*)
apply (*fact AOT*)
apply (*rule 0*)
apply (*rule thm*−*noncont*−*propos*:*3*)
apply (*rule not-noncontingent-if-contingent*)
apply (*rule contingent-neg*[*THEN* ≡*E*(*1*)])
apply (*fact AOT*)
apply (*rule 0*)

apply (*rule thm−noncont−propos*:*4*)
apply (*rule not-noncontingent-if-contingent*)
apply (*rule contingent-neg*[*THEN* ≡*E*(*1*)])
apply (*fact AOT*)
using *thm−relation−negation*:*6* ∀ *I*
∀ *E*(*1*)[*rotated*, *OF log−prop−prop*:*2*] **by** *fast*
qed

AOT-define *ContingentlyTrue* :: ‹φ ⇒ φ› (‹*ContingentlyTrue*′(-′)›)
cont−tf:*1*: ‹*ContingentlyTrue*(*p*) $\equiv_{df}$ *p* & ◇¬*p*›

AOT-define *ContingentlyFalse* :: ‹φ ⇒ φ› (‹*ContingentlyFalse*′(-′)›)
cont−tf:*2*: ‹*ContingentlyFalse*(*p*) $\equiv_{df}$ ¬*p* & ◇*p*›

AOT-theorem *cont−true−cont*:*1*:
‹*ContingentlyTrue*((*p*)) → *Contingent0*((*p*))›
proof(*rule* →*I*)
AOT-assume ‹*ContingentlyTrue*((*p*))›
AOT-hence *1*: ‹*p*› **and** *2*: ‹◇¬*p*› **using** *cont−tf*:*1*[*THEN* $\equiv_{df}$*E*] &*E* **by** *blast*+
AOT-have ‹¬*Necessary0*((*p*))›
apply (*rule contingent−properties*:*1*[*zero*][*THEN* ≡*Df*,
THEN oth−class−taut:*4*:*b*[*THEN* ≡*E*(*1*)], *THEN* ≡*E*(*2*)])
using *2 KBasic*:*11*[*THEN* ≡*E*(*2*)] **by** *blast*
moreover AOT-have ‹¬*Impossible0*((*p*))›
apply (*rule contingent−properties*:*2*[*zero*][*THEN* ≡*Df*,
THEN oth−class−taut:*4*:*b*[*THEN* ≡*E*(*1*)], *THEN* ≡*E*(*2*)])
apply (*rule conventions*:*5*[*THEN* $\equiv_{df}$*E*])
using *T*◇[*THEN* →*E*, *OF 1*]**.**
ultimately AOT-have ‹¬(*Necessary0*((*p*)) ∨ *Impossible0*((*p*)))›
using *DeMorgan*(*2*)[*THEN* ≡*E*(*2*)] &*I* **by** *blast*
AOT-thus ‹*Contingent0*((*p*))›
using *contingent−properties*:*4*[*zero*][*THEN* $\equiv_{df}$*I*] **by** *blast*
qed

AOT-theorem *cont−true−cont*:*2*:
‹*ContingentlyFalse*((*p*)) → *Contingent0*((*p*))›
proof(*rule* →*I*)
AOT-assume ‹*ContingentlyFalse*((*p*))›
AOT-hence *1*: ‹¬*p*› **and** *2*: ‹◇*p*› **using** *cont−tf*:*2*[*THEN* $\equiv_{df}$*E*] &*E* **by** *blast*+
AOT-have ‹¬*Necessary0*((*p*))›
apply (*rule contingent−properties*:*1*[*zero*][*THEN* ≡*Df*,
THEN oth−class−taut:*4*:*b*[*THEN* ≡*E*(*1*)], *THEN* ≡*E*(*2*)])
using *KBasic*:*11*[*THEN* ≡*E*(*2*)] *T*◇[*THEN* →*E*, *OF 1*] **by** *blast*
moreover AOT-have ‹¬*Impossible0*((*p*))›
apply (*rule contingent−properties*:*2*[*zero*][*THEN* ≡*Df*,
THEN oth−class−taut:*4*:*b*[*THEN* ≡*E*(*1*)], *THEN* ≡*E*(*2*)])
apply (*rule conventions*:*5*[*THEN* $\equiv_{df}$*E*])
using *2***.**
ultimately AOT-have ‹¬(*Necessary0*((*p*)) ∨ *Impossible0*((*p*)))›
using *DeMorgan*(*2*)[*THEN* ≡*E*(*2*)] &*I* **by** *blast*
AOT-thus ‹*Contingent0*((*p*))›
using *contingent−properties*:*4*[*zero*][*THEN* $\equiv_{df}$*I*] **by** *blast*
qed

AOT-theorem *cont−true−cont*:*3*:
‹*ContingentlyTrue*((*p*)) ≡ *ContingentlyFalse*(((*p*)$^{-}$))›
proof(*rule* ≡*I*; *rule* →*I*)
AOT-assume ‹*ContingentlyTrue*((*p*))›
AOT-hence *0*: ‹*p* & ◇¬*p*› **using** *cont−tf*:*1*[*THEN* $\equiv_{df}$*E*] **by** *blast*
AOT-have *1*: ‹*ContingentlyFalse*(¬*p*)›
apply (*rule cont−tf*:*2*[*THEN* $\equiv_{df}$*I*])
apply (*AOT-subst* (*reverse*) ‹¬¬*p*› *p*)
by (*auto simp*: *oth−class−taut*:*3*:*b 0*)

```
  AOT-show ‹ContingentlyFalse(((p)⁻))›
    apply (AOT-subst ‹((p)⁻)› ‹¬p›)
    by (auto simp: thm−relation−negation:3 1)
next
  AOT-assume 1: ‹ContingentlyFalse(((p)⁻))›
  AOT-have ‹ContingentlyFalse(¬p)›
    by (AOT-subst (reverse) ‹¬p› ‹((p)⁻)›)
       (auto simp: thm−relation−negation:3 1)
  AOT-hence ‹¬¬p & ◊¬p› using cont−tf:2[THEN ≡df E] by blast
  AOT-hence ‹p & ◊¬p›
    using &I &E useful−tautologies:1[THEN →E] by metis
  AOT-thus ‹ContingentlyTrue((p))›
    using cont−tf:1[THEN ≡df I] by blast
qed

AOT-theorem cont−true−cont:4:
  ‹ContingentlyFalse((p)) ≡ ContingentlyTrue(((p)⁻))›
proof(rule ≡I; rule →I)
  AOT-assume ‹ContingentlyFalse(p)›
  AOT-hence 0: ‹¬p & ◊p›
    using cont−tf:2[THEN ≡df E] by blast
  AOT-have ‹¬p & ◊¬¬p›
    by (AOT-subst (reverse) ‹¬¬p› p)
       (auto simp: oth−class−taut:3:b 0)
  AOT-hence 1: ‹ContingentlyTrue(¬p)›
    by (rule cont−tf:1[THEN ≡df I])
  AOT-show ‹ContingentlyTrue(((p)⁻))›
    by (AOT-subst ‹((p)⁻)› ‹¬p›)
       (auto simp: thm−relation−negation:3 1)
next
  AOT-assume 1: ‹ContingentlyTrue(((p)⁻))›
  AOT-have ‹ContingentlyTrue(¬p)›
    by (AOT-subst (reverse) ‹¬p› ‹((p)⁻)›)
       (auto simp add: thm−relation−negation:3 1)
  AOT-hence 2: ‹¬p & ◊¬¬p› using cont−tf:1[THEN ≡df E] by blast
  AOT-have ‹◊p›
    by (AOT-subst p ‹¬¬p›)
       (auto simp add: oth−class−taut:3:b 2[THEN &E(2)])
  AOT-hence ‹¬p & ◊p› using 2[THEN &E(1)] &I by blast
  AOT-thus ‹ContingentlyFalse(p)›
    by (rule cont−tf:2[THEN ≡df I])
qed

AOT-theorem cont−true−cont:5:
  ‹(ContingentlyTrue((p)) & Necessary0((q))) → p ≠ q›
proof (rule →I; frule &E(1); drule &E(2); rule raa−cor:1)
  AOT-assume ‹ContingentlyTrue((p))›
  AOT-hence ‹◊¬p›
    using cont−tf:1[THEN ≡df E] &E by blast
  AOT-hence 0: ‹¬□p› using KBasic:11[THEN ≡E(2)] by blast
  AOT-assume ‹Necessary0((q))›
  moreover AOT-assume ‹¬(p ≠ q)›
  AOT-hence ‹p = q›
    using =−infix[THEN ≡Df,
                  THEN oth−class−taut:4:b[THEN ≡E(1)],
                  THEN ≡E(1)]
          useful−tautologies:1[THEN →E] by blast
  ultimately AOT-have ‹Necessary0((p))› using rule=E id-sym by blast
  AOT-hence ‹□p›
    using contingent−properties:1[zero][THEN ≡df E] by blast
  AOT-thus ‹□p & ¬□p› using 0 &I by blast
qed
```

```
AOT-theorem cont–true–cont:6:
  ‹(ContingentlyFalse((p)) & Impossible0((q))) → p ≠ q›
proof (rule →I; frule &E(1); drule &E(2); rule raa–cor:1)
  AOT-assume ‹ContingentlyFalse((p))›
  AOT-hence ‹◇p›
    using cont–tf:2[THEN ≡df E] &E by blast
  AOT-hence 1: ‹¬□¬p›
    using conventions:5[THEN ≡df E] by blast
  AOT-assume ‹Impossible0((q))›
  moreover AOT-assume ‹¬(p ≠ q)›
  AOT-hence ‹p = q›
    using =–infix[THEN ≡Df,
                  THEN oth–class–taut:4:b[THEN ≡E(1)],
                  THEN ≡E(1)]
          useful–tautologies:1[THEN →E] by blast
  ultimately AOT-have ‹Impossible0((p))› using rule=E id-sym by blast
  AOT-hence ‹□¬p›
    using contingent–properties:2[zero][THEN ≡df E] by blast
  AOT-thus ‹□¬p & ¬□¬p› using 1 &I by blast
qed

AOT-act-theorem q0cf:1: ‹ContingentlyFalse(q0)›
  apply (rule cont–tf:2[THEN ≡df I])
  apply (rule =df I(2)[OF q0-def])
   apply (fact log–prop–prop:2)
  apply (rule &I)
   apply (fact no–cnac)
  by (fact qml:4[axiom-inst])

AOT-act-theorem q0cf:2: ‹ContingentlyTrue(((q0)⁻))›
  apply (rule cont–tf:1[THEN ≡df I])
  apply (rule =df I(2)[OF q0-def])
   apply (fact log–prop–prop:2)
  apply (rule &I)
   apply (rule thm–relation–negation:3
              [unvarify p, OF log–prop–prop:2, THEN ≡E(2)])
     apply (fact no–cnac)
  apply (rule rule=E[rotated,
                OF thm–relation–negation:7
                   [unvarify p, OF log–prop–prop:2, THEN id-sym]])
  apply (AOT-subst (reverse) ‹¬¬(∃x (E!x & ¬𝓐E!x))› ‹∃x (E!x & ¬𝓐E!x)›)
  by (auto simp: oth–class–taut:3:b qml:4[axiom-inst])

AOT-theorem cont–tf–thm:1: ‹∃p ContingentlyTrue((p))›
proof(rule ∨E(1)[OF exc–mid]; rule →I; rule ∃I)
  AOT-assume ‹q0›
  AOT-hence ‹q0 & ◇¬q0› using q0-prop[THEN &E(2)] &I by blast
  AOT-thus ‹ContingentlyTrue(q0)›
    by (rule cont–tf:1[THEN ≡df I])
next
  AOT-assume ‹¬q0›
  AOT-hence ‹¬q0 & ◇q0› using q0-prop[THEN &E(1)] &I by blast
  AOT-hence ‹ContingentlyFalse(q0)›
    by (rule cont–tf:2[THEN ≡df I])
  AOT-thus ‹ContingentlyTrue(((q0)⁻))›
    by (rule cont–true–cont:4[unvarify p,
                OF log–prop–prop:2, THEN ≡E(1)])
qed(auto simp: log–prop–prop:2)


AOT-theorem cont–tf–thm:2: ‹∃p ContingentlyFalse((p))›
proof(rule ∨E(1)[OF exc–mid]; rule →I; rule ∃I)
  AOT-assume ‹q0›
```

AOT-hence ‹q_0 & $\Diamond\neg q_0$› **using** *q_0-prop*[*THEN* &*E*(*2*)] &*I* **by** *blast*
AOT-hence ‹*ContingentlyTrue*(q_0)›
by (*rule cont−tf*:*1*[*THEN* $\equiv_{df}$*I*])
AOT-thus ‹*ContingentlyFalse*(((q_0)$^-$))›
by (*rule cont−true−cont*:*3*[*unvarify p*,
OF log−prop−prop:*2*, *THEN* ≡*E*(*1*)])
next
AOT-assume ‹$\neg q_0$›
AOT-hence ‹$\neg q_0$ & $\Diamond q_0$› **using** *q_0-prop*[*THEN* &*E*(*1*)] &*I* **by** *blast*
AOT-thus ‹*ContingentlyFalse*(q_0)›
by (*rule cont−tf*:*2*[*THEN* $\equiv_{df}$*I*])
qed(*auto simp*: *log−prop−prop*:*2*)

AOT-theorem *property−facts1*:*1*: ‹∃ *F*∃ *x* ([*F*]*x* & ◊¬[*F*]*x*)›
proof −
fix *x*
AOT-obtain p_1 **where** ‹*ContingentlyTrue*((p_1))›
using *cont−tf−thm*:*1* ∃ *E*[*rotated*] **by** *blast*
AOT-hence *1*: ‹p_1 & $\Diamond\neg p_1$› **using** *cont−tf*:*1*[*THEN* $\equiv_{df}$*E*] **by** *blast*
AOT-modally-strict {
AOT-have ‹*for arbitrary p*: $\vdash_\Box$ ([λ*z p*]*x* ≡ *p*)›
by (*rule beta−C−cor*:*3*[*THEN* ∀ *E*(*2*)]) *cqt-2-lambda-inst-prover*
AOT-hence ‹*for arbitrary p*: $\vdash_\Box$ □ ([λ*z p*]*x* ≡ *p*)›
by (*rule RN*)
AOT-hence ‹∀ *p* □([λ*z p*]*x* ≡ *p*)› **using** *GEN* **by** *fast*
AOT-hence ‹□([λ*z* p_1]*x* ≡ p_1)› **using** ∀ *E* **by** *fast*
} **note** *2* = *this*
AOT-hence ‹□([λ*z* p_1]*x* ≡ p_1)› **using** ∀ *E* **by** *blast*
AOT-hence ‹[λ*z* p_1]*x*›
using *1*[*THEN* &*E*(*1*)] *qml*:*2*[*axiom-inst*, *THEN* →*E*] ≡*E*(*2*) **by** *blast*
moreover AOT-have ‹◊¬[λ*z* p_1]*x*›
using *2*[*THEN qml*:*2*[*axiom-inst*, *THEN* →*E*]]
apply (*AOT-subst* ‹[λ*z* p_1]*x*› ‹p_1›)
using *1*[*THEN* &*E*(*2*)] **by** *blast*
ultimately AOT-have ‹[λ*z* p_1]*x* & ◊¬[λ*z* p_1]*x*› **using** &*I* **by** *blast*
AOT-hence ‹∃ *x* ([λ*z* p_1]*x* & ◊¬[λ*z* p_1]*x*)› **using** ∃ *I*(*2*) **by** *fast*
moreover AOT-have ‹[λ*z* p_1]↓› **by** *cqt*:*2*[*lambda*]
ultimately AOT-show ‹∃ *F*∃ *x* ([*F*]*x* & ◊¬[*F*]*x*)› **by** (*rule* ∃ *I*(*1*))
qed

AOT-theorem *property−facts1*:*2*: ‹∃ *F*∃ *x* (¬[*F*]*x* & ◊[*F*]*x*)›
proof −
fix *x*
AOT-obtain p_1 **where** ‹*ContingentlyFalse*((p_1))›
using *cont−tf−thm*:*2* ∃ *E*[*rotated*] **by** *blast*
AOT-hence *1*: ‹$\neg p_1$ & $\Diamond p_1$› **using** *cont−tf*:*2*[*THEN* $\equiv_{df}$*E*] **by** *blast*
AOT-modally-strict {
AOT-have ‹*for arbitrary p*: $\vdash_\Box$ ([λ*z p*]*x* ≡ *p*)›
by (*rule beta−C−cor*:*3*[*THEN* ∀ *E*(*2*)]) *cqt-2-lambda-inst-prover*
AOT-hence ‹*for arbitrary p*: $\vdash_\Box$ (¬[λ*z p*]*x* ≡ ¬*p*)›
using *oth−class−taut*:*4*:*b* ≡*E* **by** *blast*
AOT-hence ‹*for arbitrary p*: $\vdash_\Box$ □(¬[λ*z p*]*x* ≡ ¬*p*)›
by (*rule RN*)
AOT-hence ‹∀ *p* □(¬[λ*z p*]*x* ≡ ¬*p*)› **using** *GEN* **by** *fast*
AOT-hence ‹□(¬[λ*z* p_1]*x* ≡ ¬p_1)› **using** ∀ *E* **by** *fast*
} **note** *2* = *this*
AOT-hence ‹□(¬[λ*z* p_1]*x* ≡ ¬p_1)› **using** ∀ *E* **by** *blast*
AOT-hence *3*: ‹¬[λ*z* p_1]*x*›
using *1*[*THEN* &*E*(*1*)] *qml*:*2*[*axiom-inst*, *THEN* →*E*] ≡*E*(*2*) **by** *blast*
AOT-modally-strict {
AOT-have ‹*for arbitrary p*: $\vdash_\Box$ ([λ*z p*]*x* ≡ *p*)›
by (*rule beta−C−cor*:*3*[*THEN* ∀ *E*(*2*)]) *cqt-2-lambda-inst-prover*
AOT-hence ‹*for arbitrary p*: $\vdash_\Box$ □([λ*z p*]*x* ≡ *p*)›

by (*rule RN*)
AOT-hence ‹$\forall p\ \Box([\lambda z\ p]x \equiv p)$› **using** *GEN* **by** *fast*
AOT-hence ‹$\Box([\lambda z\ p_1]x \equiv p_1)$› **using** $\forall E$ **by** *fast*
} **note** *4* = *this*
AOT-have ‹$\Diamond[\lambda z\ p_1]x$›
using *4*[*THEN qml*:*2*[*axiom-inst*, *THEN* →*E*]]
apply (*AOT-subst* ‹$[\lambda z\ p_1]x$› ‹p_1›)
using *1*[*THEN* &*E*(*2*)] **by** *blast*
AOT-hence ‹$\neg[\lambda z\ p_1]x\ \&\ \Diamond[\lambda z\ p_1]x$› **using** *3* &*I* **by** *blast*
AOT-hence ‹$\exists x\ (\neg[\lambda z\ p_1]x\ \&\ \Diamond[\lambda z\ p_1]x)$› **using** $\exists I(2)$ **by** *fast*
moreover AOT-have ‹$[\lambda z\ p_1]\downarrow$› **by** *cqt*:*2*[*lambda*]
ultimately AOT-show ‹$\exists F \exists x\ (\neg[F]x\ \&\ \Diamond[F]x)$› **by** (*rule* $\exists I(1)$)
qed

context
begin

private AOT-lemma *eqnotnec-123-Aux-ζ*: ‹$[L]x \equiv (E!x \rightarrow E!x)$›
apply (*rule* $=_{df}I(2)$[*OF L-def*])
apply *cqt*:*2*[*lambda*]
apply (*rule beta−C−meta*[*THEN* →*E*])
by *cqt*:*2*[*lambda*]

private AOT-lemma *eqnotnec-123-Aux-ω*: ‹$[\lambda z\ \varphi]x \equiv \varphi$›
by (*rule beta−C−meta*[*THEN* →*E*]) *cqt*:*2*[*lambda*]

private AOT-lemma *eqnotnec-123-Aux-ϑ*: ‹$\varphi \equiv \forall x([L]x \equiv [\lambda z\ \varphi]x)$›
proof(*rule* ≡*I*; *rule* →*I*; (*rule* ∀ *I*)?)
fix *x*
AOT-assume *1*: ‹φ›
AOT-have ‹$[L]x \equiv (E!x \rightarrow E!x)$› **using** *eqnotnec-123-Aux-ζ*.
also AOT-have ‹$\ldots \equiv \varphi$›
using *if−p−then−p 1* ≡*I* →*I* **by** *simp*
also AOT-have ‹$\ldots \equiv [\lambda z\ \varphi]x$›
using *Commutativity of* ≡[*THEN* ≡*E*(*1*)] *eqnotnec-123-Aux-ω* **by** *blast*
finally AOT-show ‹$[L]x \equiv [\lambda z\ \varphi]x$›.
next
fix *x*
AOT-assume ‹$\forall x([L]x \equiv [\lambda z\ \varphi]x)$›
AOT-hence ‹$[L]x \equiv [\lambda z\ \varphi]x$› **using** ∀ *E* **by** *blast*
also AOT-have ‹$\ldots \equiv \varphi$› **using** *eqnotnec-123-Aux-ω*.
finally AOT-have ‹$\varphi \equiv [L]x$›
using *Commutativity of* ≡[*THEN* ≡*E*(*1*)] **by** *blast*
also AOT-have ‹$\ldots \equiv E!x \rightarrow E!x$› **using** *eqnotnec-123-Aux-ζ*.
finally AOT-show ‹φ› **using** ≡*E if−p−then−p* **by** *fast*
qed
private lemmas *eqnotnec-123-Aux-ξ* =
eqnotnec-123-Aux-ϑ[*THEN oth−class−taut*:*4*:*b*[*THEN* ≡*E*(*1*)],
THEN conventions:*3*[*THEN* ≡*Df*, *THEN* ≡*E*(*1*), *THEN* &*E*(*1*)],
THEN RM◇]
private lemmas *eqnotnec-123-Aux-ξ'* =
eqnotnec-123-Aux-ϑ[
THEN conventions:*3*[*THEN* ≡*Df*, *THEN* ≡*E*(*1*), *THEN* &*E*(*1*)],
THEN RM◇]

AOT-theorem *eqnotnec*:*1*: ‹$\exists F \exists G(\forall x([F]x \equiv [G]x)\ \&\ \Diamond\neg\forall x([F]x \equiv [G]x))$›
proof−
AOT-obtain p_1 **where** ‹*ContingentlyTrue*(p_1)›
using *cont−tf−thm*:*1* ∃ *E*[*rotated*] **by** *blast*
AOT-hence ‹$p_1\ \&\ \Diamond\neg p_1$› **using** *cont−tf*:*1*[*THEN* $\equiv_{df}E$] **by** *blast*
AOT-hence ‹$\forall x\ ([L]x \equiv [\lambda z\ p_1]x)\ \&\ \Diamond\neg\forall x([L]x \equiv [\lambda z\ p_1]x)$›
apply − **apply** (*rule* &*I*)
using &*E eqnotnec-123-Aux-ϑ*[*THEN* ≡*E*(*1*)]

eqnotnec-123-Aux-ξ →*E* **by** *fast+*
AOT-hence ‹∃ *G* (∀ *x*([*L*]*x* ≡ [*G*]*x*) & ◇¬∀ *x*([*L*]*x* ≡ [*G*]*x*))›
by (*rule* ∃ *I*) *cqt*:*2*[*lambda*]
AOT-thus ‹∃ *F*∃ *G* (∀ *x*([*F*]*x* ≡ [*G*]*x*) & ◇¬∀ *x*([*F*]*x* ≡ [*G*]*x*))›
apply (*rule* ∃ *I*)
by (*rule* $=_{df}$*I*(*2*)[*OF L-def*]) *cqt*:*2*[*lambda*]+
qed

AOT-theorem *eqnotnec*:*2*: ‹∃ *F*∃ *G*(¬∀ *x*([*F*]*x* ≡ [*G*]*x*) & ◇∀ *x*([*F*]*x* ≡ [*G*]*x*))›
proof−
AOT-obtain p_1 **where** ‹*ContingentlyFalse*(p_1)›
using *cont*−*tf*−*thm*:*2* ∃ *E*[*rotated*] **by** *blast*
AOT-hence ‹¬p_1 & ◇p_1› **using** *cont*−*tf*:*2*[*THEN* $\equiv_{df}$*E*] **by** *blast*
AOT-hence ‹¬∀ *x* ([*L*]*x* ≡ [λ*z* p_1]*x*) & ◇∀ *x*([*L*]*x* ≡ [λ*z* p_1]*x*)›
apply − **apply** (*rule* &*I*)
using *eqnotnec-123-Aux-ϑ*[*THEN oth*−*class*−*taut*:*4*:*b*[*THEN* ≡*E*(*1*)],
THEN ≡*E*(*1*)]
&*E* *eqnotnec-123-Aux-ξ*′ →*E* **by** *fast+*
AOT-hence ‹∃ *G* (¬∀ *x*([*L*]*x* ≡ [*G*]*x*) & ◇∀ *x*([*L*]*x* ≡ [*G*]*x*))›
by (*rule* ∃ *I*) *cqt*:*2*[*lambda*]
AOT-thus ‹∃ *F*∃ *G* (¬∀ *x*([*F*]*x* ≡ [*G*]*x*) & ◇∀ *x*([*F*]*x* ≡ [*G*]*x*))›
apply (*rule* ∃ *I*)
by (*rule* $=_{df}$*I*(*2*)[*OF L-def*]) *cqt*:*2*[*lambda*]+
qed

AOT-theorem *eqnotnec*:*3*: ‹∃ *F*∃ *G*(𝓐¬∀ *x*([*F*]*x* ≡ [*G*]*x*) & ◇∀ *x*([*F*]*x* ≡ [*G*]*x*))›
proof−
AOT-have ‹¬𝓐q_0›
apply (*rule* $=_{df}$*I*(*2*)[*OF* q_0-*def*])
apply (*fact log*−*prop*−*prop*:*2*)
by (*fact AOT*)
AOT-hence ‹𝓐¬q_0›
using *logic*−*actual*−*nec*:*1*[*axiom-inst*, *THEN* ≡*E*(*2*)] **by** *blast*
AOT-hence ‹𝓐¬∀ *x* ([*L*]*x* ≡ [λ*z* q_0]*x*)›
using *eqnotnec-123-Aux-ϑ*[*THEN oth*−*class*−*taut*:*4*:*b*[*THEN* ≡*E*(*1*)],
THEN conventions:*3*[*THEN* ≡*Df*, *THEN* ≡*E*(*1*), *THEN* &*E*(*1*)],
THEN RA[*2*], *THEN act*−*cond*[*THEN* →*E*], *THEN* →*E*] **by** *blast*
moreover AOT-have ‹◇∀ *x* ([*L*]*x* ≡ [λ*z* q_0]*x*)›
using *eqnotnec-123-Aux-ξ*′[*THEN* →*E*] q_0-*prop*[*THEN* &*E*(*1*)] **by** *blast*
ultimately AOT-have ‹𝓐¬∀ *x* ([*L*]*x* ≡ [λ*z* q_0]*x*) & ◇∀ *x* ([*L*]*x* ≡ [λ*z* q_0]*x*)›
using &*I* **by** *blast*
AOT-hence ‹∃ *G* (𝓐¬∀ *x*([*L*]*x* ≡ [*G*]*x*) & ◇∀ *x*([*L*]*x* ≡ [*G*]*x*))›
by (*rule* ∃ *I*) *cqt*:*2*[*lambda*]
AOT-thus ‹∃ *F*∃ *G* (𝓐¬∀ *x*([*F*]*x* ≡ [*G*]*x*) & ◇∀ *x*([*F*]*x* ≡ [*G*]*x*))›
apply (*rule* ∃ *I*)
by (*rule* $=_{df}$*I*(*2*)[*OF L-def*]) *cqt*:*2*[*lambda*]+
qed

end

AOT-theorem *eqnotnec*:*4*: ‹∀ *F*∃ *G*(∀ *x*([*F*]*x* ≡ [*G*]*x*) & ◇¬∀ *x*([*F*]*x* ≡ [*G*]*x*))›
proof(*rule GEN*)
fix *F*
AOT-have *Aux-A*: ‹$\vdash_\Box$ ψ → ∀ *x*([*F*]*x* ≡ [λ*z* [*F*]*z* & ψ]*x*)› **for** ψ
proof(*rule* →*I*; *rule GEN*)
AOT-modally-strict {
fix *x*
AOT-assume *0*: ‹ψ›
AOT-have ‹[λ*z* [*F*]*z* & ψ]*x* ≡ [*F*]*x* & ψ›
by (*rule beta*−*C*−*meta*[*THEN* →*E*]) *cqt*:*2*[*lambda*]
also AOT-have ‹... ≡ [*F*]*x*›
apply (*rule* ≡*I*; *rule* →*I*)
using ∨*E*(*3*)[*rotated*, *OF useful*−*tautologies*:*2*[*THEN* →*E*], *OF 0*] &*E*

```
    apply blast
    using 0 &I by blast
  finally AOT-show ‹[F]x ≡ [λz [F]z & ψ]x›
    using Commutativity of ≡[THEN ≡E(1)] by blast
  }
qed

AOT-have Aux-B: ‹⊢□ ψ → ∀x([F]x ≡ [λz [F]z & ψ ∨ ¬ψ]x)› for ψ
proof (rule →I; rule GEN)
  AOT-modally-strict {
    fix x
    AOT-assume 0: ‹ψ›
    AOT-have ‹[λz ([F]z & ψ) ∨ ¬ψ]x ≡ (([F]x & ψ) ∨ ¬ψ)›
      by (rule beta−C−meta[THEN →E]) cqt:2[lambda]
    also AOT-have ‹... ≡ [F]x›
      apply (rule ≡I; rule →I)
      using ∨E(3)[rotated, OF useful−tautologies:2[THEN →E], OF 0]
            &E
       apply blast
      apply (rule ∨I(1)) using 0 &I by blast
    finally AOT-show ‹[F]x ≡ [λz ([F]z & ψ) ∨ ¬ψ]x›
      using Commutativity of ≡[THEN ≡E(1)] by blast
  }
qed

AOT-have Aux-C:
  ‹⊢□ ◇¬ψ → ◇¬∀z([λz [F]z & ψ]z ≡ [λz [F]z & ψ ∨ ¬ψ]z)› for ψ
proof(rule RM◇; rule →I; rule raa−cor:2)
AOT-modally-strict {
    AOT-assume 0: ‹¬ψ›
    AOT-assume ‹∀z ([λz [F]z & ψ]z ≡ [λz [F]z & ψ ∨ ¬ψ]z)›
    AOT-hence ‹[λz [F]z & ψ]z ≡ [λz [F]z & ψ ∨ ¬ψ]z› for z
      using ∀E by blast
    moreover AOT-have ‹[λz [F]z & ψ]z ≡ [F]z & ψ› for z
        by (rule beta−C−meta[THEN →E]) cqt:2[lambda]
    moreover AOT-have ‹[λz ([F]z & ψ) ∨ ¬ψ]z ≡ (([F]z & ψ) ∨ ¬ψ)› for z
      by (rule beta−C−meta[THEN →E]) cqt:2[lambda]
    ultimately AOT-have ‹[F]z & ψ ≡ (([F]z & ψ) ∨ ¬ψ)› for z
      using Commutativity of ≡[THEN ≡E(1)] ≡E(5) by meson
    moreover AOT-have ‹(([F]z & ψ) ∨ ¬ψ)› for z using 0 ∨I by blast
    ultimately AOT-have ‹ψ› using ≡E &E by metis
    AOT-thus ‹ψ & ¬ψ› using 0 &I by blast
  }
qed

AOT-have Aux-D: ‹□∀z ([F]z ≡ [λz [F]z & ψ]z) →
    (◇¬∀x ([λz [F]z & ψ]x ≡ [λz [F]z & ψ ∨ ¬ψ]x) ≡
     ◇¬∀x ([F]x ≡ [λz [F]z & ψ ∨ ¬ψ]x))› for ψ
proof (rule →I)
  AOT-assume A: ‹□∀z([F]z ≡ [λz [F]z & ψ]z)›
  AOT-show ‹◇¬∀x ([λz [F]z & ψ]x ≡ [λz [F]z & ψ ∨ ¬ψ]x) ≡
            ◇¬∀x ([F]x ≡ [λz [F]z & ψ ∨ ¬ψ]x)›
  proof(rule ≡I; rule KBasic:13[THEN →E];
        rule RN[prem][where Γ={«∀z([F]z ≡ [λz [F]z & ψ]z)»}, simplified];
        (rule useful−tautologies:5[THEN →E]; rule →I)?)
    AOT-modally-strict {
      AOT-assume ‹∀z ([F]z ≡ [λz [F]z & ψ]z)›
      AOT-hence 1: ‹[F]z ≡ [λz [F]z & ψ]z› for z
        using ∀E by blast
      AOT-assume ‹∀x ([F]x ≡ [λz [F]z & ψ ∨ ¬ψ]x)›
      AOT-hence 2: ‹[F]z ≡ [λz [F]z & ψ ∨ ¬ψ]z› for z
        using ∀E by blast
      AOT-have ‹[λz [F]z & ψ]z ≡ [λz [F]z & ψ ∨ ¬ψ]z› for z
```

```
        using ≡E 1 2 by meson
      AOT-thus ‹∀x ([λz [F]z & ψ]x ≡ [λz [F]z & ψ ∨ ¬ψ]x)›
        by (rule GEN)
    }
  next
    AOT-modally-strict {
      AOT-assume ‹∀z ([F]z ≡ [λz [F]z & ψ]z)›
      AOT-hence 1: ‹[F]z ≡ [λz [F]z & ψ]z› for z
        using ∀E by blast
      AOT-assume ‹∀x ([λz [F]z & ψ]x ≡ [λz [F]z & ψ ∨ ¬ψ]x)›
      AOT-hence 2: ‹[λz [F]z & ψ]z ≡ [λz [F]z & ψ ∨ ¬ψ]z› for z
        using ∀E by blast
      AOT-have ‹[F]z ≡ [λz [F]z & ψ ∨ ¬ψ]z› for z
        using 1 2 ≡E by meson
      AOT-thus ‹ ∀x ([F]x ≡ [λz [F]z & ψ ∨ ¬ψ]x)›
        by (rule GEN)
    }
  qed(auto simp: A)
qed

AOT-obtain p₁ where p₁-prop: ‹p₁ & ◇¬p₁›
  using cont−tf−thm:1 ∃E[rotated]
      cont−tf:1[THEN ≡df E] by blast
{
  AOT-assume 1: ‹□∀x([F]x ≡ [λz [F]z & p₁]x)›
  AOT-have 2: ‹∀x([F]x ≡ [λz [F]z & p₁ ∨ ¬p₁]x)›
    using Aux-B[THEN →E, OF p₁-prop[THEN &E(1)]].
  AOT-have ‹◇¬∀x([λz [F]z & p₁]x ≡ [λz [F]z & p₁ ∨ ¬p₁]x)›
    using Aux-C[THEN →E, OF p₁-prop[THEN &E(2)]].
  AOT-hence 3: ‹◇¬∀x([F]x ≡ [λz [F]z & p₁ ∨ ¬p₁]x)›
    using Aux-D[THEN →E, OF 1, THEN ≡E(1)] by blast
  AOT-hence ‹∀x([F]x ≡ [λz [F]z & p₁ ∨ ¬p₁]x) &
          ◇¬∀x([F]x ≡ [λz [F]z & p₁ ∨ ¬p₁]x)›
    using 2 &I by blast
  AOT-hence ‹∃G (∀x ([F]x ≡ [G]x) & ◇¬∀x([F]x ≡ [G]x))›
    by (rule ∃I(1)) cqt:2[lambda]
}
moreover {
  AOT-assume 2: ‹¬□∀x([F]x ≡ [λz [F]z & p₁]x)›
  AOT-hence ‹◇¬∀x([F]x ≡ [λz [F]z & p₁]x)›
    using KBasic:11[THEN ≡E(1)] by blast
  AOT-hence ‹∀x ([F]x ≡ [λz [F]z & p₁]x) & ◇¬∀x([F]x ≡ [λz [F]z & p₁]x)›
    using Aux-A[THEN →E, OF p₁-prop[THEN &E(1)]] &I by blast
  AOT-hence ‹∃G (∀x ([F]x ≡ [G]x) & ◇¬∀x([F]x ≡ [G]x))›
    by (rule ∃I(1)) cqt:2[lambda]
}
ultimately AOT-show ‹∃G (∀x ([F]x ≡ [G]x) & ◇¬∀x([F]x ≡ [G]x))›
  using ∨E(1)[OF exc−mid] →I by blast
qed

AOT-theorem eqnotnec:5: ‹∀F∃G(¬∀x([F]x ≡ [G]x) & ◇∀x([F]x ≡ [G]x))›
proof(rule GEN)
  fix F
  AOT-have Aux-A: ‹⊢□ ◇ψ → ◇∀x([F]x ≡ [λz [F]z & ψ]x)› for ψ
  proof(rule RM◇; rule →I; rule GEN)
    AOT-modally-strict {
    fix x
    AOT-assume 0: ‹ψ›
    AOT-have ‹[λz [F]z & ψ]x ≡ [F]x & ψ›
      by (rule beta−C−meta[THEN →E]) cqt:2[lambda]
    also AOT-have ‹... ≡ [F]x›
      apply (rule ≡I; rule →I)
      using ∨E(3)[rotated, OF useful−tautologies:2[THEN →E], OF 0] &E
```

apply *blast*
using *0* &*I* **by** *blast*
finally AOT-show ‹$[F]x \equiv [\lambda z\ [F]z\ \&\ \psi]x$›
using *Commutativity of* ≡[*THEN* ≡*E*(*1*)] **by** *blast*
}
qed

AOT-have *Aux-B*: ‹$\vdash_\Box \Diamond\psi \rightarrow \Diamond\forall x([F]x \equiv [\lambda z\ [F]z\ \&\ \psi \vee \neg\psi]x)$› **for** ψ
proof (*rule RM*◇; *rule* →*I*; *rule GEN*)
AOT-modally-strict {
fix x
AOT-assume *0*: ‹ψ›
AOT-have ‹$[\lambda z\ ([F]z\ \&\ \psi) \vee \neg\psi]x \equiv (([F]x\ \&\ \psi) \vee \neg\psi)$›
by (*rule beta*−*C*−*meta*[*THEN* →*E*]) *cqt*:*2*[*lambda*]
also AOT-have ‹... ≡ $[F]x$›
apply (*rule* ≡*I*; *rule* →*I*)
using ∨*E*(*3*)[*rotated*, *OF useful*−*tautologies*:*2*[*THEN* →*E*], *OF 0*] &*E*
apply *blast*
apply (*rule* ∨*I*(*1*)) **using** *0* &*I* **by** *blast*
finally AOT-show ‹$[F]x \equiv [\lambda z\ ([F]z\ \&\ \psi) \vee \neg\psi]x$›
using *Commutativity of* ≡[*THEN* ≡*E*(*1*)] **by** *blast*
}
qed

AOT-have *Aux-C*: ‹$\vdash_\Box \neg\psi \rightarrow \neg\forall z([\lambda z\ [F]z\ \&\ \psi]z \equiv [\lambda z\ [F]z\ \&\ \psi \vee \neg\psi]z)$› **for** ψ
proof(*rule* →*I*; *rule raa*−*cor*:*2*)
AOT-modally-strict {
AOT-assume *0*: ‹$\neg\psi$›
AOT-assume ‹$\forall z\ ([\lambda z\ [F]z\ \&\ \psi]z \equiv [\lambda z\ [F]z\ \&\ \psi \vee \neg\psi]z)$›
AOT-hence ‹$[\lambda z\ [F]z\ \&\ \psi]z \equiv [\lambda z\ [F]z\ \&\ \psi \vee \neg\psi]z$› **for** z
using ∀*E* **by** *blast*
moreover AOT-have ‹$[\lambda z\ [F]z\ \&\ \psi]z \equiv [F]z\ \&\ \psi$› **for** z
by (*rule beta*−*C*−*meta*[*THEN* →*E*]) *cqt*:*2*[*lambda*]
moreover AOT-have ‹$[\lambda z\ ([F]z\ \&\ \psi) \vee \neg\psi]z \equiv (([F]z\ \&\ \psi) \vee \neg\psi)$› **for** z
by (*rule beta*−*C*−*meta*[*THEN* →*E*]) *cqt*:*2*[*lambda*]
ultimately AOT-have ‹$[F]z\ \&\ \psi \equiv (([F]z\ \&\ \psi) \vee \neg\psi)$› **for** z
using *Commutativity of* ≡[*THEN* ≡*E*(*1*)] ≡*E*(*5*) **by** *meson*
moreover AOT-have ‹$(([F]z\ \&\ \psi) \vee \neg\psi)$› **for** z
using *0* ∨*I* **by** *blast*
ultimately AOT-have ‹ψ› **using** ≡*E* &*E* **by** *metis*
AOT-thus ‹$\psi\ \&\ \neg\psi$› **using** *0* &*I* **by** *blast*
}
qed

AOT-have *Aux-D*: ‹$\forall z\ ([F]z \equiv [\lambda z\ [F]z\ \&\ \psi]z) \rightarrow$
$(\neg\forall x\ ([\lambda z\ [F]z\ \&\ \psi]x \equiv [\lambda z\ [F]z\ \&\ \psi \vee \neg\psi]x) \equiv$
$\neg\forall x\ ([F]x \equiv [\lambda z\ [F]z\ \&\ \psi \vee \neg\psi]x))$› **for** ψ
proof (*rule* →*I*; *rule* ≡*I*;
(*rule useful*−*tautologies*:*5*[*THEN* →*E*]; *rule* →*I*)*?*)
AOT-modally-strict {
AOT-assume ‹$\forall z\ ([F]z \equiv [\lambda z\ [F]z\ \&\ \psi]z)$›
AOT-hence *1*: ‹$[F]z \equiv [\lambda z\ [F]z\ \&\ \psi]z$› **for** z
using ∀*E* **by** *blast*
AOT-assume ‹$\forall x\ ([F]x \equiv [\lambda z\ [F]z\ \&\ \psi \vee \neg\psi]x)$›
AOT-hence *2*: ‹$[F]z \equiv [\lambda z\ [F]z\ \&\ \psi \vee \neg\psi]z$› **for** z
using ∀*E* **by** *blast*
AOT-have ‹$[\lambda z\ [F]z\ \&\ \psi]z \equiv [\lambda z\ [F]z\ \&\ \psi \vee \neg\psi]z$› **for** z
using ≡*E 1 2* **by** *meson*
AOT-thus ‹$\forall x\ ([\lambda z\ [F]z\ \&\ \psi]x \equiv [\lambda z\ [F]z\ \&\ \psi \vee \neg\psi]x)$›
by (*rule GEN*)
}
next
AOT-modally-strict {

```
    AOT-assume ‹∀z ([F]z ≡ [λz [F]z & ψ]z)›
    AOT-hence 1: ‹[F]z ≡ [λz [F]z & ψ]z› for z
      using ∀E by blast
    AOT-assume ‹∀x ([λz [F]z & ψ]x ≡ [λz [F]z & ψ ∨ ¬ψ]x)›
    AOT-hence 2: ‹[λz [F]z & ψ]z ≡ [λz [F]z & ψ ∨ ¬ψ]z› for z
      using ∀E by blast
    AOT-have ‹[F]z ≡ [λz [F]z & ψ ∨ ¬ψ]z› for z
      using 1 2 ≡E by meson
    AOT-thus ‹ ∀x ([F]x ≡ [λz [F]z & ψ ∨ ¬ψ]x)›
      by (rule GEN)
  }
qed

AOT-obtain p_1 where p_1-prop: ‹¬p_1 & ◇p_1›
  using cont−tf−thm:2 ∃E[rotated] cont−tf:2[THEN ≡_df E] by blast
{
  AOT-assume 1: ‹∀x([F]x ≡ [λz [F]z & p_1]x)›
  AOT-have 2: ‹◇∀x([F]x ≡ [λz [F]z & p_1 ∨ ¬p_1]x)›
    using Aux-B[THEN →E, OF p_1-prop[THEN &E(2)]].
  AOT-have ‹¬∀x([λz [F]z & p_1]x ≡ [λz [F]z & p_1 ∨ ¬p_1]x)›
    using Aux-C[THEN →E, OF p_1-prop[THEN &E(1)]].
  AOT-hence 3: ‹¬∀x([F]x ≡ [λz [F]z & p_1 ∨ ¬p_1]x)›
    using Aux-D[THEN →E, OF 1, THEN ≡E(1)] by blast
  AOT-hence ‹¬∀x([F]x ≡ [λz [F]z & p_1 ∨ ¬p_1]x) &
             ◇∀x([F]x ≡ [λz [F]z & p_1 ∨ ¬p_1]x)›
    using 2 &I by blast
  AOT-hence ‹∃G (¬∀x ([F]x ≡ [G]x) & ◇∀x([F]x ≡ [G]x))›
    by (rule ∃I(1)) cqt:2[lambda]
}
moreover {
  AOT-assume 2: ‹¬∀x([F]x ≡ [λz [F]z & p_1]x)›
  AOT-hence ‹¬∀x([F]x ≡ [λz [F]z & p_1]x)›
    using KBasic:11[THEN ≡E(1)] by blast
  AOT-hence ‹¬∀x ([F]x ≡ [λz [F]z & p_1]x) &
             ◇∀x([F]x ≡ [λz [F]z & p_1]x)›
    using Aux-A[THEN →E, OF p_1-prop[THEN &E(2)]] &I by blast
  AOT-hence ‹∃G (¬∀x ([F]x ≡ [G]x) & ◇∀x([F]x ≡ [G]x))›
    by (rule ∃I(1)) cqt:2[lambda]
}
ultimately AOT-show ‹∃G (¬∀x ([F]x ≡ [G]x) & ◇∀x([F]x ≡ [G]x))›
  using ∨E(1)[OF exc−mid] →I by blast
qed

AOT-theorem eqnotnec:6: ‹∀F∃G(𝒜¬∀x([F]x ≡ [G]x) & ◇∀x([F]x ≡ [G]x))›
proof(rule GEN)
  fix F
  AOT-have Aux-A: ‹⊢□ ◇ψ → ◇∀x([F]x ≡ [λz [F]z & ψ]x)› for ψ
  proof(rule RM◇; rule →I; rule GEN)
    AOT-modally-strict {
    fix x
    AOT-assume 0: ‹ψ›
    AOT-have ‹[λz [F]z & ψ]x ≡ [F]x & ψ›
      by (rule beta−C−meta[THEN →E]) cqt:2[lambda]
    also AOT-have ‹... ≡ [F]x›
      apply (rule ≡I; rule →I)
      using ∨E(3)[rotated, OF useful−tautologies:2[THEN →E], OF 0]
            &E
       apply blast
      using 0 &I by blast
    finally AOT-show ‹[F]x ≡ [λz [F]z & ψ]x›
      using Commutativity of ≡[THEN ≡E(1)] by blast
    }
  qed
```

```
AOT-have Aux-B: ‹⊢□ ◇ψ → ◇∀x([F]x ≡ [λz [F]z & ψ ∨ ¬ψ]x)› for ψ
proof (rule RM◇; rule →I; rule GEN)
  AOT-modally-strict {
    fix x
    AOT-assume 0: ‹ψ›
    AOT-have ‹[λz ([F]z & ψ) ∨ ¬ψ]x ≡ (([F]x & ψ) ∨ ¬ψ)›
      by (rule beta−C−meta[THEN →E]) cqt:2[lambda]
    also AOT-have ‹... ≡ [F]x›
      apply (rule ≡I; rule →I)
      using ∨E(3)[rotated, OF useful−tautologies:2[THEN →E], OF 0] &E
       apply blast
      apply (rule ∨I(1)) using 0 &I by blast
    finally AOT-show ‹[F]x ≡ [λz ([F]z & ψ) ∨ ¬ψ]x›
      using Commutativity of ≡[THEN ≡E(1)] by blast
  }
qed

AOT-have Aux-C:
  ‹⊢□ 𝓐¬ψ → 𝓐¬∀z([λz [F]z & ψ]z ≡ [λz [F]z & ψ ∨ ¬ψ]z)› for ψ
proof(rule act−cond[THEN →E]; rule RA[2]; rule →I; rule raa−cor:2)
AOT-modally-strict {
    AOT-assume 0: ‹¬ψ›
    AOT-assume ‹∀z ([λz [F]z & ψ]z ≡ [λz [F]z & ψ ∨ ¬ψ]z)›
    AOT-hence ‹[λz [F]z & ψ]z ≡ [λz [F]z & ψ ∨ ¬ψ]z› for z
      using ∀E by blast
    moreover AOT-have ‹[λz [F]z & ψ]z ≡ [F]z & ψ› for z
        by (rule beta−C−meta[THEN →E]) cqt:2[lambda]
    moreover AOT-have ‹[λz ([F]z & ψ) ∨ ¬ψ]z ≡ (([F]z & ψ) ∨ ¬ψ)› for z
      by (rule beta−C−meta[THEN →E]) cqt:2[lambda]
    ultimately AOT-have ‹[F]z & ψ ≡ (([F]z & ψ) ∨ ¬ψ)› for z
      using Commutativity of ≡[THEN ≡E(1)] ≡E(5) by meson
    moreover AOT-have ‹(([F]z & ψ) ∨ ¬ψ)› for z
      using 0 ∨I by blast
    ultimately AOT-have ‹ψ› using ≡E &E by metis
    AOT-thus ‹ψ & ¬ψ› using 0 &I by blast
  }
qed

AOT-have ‹□(∀z ([F]z ≡ [λz [F]z & ψ]z) →
  (¬∀x ([λz [F]z & ψ]x ≡ [λz [F]z & ψ ∨ ¬ψ]x) ≡
   ¬∀x ([F]x ≡ [λz [F]z & ψ ∨ ¬ψ]x)))› for ψ
proof (rule RN; rule →I)
  AOT-modally-strict {
      AOT-assume ‹∀z ([F]z ≡ [λz [F]z & ψ]z)›
      AOT-thus ‹¬∀x ([λz [F]z & ψ]x ≡ [λz [F]z & ψ ∨ ¬ψ]x) ≡
                ¬∀x ([F]x ≡ [λz [F]z & ψ ∨ ¬ψ]x)›
        apply −
      proof(rule ≡I; (rule useful−tautologies:5[THEN →E]; rule →I)?)
        AOT-assume ‹∀z ([F]z ≡ [λz [F]z & ψ]z)›
        AOT-hence 1: ‹[F]z ≡ [λz [F]z & ψ]z› for z
          using ∀E by blast
        AOT-assume ‹∀x ([F]x ≡ [λz [F]z & ψ ∨ ¬ψ]x)›
        AOT-hence 2: ‹[F]z ≡ [λz [F]z & ψ ∨ ¬ψ]z› for z
          using ∀E by blast
        AOT-have ‹[λz [F]z & ψ]z ≡ [λz [F]z & ψ ∨ ¬ψ]z› for z
          using ≡E 1 2 by meson
        AOT-thus ‹∀x ([λz [F]z & ψ]x ≡ [λz [F]z & ψ ∨ ¬ψ]x)›
          by (rule GEN)
      next
          AOT-assume ‹∀z ([F]z ≡ [λz [F]z & ψ]z)›
          AOT-hence 1: ‹[F]z ≡ [λz [F]z & ψ]z› for z
            using ∀E by blast
```

AOT-assume ‹$\forall x$ ([λz [F]z & ψ]x ≡ [λz [F]z & ψ ∨ ¬ψ]x)›
AOT-hence *2*: ‹[λz [F]z & ψ]z ≡ [λz [F]z & ψ ∨ ¬ψ]z› **for** z
using ∀ *E* **by** *blast*
AOT-have ‹[F]z ≡ [λz [F]z & ψ ∨ ¬ψ]z› **for** z
using *1 2* ≡*E* **by** *meson*
AOT-thus ‹ $\forall x$ ([F]x ≡ [λz [F]z & ψ ∨ ¬ψ]x)›
by (*rule GEN*)
qed
}
qed
AOT-hence ‹**𝓐**(∀ z ([F]z ≡ [λz [F]z & ψ]z) →
(¬∀ x ([λz [F]z & ψ]x ≡ [λz [F]z & ψ ∨ ¬ψ]x) ≡
¬∀ x ([F]x ≡ [λz [F]z & ψ ∨ ¬ψ]x)))› **for** ψ
using *nec−imp−act*[*THEN* →*E*] **by** *blast*
AOT-hence ‹**𝓐**∀ z ([F]z ≡ [λz [F]z & ψ]z) →
𝓐(¬∀ x ([λz [F]z & ψ]x ≡ [λz [F]z & ψ ∨ ¬ψ]x) ≡
¬∀ x ([F]x ≡ [λz [F]z & ψ ∨ ¬ψ]x))› **for** ψ
using *act−cond*[*THEN* →*E*] **by** *blast*
AOT-hence *Aux-D*: ‹**𝓐**∀ z ([F]z ≡ [λz [F]z & ψ]z) →
(**𝓐**¬∀ x ([λz [F]z & ψ]x ≡ [λz [F]z & ψ ∨ ¬ψ]x) ≡
𝓐¬∀ x ([F]x ≡ [λz [F]z & ψ ∨ ¬ψ]x))› **for** ψ
by (*auto intro*!: →*I Act−Basic*:*5*[*THEN* ≡*E*(*1*)] *dest*!: →*E*)

AOT-have ‹¬**𝓐**q_0›
apply (*rule* $=_{df}$*I*(*2*)[*OF* q_0*-def*])
apply (*fact log−prop−prop*:*2*)
by (*fact AOT*)
AOT-hence q_0*-prop-1*: ‹**𝓐**¬q_0›
using *logic−actual−nec*:*1*[*axiom-inst*, *THEN* ≡*E*(*2*)] **by** *blast*
{
AOT-assume *1*: ‹**𝓐**∀ x([F]x ≡ [λz [F]z & q_0]x)›
AOT-have *2*: ‹◇∀ x([F]x ≡ [λz [F]z & q_0 ∨ ¬q_0]x)›
using *Aux-B*[*THEN* →*E*, *OF* q_0*-prop*[*THEN* &*E*(*1*)]]**.**
AOT-have ‹**𝓐**¬∀ x([λz [F]z & q_0]x ≡ [λz [F]z & q_0 ∨ ¬q_0]x)›
using *Aux-C*[*THEN* →*E*, *OF* q_0*-prop-1*]**.**
AOT-hence *3*: ‹**𝓐**¬∀ x([F]x ≡ [λz [F]z & q_0 ∨ ¬q_0]x)›
using *Aux-D*[*THEN* →*E*, *OF 1*, *THEN* ≡*E*(*1*)] **by** *blast*
AOT-hence ‹**𝓐**¬∀ x([F]x ≡ [λz [F]z & q_0 ∨ ¬q_0]x) &
◇∀ x([F]x ≡ [λz [F]z & q_0 ∨ ¬q_0]x)›
using *2* &*I* **by** *blast*
AOT-hence ‹∃ G (**𝓐**¬∀ x ([F]x ≡ [G]x) & ◇∀ x([F]x ≡ [G]x))›
by (*rule* ∃ *I*(*1*)) *cqt*:*2*[*lambda*]
}
moreover {
AOT-assume *2*: ‹¬**𝓐**∀ x([F]x ≡ [λz [F]z & q_0]x)›
AOT-hence ‹**𝓐**¬∀ x([F]x ≡ [λz [F]z & q_0]x)›
using *logic−actual−nec*:*1*[*axiom-inst*, *THEN* ≡*E*(*2*)] **by** *blast*
AOT-hence ‹**𝓐**¬∀ x ([F]x ≡ [λz [F]z & q_0]x) & ◇∀ x([F]x ≡ [λz [F]z & q_0]x)›
using *Aux-A*[*THEN* →*E*, *OF* q_0*-prop*[*THEN* &*E*(*1*)]] &*I* **by** *blast*
AOT-hence ‹∃ G (**𝓐**¬∀ x ([F]x ≡ [G]x) & ◇∀ x([F]x ≡ [G]x))›
by (*rule* ∃ *I*(*1*)) *cqt*:*2*[*lambda*]
}
ultimately AOT-show ‹∃ G (**𝓐**¬∀ x ([F]x ≡ [G]x) & ◇∀ x([F]x ≡ [G]x))›
using ∨*E*(*1*)[*OF exc−mid*] →*I* **by** *blast*
qed

AOT-theorem *oa−contingent*:*1*: ‹$O!$ ≠ $A!$›
proof(*rule* $\equiv_{df}$*I*[*OF* =−*infix*]; *rule raa−cor*:*2*)
fix x
AOT-assume *1*: ‹$O!$ = $A!$›
AOT-hence ‹[λx ◇$E!x$] = $A!$›
by (*rule* $=_{df}$*E*(*2*)[*OF AOT-ordinary*, *rotated*]) *cqt*:*2*[*lambda*]
AOT-hence ‹[λx ◇$E!x$] = [λx ¬◇$E!x$]›

by (*rule* $=_{df}E(2)$[*OF AOT-abstract*, *rotated*]) *cqt*:*2*[*lambda*]
moreover AOT-have ‹$[\lambda x\ \Diamond E!x]x \equiv \Diamond E!x$›
by (*rule beta−C−meta*[*THEN* →*E*]) *cqt*:*2*[*lambda*]
ultimately AOT-have ‹$[\lambda x\ \neg\Diamond E!x]x \equiv \Diamond E!x$›
using *rule*=*E* **by** *fast*
moreover AOT-have ‹$[\lambda x\ \neg\Diamond E!x]x \equiv \neg\Diamond E!x$›
by (*rule beta−C−meta*[*THEN* →*E*]) *cqt*:*2*[*lambda*]
ultimately AOT-have ‹$\Diamond E!x \equiv \neg\Diamond E!x$›
using ≡*E*(*6*) *Commutativity of* ≡[*THEN* ≡*E*(*1*)] **by** *blast*
AOT-thus $(\Diamond E!x \equiv \neg\Diamond E!x)\ \&\ \neg(\Diamond E!x \equiv \neg\Diamond E!x)$
using *oth−class−taut*:*3*:*c* &*I* **by** *blast*
qed

AOT-theorem *oa−contingent*:*2*: ‹$O!x \equiv \neg A!x$›
proof −
AOT-have ‹$O!x \equiv [\lambda x\ \Diamond E!x]x$›
apply (*rule* ≡*I*; *rule* →*I*)
apply (*rule* $=_{df}E(2)$[*OF AOT-ordinary*])
apply *cqt*:*2*[*lambda*]
apply *argo*
apply (*rule* $=_{df}I(2)$[*OF AOT-ordinary*])
apply *cqt*:*2*[*lambda*]
by *argo*
also AOT-have ‹$\ldots \equiv \Diamond E!x$›
by (*rule beta−C−meta*[*THEN* →*E*]) *cqt*:*2*[*lambda*]
also AOT-have ‹$\ldots \equiv \neg\neg\Diamond E!x$›
using *oth−class−taut*:*3*:*b*.
also AOT-have ‹$\ldots \equiv \neg[\lambda x\ \neg\Diamond E!x]x$›
by (*rule beta−C−meta*[*THEN* →*E*,
THEN oth−class−taut:*4*:*b*[*THEN* ≡*E*(*1*)], *symmetric*])
cqt:*2*
also AOT-have ‹$\ldots \equiv \neg A!x$›
apply (*rule* ≡*I*; *rule* →*I*)
apply (*rule* $=_{df}I(2)$[*OF AOT-abstract*])
apply *cqt*:*2*[*lambda*]
apply *argo*
apply (*rule* $=_{df}E(2)$[*OF AOT-abstract*])
apply *cqt*:*2*[*lambda*]
by *argo*
finally show *?thesis*.
qed

AOT-theorem *oa−contingent*:*3*: ‹$A!x \equiv \neg O!x$›
by (*AOT-subst* ‹$A!x$› ‹$\neg\neg A!x$›)
(*auto simp add*: *oth−class−taut*:*3*:*b oa−contingent*:*2*[*THEN*
oth−class−taut:*4*:*b*[*THEN* ≡*E*(*1*)], *symmetric*])

AOT-theorem *oa−contingent*:*4*: ‹*Contingent*(*O*!)›
proof (*rule thm−cont−prop*:*2*[*unvarify F*, *OF oa−exist*:*1*, *THEN* ≡*E*(*2*)];
rule &*I*)
AOT-have ‹$\Diamond\exists x\ E!x$› **using** *thm−cont−e*:*3* **.**
AOT-hence ‹$\exists x\ \Diamond E!x$› **using** *BF*◇[*THEN* →*E*] **by** *blast*
then AOT-obtain *a* **where** ‹$\Diamond E!a$› **using** ∃*E*[*rotated*] **by** *blast*
AOT-hence ‹$[\lambda x\ \Diamond E!x]a$›
by (*rule beta−C−meta*[*THEN* →*E*, *THEN* ≡*E*(*2*), *rotated*]) *cqt*:*2*
AOT-hence ‹$O!a$›
by (*rule* $=_{df}I(2)$[*OF AOT-ordinary*, *rotated*]) *cqt*:*2*
AOT-hence ‹$\exists x\ O!x$› **using** ∃*I* **by** *blast*
AOT-thus ‹$\Diamond\exists x\ O!x$› **using** *T*◇[*THEN* →*E*] **by** *blast*
next
AOT-obtain *a* **where** ‹$A!a$›
using *A−objects*[*axiom-inst*] ∃*E*[*rotated*] &*E* **by** *blast*
AOT-hence ‹$\neg O!a$› **using** *oa−contingent*:*3*[*THEN* ≡*E*(*1*)] **by** *blast*

```
  AOT-hence ‹∃ x ¬O!x› using ∃ I by fast
  AOT-thus ‹◇∃ x ¬O!x› using T◇[THEN →E] by blast
qed

AOT-theorem oa−contingent:5: ‹Contingent(A!)›
proof (rule thm−cont−prop:2[unvarify F, OF oa−exist:2, THEN ≡E(2)];
       rule &I)
  AOT-obtain a where ‹A!a›
    using A−objects[axiom-inst] ∃ E[rotated] &E by blast
  AOT-hence ‹∃ x A!x› using ∃ I by fast
  AOT-thus ‹◇∃ x A!x› using T◇[THEN →E] by blast
next
  AOT-have ‹◇∃ x E!x› using thm−cont−e:3 .
  AOT-hence ‹∃ x ◇E!x› using BF◇[THEN →E] by blast
  then AOT-obtain a where ‹◇E!a› using ∃ E[rotated] by blast
  AOT-hence ‹[λx ◇E!x]a›
    by (rule beta−C−meta[THEN →E, THEN ≡E(2), rotated]) cqt:2[lambda]
  AOT-hence ‹O!a›
    by (rule =df I(2)[OF AOT-ordinary, rotated]) cqt:2[lambda]
  AOT-hence ‹¬A!a› using oa−contingent:2[THEN ≡E(1)] by blast
  AOT-hence ‹∃ x ¬A!x› using ∃ I by fast
  AOT-thus ‹◇∃ x ¬A!x› using T◇[THEN →E] by blast
qed

AOT-theorem oa−contingent:7: ‹O!⁻x ≡ ¬A!⁻x›
proof −
  AOT-have ‹O!x ≡ ¬A!x›
    using oa−contingent:2 by blast
  also AOT-have ‹. . . ≡ A!⁻x›
    using thm−relation−negation:1[symmetric, unvarify F, OF oa−exist:2].
  finally AOT-have 1: ‹O!x ≡ A!⁻x›.

  AOT-have ‹A!x ≡ ¬O!x›
    using oa−contingent:3 by blast
  also AOT-have ‹. . . ≡ O!⁻x›
    using thm−relation−negation:1[symmetric, unvarify F, OF oa−exist:1].
  finally AOT-have 2: ‹A!x ≡ O!⁻x›.

  AOT-show ‹O!⁻x ≡ ¬A!⁻x›
    using 1[THEN oth−class−taut:4:b[THEN ≡E(1)]]
          oa−contingent:3[of - x] 2[symmetric]
          ≡E(5) by blast
qed

AOT-theorem oa−contingent:6: ‹O!⁻ ≠ A!⁻›
proof (rule =−infix[THEN ≡df I]; rule raa−cor:2)
  AOT-assume 1: ‹O!⁻ = A!⁻›
  fix x
  AOT-have ‹A!⁻x ≡ O!⁻x›
    apply (rule rule=E[rotated, OF 1])
    by (fact oth−class−taut:3:a)
  AOT-hence ‹A!⁻x ≡ ¬A!⁻x›
    using oa−contingent:7 ≡E by fast
  AOT-thus ‹(A!⁻x ≡ ¬A!⁻x) & ¬(A!⁻x ≡ ¬A!⁻x)›
    using oth−class−taut:3:c &I by blast
qed

AOT-theorem oa−contingent:8: ‹Contingent(O!⁻)›
  using thm−cont−prop:3[unvarify F, OF oa−exist:1, THEN ≡E(1),
          OF oa−contingent:4].

AOT-theorem oa−contingent:9: ‹Contingent(A!⁻)›
  using thm−cont−prop:3[unvarify F, OF oa−exist:2, THEN ≡E(1),
```

OF oa−contingent:5].

AOT-define *WeaklyContingent* :: ‹Π ⇒ φ› (‹*WeaklyContingent*′(-′)›)
df−cont−nec:
‹*WeaklyContingent*([*F*]) $\equiv_{df}$ *Contingent*([*F*]) & ∀ *x* (◇[*F*]*x* → □[*F*]*x*)›

AOT-theorem *cont−nec−fact1*:*1*:
‹*WeaklyContingent*([*F*]) ≡ *WeaklyContingent*([*F*]⁻)›
proof −
AOT-have ‹*WeaklyContingent*([*F*]) ≡ *Contingent*([*F*]) & ∀ *x* (◇[*F*]*x* → □[*F*]*x*)›
using *df−cont−nec*[*THEN* ≡*Df*] **by** *blast*
also AOT-have ‹... ≡ *Contingent*([*F*]⁻) & ∀ *x* (◇[*F*]*x* → □[*F*]*x*)›
apply (*rule oth−class−taut*:*8*:*f*[*THEN* ≡*E*(*2*)]; *rule* →*I*)
using *thm−cont−prop*:*3*.
also AOT-have ‹... ≡ *Contingent*([*F*]⁻) & ∀ *x* (◇[*F*]⁻*x* → □[*F*]⁻*x*)›
proof (*rule oth−class−taut*:*8*:*e*[*THEN* ≡*E*(*2*)];
rule →*I*; *rule* ≡*I*; *rule* →*I*; *rule GEN*; *rule* →*I*)
fix *x*
AOT-assume *0*: ‹∀ *x* (◇[*F*]*x* → □[*F*]*x*)›
AOT-assume *1*: ‹◇[*F*]⁻*x*›
AOT-have ‹◇¬[*F*]*x*›
by (*AOT-subst* (*reverse*) ‹¬[*F*]*x*› ‹[*F*]⁻*x*›)
(*auto simp add*: *thm−relation−negation*:*1 1*)
AOT-hence *2*: ‹¬□[*F*]*x*›
using *KBasic*:*11*[*THEN* ≡*E*(*2*)] **by** *blast*
AOT-show ‹□[*F*]⁻*x*›
proof (*rule raa−cor*:*1*)
AOT-assume *3*: ‹¬□[*F*]⁻*x*›
AOT-have ‹¬□¬[*F*]*x*›
by (*AOT-subst* (*reverse*) ‹¬[*F*]*x*› ‹[*F*]⁻*x*›)
(*auto simp add*: *thm−relation−negation*:*1 3*)
AOT-hence ‹◇[*F*]*x*›
using *conventions*:*5*[*THEN* $\equiv_{df}$*I*] **by** *simp*
AOT-hence ‹□[*F*]*x*› **using** *0* ∀ *E* →*E* **by** *fast*
AOT-thus ‹□[*F*]*x* & ¬□[*F*]*x*› **using** &*I 2* **by** *blast*
qed
next
fix *x*
AOT-assume *0*: ‹∀ *x* (◇[*F*]⁻*x* → □[*F*]⁻*x*)›
AOT-assume *1*: ‹◇[*F*]*x*›
AOT-have ‹◇¬[*F*]⁻*x*›
by (*AOT-subst* ‹¬[*F*]⁻*x*› ‹[*F*]*x*›)
(*auto simp*: *thm−relation−negation*:*2 1*)
AOT-hence *2*: ‹¬□[*F*]⁻*x*›
using *KBasic*:*11*[*THEN* ≡*E*(*2*)] **by** *blast*
AOT-show ‹□[*F*]*x*›
proof (*rule raa−cor*:*1*)
AOT-assume *3*: ‹¬□[*F*]*x*›
AOT-have ‹¬□¬[*F*]⁻*x*›
by (*AOT-subst* ‹¬[*F*]⁻*x*› ‹[*F*]*x*›)
(*auto simp add*: *thm−relation−negation*:*2 3*)
AOT-hence ‹◇[*F*]⁻*x*›
using *conventions*:*5*[*THEN* $\equiv_{df}$*I*] **by** *simp*
AOT-hence ‹□[*F*]⁻*x*› **using** *0* ∀ *E* →*E* **by** *fast*
AOT-thus ‹□[*F*]⁻*x* & ¬□[*F*]⁻*x*› **using** &*I 2* **by** *blast*
qed
qed
also AOT-have ‹... ≡ *WeaklyContingent*([*F*]⁻)›
using *df−cont−nec*[*THEN* ≡*Df*, *symmetric*] **by** *blast*
finally show *?thesis*.
qed

AOT-theorem *cont−nec−fact1*:*2*:

‹(*WeaklyContingent*([*F*]) & ¬*WeaklyContingent*([*G*])) → *F* ≠ *G*›
proof (*rule* →*I*; *rule* =−*infix*[*THEN* $\equiv_{df}$*I*]; *rule raa*−*cor*:*2*)
  AOT-assume *1*: ‹*WeaklyContingent*([*F*]) & ¬*WeaklyContingent*([*G*])›
  AOT-hence ‹*WeaklyContingent*([*F*])› **using** &*E* **by** *blast*
  moreover AOT-assume ‹*F* = *G*›
  ultimately AOT-have ‹*WeaklyContingent*([*G*])›
    using *rule*=*E* **by** *blast*
  AOT-thus ‹*WeaklyContingent*([*G*]) & ¬*WeaklyContingent*([*G*])›
    using *1* &*I* &*E* **by** *blast*
qed

AOT-theorem *cont*−*nec*−*fact2*:*1*: ‹*WeaklyContingent*(*O*!)›
proof (*rule df*−*cont*−*nec*[*THEN* $\equiv_{df}$*I*]; *rule* &*I*)
  AOT-show ‹*Contingent*(*O*!)›
    using *oa*−*contingent*:*4*.
next
  AOT-show ‹∀ *x* (◊[*O*!]*x* → □[*O*!]*x*)›
    apply (*rule GEN*; *rule* →*I*)
    using *oa*−*facts*:*5*[*THEN* ≡*E*(*1*)] **by** *blast*
qed

AOT-theorem *cont*−*nec*−*fact2*:*2*: ‹*WeaklyContingent*(*A*!)›
proof (*rule df*−*cont*−*nec*[*THEN* $\equiv_{df}$*I*]; *rule* &*I*)
  AOT-show ‹*Contingent*(*A*!)›
    using *oa*−*contingent*:*5*.
next
  AOT-show ‹∀ *x* (◊[*A*!]*x* → □[*A*!]*x*)›
    apply (*rule GEN*; *rule* →*I*)
    using *oa*−*facts*:*6*[*THEN* ≡*E*(*1*)] **by** *blast*
qed

AOT-theorem *cont*−*nec*−*fact2*:*3*: ‹¬*WeaklyContingent*(*E*!)›
proof (*rule df*−*cont*−*nec*[*THEN* ≡*Df*,
                THEN oth−*class*−*taut*:*4*:*b*[*THEN* ≡*E*(*1*)],
                THEN ≡*E*(*2*)];
      rule DeMorgan(*1*)[*THEN* ≡*E*(*2*)]; *rule* ∨*I*(*2*); *rule raa*−*cor*:*2*)
  AOT-have ‹◊∃ *x* (*E*!*x* & ¬𝓐*E*!*x*)› **using** *qml*:*4*[*axiom-inst*].
  AOT-hence ‹∃ *x* ◊(*E*!*x* & ¬𝓐*E*!*x*)› **using** *BF*◊[*THEN* →*E*] **by** *blast*
  then AOT-obtain *a* **where** ‹◊(*E*!*a* & ¬𝓐*E*!*a*)› **using** ∃ *E*[*rotated*] **by** *blast*
  AOT-hence *1*: ‹◊*E*!*a* & ◊¬𝓐*E*!*a*› **using** *KBasic2*:*3*[*THEN* →*E*] **by** *simp*
  moreover AOT-assume ‹∀ *x* (◊[*E*!]*x* → □[*E*!]*x*)›
  ultimately AOT-have ‹□*E*!*a*› **using** &*E* ∀ *E* →*E* **by** *fast*
  AOT-hence ‹𝓐*E*!*a*› **using** *nec*−*imp*−*act*[*THEN* →*E*] **by** *blast*
  AOT-hence ‹□𝓐*E*!*a*› **using** *qml*−*act*:*1*[*axiom-inst*, *THEN* →*E*] **by** *blast*
  moreover AOT-have ‹¬□𝓐*E*!*a*›
    using *KBasic*:*11*[*THEN* ≡*E*(*2*)] *1*[*THEN* &*E*(*2*)] **by** *meson*
  ultimately AOT-have ‹□𝓐*E*!*a* & ¬□𝓐*E*!*a*› **using** &*I* **by** *blast*
  AOT-thus ‹*p* & ¬*p*› **for** *p* **using** *raa*−*cor*:*1* **by** *blast*
qed

AOT-theorem *cont*−*nec*−*fact2*:*4*: ‹¬*WeaklyContingent*(*L*)›
  apply (*rule df*−*cont*−*nec*[*THEN* ≡*Df*,
                THEN oth−*class*−*taut*:*4*:*b*[*THEN* ≡*E*(*1*)],
                THEN ≡*E*(*2*)];
      rule DeMorgan(*1*)[*THEN* ≡*E*(*2*)]; *rule* ∨*I*(*1*))
  apply (*rule contingent*−*properties*:*4*
        [*THEN* ≡*Df*,
         THEN oth−*class*−*taut*:*4*:*b*[*THEN* ≡*E*(*1*)],
         THEN ≡*E*(*2*)])
  apply (*rule DeMorgan*(*1*)[*THEN* ≡*E*(*2*)];
      rule ∨*I*(*2*);
      rule useful−*tautologies*:*2*[*THEN* →*E*])

```
  using thm−noncont−e−e:3[THEN contingent−properties:3[THEN ≡df E]].

AOT-theorem cont−nec−fact2:5: ‹O! ≠ E! & O! ≠ E!⁻ & O! ≠ L & O! ≠ L⁻›
proof −
  AOT-have 1: ‹L↓›
    by (rule =df I(2)[OF L-def]) cqt:2[lambda]+
  {
    fix φ and Π Π' :: ‹<κ>›
    AOT-have A: ‹¬(φ{Π'} ≡ φ{Π})› if ‹φ{Π}› and ‹¬φ{Π'}›
    proof (rule raa−cor:2)
      AOT-assume ‹φ{Π'} ≡ φ{Π}›
      AOT-hence ‹φ{Π'}› using that(1) ≡E by blast
      AOT-thus ‹φ{Π'} & ¬φ{Π'}› using that(2) &I by blast
    qed
    AOT-have ‹Π' ≠ Π› if ‹Π↓› and ‹Π'↓› and ‹φ{Π}› and ‹¬φ{Π'}›
      using pos−not−equiv−ne:4[unvarify F G, THEN →E,
                              OF that(1,2), OF A[OF that(3, 4)]].
  } note 0 = this
  show ?thesis
    apply(safe intro!: &I; rule 0)
    apply cqt:2
    using oa−exist:1 apply blast
    using cont−nec−fact2:3 apply fast
    apply (rule useful−tautologies:2[THEN →E])
    using cont−nec−fact2:1 apply fast
    using rel−neg−T:3 apply fast
    using oa−exist:1 apply blast
    using cont−nec−fact1:1[THEN oth−class−taut:4:b[THEN ≡E(1)],
          THEN ≡E(1), rotated, OF cont−nec−fact2:3] apply fast
    apply (rule useful−tautologies:2[THEN →E])
    using cont−nec−fact2:1 apply blast
    apply (rule =df I(2)[OF L-def]; cqt:2[lambda])
    using oa−exist:1 apply fast
    using cont−nec−fact2:4 apply fast
    apply (rule useful−tautologies:2[THEN →E])
    using cont−nec−fact2:1 apply fast
    using rel−neg−T:3 apply fast
    using oa−exist:1 apply fast
     apply (rule cont−nec−fact1:1[unvarify F,
                 THEN oth−class−taut:4:b[THEN ≡E(1)],
                 THEN ≡E(1), rotated, OF cont−nec−fact2:4])
    apply (rule =df I(2)[OF L-def]; cqt:2[lambda])
    apply (rule useful−tautologies:2[THEN →E])
    using cont−nec−fact2:1 by blast
qed

AOT-theorem cont−nec−fact2:6: ‹A! ≠ E! & A! ≠ E!⁻ & A! ≠ L & A! ≠ L⁻›
proof −
  AOT-have 1: ‹L↓›
    by (rule =df I(2)[OF L-def]) cqt:2[lambda]+
  {
    fix φ and Π Π' :: ‹<κ>›
    AOT-have A: ‹¬(φ{Π'} ≡ φ{Π})› if ‹φ{Π}› and ‹¬φ{Π'}›
    proof (rule raa−cor:2)
      AOT-assume ‹φ{Π'} ≡ φ{Π}›
      AOT-hence ‹φ{Π'}› using that(1) ≡E by blast
      AOT-thus ‹φ{Π'} & ¬φ{Π'}› using that(2) &I by blast
    qed
    AOT-have ‹Π' ≠ Π› if ‹Π↓› and ‹Π'↓› and ‹φ{Π}› and ‹¬φ{Π'}›
      using pos−not−equiv−ne:4[unvarify F G, THEN →E,
            OF that(1,2), OF A[OF that(3, 4)]].
  } note 0 = this
  show ?thesis
```

apply(*safe intro*!: &I; *rule 0*)
apply *cqt*:*2*
using *oa−exist*:*2* **apply** *blast*
using *cont−nec−fact2*:*3* **apply** *fast*
apply (*rule useful−tautologies*:*2*[*THEN* →*E*])
using *cont−nec−fact2*:*2* **apply** *fast*
using *rel−neg−T*:*3* **apply** *fast*
using *oa−exist*:*2* **apply** *blast*
using *cont−nec−fact1*:*1*[*THEN oth−class−taut*:*4*:*b*[*THEN* ≡*E*(*1*)],
THEN ≡*E*(*1*), *rotated*, *OF cont−nec−fact2*:*3*] **apply** *fast*
apply (*rule useful−tautologies*:*2*[*THEN* →*E*])
using *cont−nec−fact2*:*2* **apply** *blast*
apply (*rule* $=_{df}$*I*(*2*)[*OF L-def*]; *cqt*:*2*[*lambda*])
using *oa−exist*:*2* **apply** *fast*
using *cont−nec−fact2*:*4* **apply** *fast*
apply (*rule useful−tautologies*:*2*[*THEN* →*E*])
using *cont−nec−fact2*:*2* **apply** *fast*
using *rel−neg−T*:*3* **apply** *fast*
using *oa−exist*:*2* **apply** *fast*
apply (*rule cont−nec−fact1*:*1*[*unvarify F*,
THEN oth−class−taut:*4*:*b*[*THEN* ≡*E*(*1*)],
THEN ≡*E*(*1*), *rotated*, *OF cont−nec−fact2*:*4*])
apply (*rule* $=_{df}$*I*(*2*)[*OF L-def*]; *cqt*:*2*[*lambda*])
apply (*rule useful−tautologies*:*2*[*THEN* →*E*])
using *cont−nec−fact2*:*2* **by** *blast*
qed

AOT-define *necessary-or-contingently-false* :: ‹$\varphi \Rightarrow \varphi$› (‹**Δ**-› [*49*] *54*)
‹$\mathbf{\Delta} p \equiv_{df} \Box p \vee (\neg \boldsymbol{\mathcal{A}} p$ & $\Diamond p)$›

AOT-theorem *sixteen*:
shows ‹$\exists F_1 \exists F_2 \exists F_3 \exists F_4 \exists F_5 \exists F_6 \exists F_7 \exists F_8 \exists F_9 \exists F_{10} \exists F_{11} \exists F_{12} \exists F_{13} \exists F_{14} \exists F_{15} \exists F_{16}$ (
«F_1::<κ>» $\neq F_2$ & $F_1 \neq F_3$ & $F_1 \neq F_4$ & $F_1 \neq F_5$ & $F_1 \neq F_6$ & $F_1 \neq F_7$ &
$F_1 \neq F_8$ & $F_1 \neq F_9$ & $F_1 \neq F_{10}$ & $F_1 \neq F_{11}$ & $F_1 \neq F_{12}$ & $F_1 \neq F_{13}$ &
$F_1 \neq F_{14}$ & $F_1 \neq F_{15}$ & $F_1 \neq F_{16}$ &
$F_2 \neq F_3$ & $F_2 \neq F_4$ & $F_2 \neq F_5$ & $F_2 \neq F_6$ & $F_2 \neq F_7$ & $F_2 \neq F_8$ &
$F_2 \neq F_9$ & $F_2 \neq F_{10}$ & $F_2 \neq F_{11}$ & $F_2 \neq F_{12}$ & $F_2 \neq F_{13}$ & $F_2 \neq F_{14}$ &
$F_2 \neq F_{15}$ & $F_2 \neq F_{16}$ &
$F_3 \neq F_4$ & $F_3 \neq F_5$ & $F_3 \neq F_6$ & $F_3 \neq F_7$ & $F_3 \neq F_8$ & $F_3 \neq F_9$ & $F_3 \neq F_{10}$ &
$F_3 \neq F_{11}$ & $F_3 \neq F_{12}$ & $F_3 \neq F_{13}$ & $F_3 \neq F_{14}$ & $F_3 \neq F_{15}$ & $F_3 \neq F_{16}$ &
$F_4 \neq F_5$ & $F_4 \neq F_6$ & $F_4 \neq F_7$ & $F_4 \neq F_8$ & $F_4 \neq F_9$ & $F_4 \neq F_{10}$ & $F_4 \neq F_{11}$ &
$F_4 \neq F_{12}$ & $F_4 \neq F_{13}$ & $F_4 \neq F_{14}$ & $F_4 \neq F_{15}$ & $F_4 \neq F_{16}$ &
$F_5 \neq F_6$ & $F_5 \neq F_7$ & $F_5 \neq F_8$ & $F_5 \neq F_9$ & $F_5 \neq F_{10}$ & $F_5 \neq F_{11}$ & $F_5 \neq F_{12}$ &
$F_5 \neq F_{13}$ & $F_5 \neq F_{14}$ & $F_5 \neq F_{15}$ & $F_5 \neq F_{16}$ &
$F_6 \neq F_7$ & $F_6 \neq F_8$ & $F_6 \neq F_9$ & $F_6 \neq F_{10}$ & $F_6 \neq F_{11}$ & $F_6 \neq F_{12}$ & $F_6 \neq F_{13}$ &
$F_6 \neq F_{14}$ & $F_6 \neq F_{15}$ & $F_6 \neq F_{16}$ &
$F_7 \neq F_8$ & $F_7 \neq F_9$ & $F_7 \neq F_{10}$ & $F_7 \neq F_{11}$ & $F_7 \neq F_{12}$ & $F_7 \neq F_{13}$ & $F_7 \neq F_{14}$ &
$F_7 \neq F_{15}$ & $F_7 \neq F_{16}$ &
$F_8 \neq F_9$ & $F_8 \neq F_{10}$ & $F_8 \neq F_{11}$ & $F_8 \neq F_{12}$ & $F_8 \neq F_{13}$ & $F_8 \neq F_{14}$ & $F_8 \neq F_{15}$ &
$F_8 \neq F_{16}$ &
$F_9 \neq F_{10}$ & $F_9 \neq F_{11}$ & $F_9 \neq F_{12}$ & $F_9 \neq F_{13}$ & $F_9 \neq F_{14}$ & $F_9 \neq F_{15}$ & $F_9 \neq F_{16}$ &
$F_{10} \neq F_{11}$ & $F_{10} \neq F_{12}$ & $F_{10} \neq F_{13}$ & $F_{10} \neq F_{14}$ & $F_{10} \neq F_{15}$ & $F_{10} \neq F_{16}$ &
$F_{11} \neq F_{12}$ & $F_{11} \neq F_{13}$ & $F_{11} \neq F_{14}$ & $F_{11} \neq F_{15}$ & $F_{11} \neq F_{16}$ &
$F_{12} \neq F_{13}$ & $F_{12} \neq F_{14}$ & $F_{12} \neq F_{15}$ & $F_{12} \neq F_{16}$ &
$F_{13} \neq F_{14}$ & $F_{13} \neq F_{15}$ & $F_{13} \neq F_{16}$ &
$F_{14} \neq F_{15}$ & $F_{14} \neq F_{16}$ &
$F_{15} \neq F_{16}$)›
proof −
AOT-have *Delta-pos*: ‹$\mathbf{\Delta}\varphi \rightarrow \Diamond\varphi$› **for** φ
proof(*rule* →*I*)
AOT-assume ‹$\mathbf{\Delta}\varphi$›
AOT-hence ‹$\Box\varphi \vee (\neg\boldsymbol{\mathcal{A}}\varphi$ & $\Diamond\varphi)$›
using $\equiv_{df}$*E*[*OF necessary-or-contingently-false*] **by** *blast*

```
  moreover {
    AOT-assume ‹□φ›
    AOT-hence ‹◇φ›
      by (metis B◇ T◇ vdash−properties:10)
  }
  moreover {
    AOT-assume ‹¬𝓐φ & ◇φ›
    AOT-hence ‹◇φ›
      using &E by blast
  }
  ultimately AOT-show ‹◇φ›
    by (metis ∨E(2) raa−cor:1)
qed

AOT-have act-and-not-nec-not-delta: ‹¬Δφ› if ‹𝓐φ› and ‹¬□φ› for φ
  using ≡df E &E(1) ∨E(2) necessary-or-contingently-false
        raa−cor:3 that(1,2) by blast
AOT-have act-and-pos-not-not-delta: ‹¬Δφ› if ‹𝓐φ› and ‹◇¬φ› for φ
  using KBasic:11 act-and-not-nec-not-delta ≡E(2) that(1,2) by blast
AOT-have impossible-delta: ‹¬Δφ› if ‹¬◇φ› for φ
  using Delta-pos modus−tollens:1 that by blast
AOT-have not-act-and-pos-delta: ‹Δφ› if ‹¬𝓐φ› and ‹◇φ› for φ
  by (meson ≡df I &I ∨I(2) necessary-or-contingently-false that(1,2))
AOT-have nec-delta: ‹Δφ› if ‹□φ› for φ
  using ≡df I ∨I(1) necessary-or-contingently-false that by blast

AOT-obtain a where a-prop: ‹A!a›
  using A−objects[axiom-inst] ∃E[rotated] &E by blast
AOT-obtain b where b-prop: ‹◇[E!]b & ¬𝓐[E!]b›
  using pos−not−pna:3 using ∃E[rotated] by blast

AOT-have b-ord: ‹[O!]b›
proof(rule =df I(2)[OF AOT-ordinary])
  AOT-show ‹[λx ◇[E!]x]↓› by cqt:2[lambda]
next
  AOT-show ‹[λx ◇[E!]x]b›
  proof (rule β←C(1); (cqt:2[lambda])?)
    AOT-show ‹b↓› by (rule cqt:2[const-var][axiom-inst])
    AOT-show ‹◇[E!]b› by (fact b-prop[THEN &E(1)])
  qed
qed

AOT-have nec-not-L-neg: ‹□¬[L⁻]x› for x
  using thm−noncont−e−e:2 contingent−properties:2[THEN ≡df E] &E
        CBF[THEN →E] ∀E by blast
AOT-have nec-L: ‹□[L]x› for x
  using thm−noncont−e−e:1 contingent−properties:1[THEN ≡df E]
    CBF[THEN →E] ∀E by blast

AOT-have act-ord-b: ‹𝓐[O!]b›
  using b-ord ≡E(1) oa−facts:7 by blast
AOT-have delta-ord-b: ‹Δ[O!]b›
  by (meson ≡df I b-ord ∨I(1) necessary-or-contingently-false
            oa−facts:1 →E)
AOT-have not-act-ord-a: ‹¬𝓐[O!]a›
  by (meson a-prop ≡E(1) ≡E(3) oa−contingent:3 oa−facts:7)
AOT-have not-delta-ord-a: ‹¬Δ[O!]a›
  by (metis Delta-pos ≡E(4) not-act-ord-a oa−facts:3 oa−facts:7
            reductio−aa:1 →E)

AOT-have not-act-abs-b: ‹¬𝓐[A!]b›
  by (meson b-ord ≡E(1) ≡E(3) oa−contingent:2 oa−facts:8)
AOT-have not-delta-abs-b: ‹¬Δ[A!]b›
```

proof(*rule raa−cor*:*2*)
  AOT-assume ‹$\boldsymbol{\Delta}[A!]b$›
  AOT-hence ‹$\Diamond[A!]b$›
    by (*metis Delta-pos vdash−properties*:*10*)
  AOT-thus ‹$[A!]b$ & $\neg[A!]b$›
    by (*metis b-ord* &*I* ≡*E*(*1*) *oa−contingent*:*2*
        oa−facts:*4* →*E*)
qed
AOT-have *act-abs-a*: ‹$\boldsymbol{\mathcal{A}}[A!]a$›
  using *a-prop* ≡*E*(*1*) *oa−facts*:*8* **by** *blast*
AOT-have *delta-abs-a*: ‹$\boldsymbol{\Delta}[A!]a$›
  by (*metis* $\equiv_{df}I$ *a-prop oa−facts*:*2* →*E* ∨*I*(*1*)
      necessary-or-contingently-false)

AOT-have *not-act-concrete-b*: ‹$\neg\boldsymbol{\mathcal{A}}[E!]b$›
  using *b-prop* &*E*(*2*) **by** *blast*
AOT-have *delta-concrete-b*: ‹$\boldsymbol{\Delta}[E!]b$›
proof (*rule* $\equiv_{df}I$[*OF necessary-or-contingently-false*];
    rule ∨*I*(*2*); *rule* &*I*)
  AOT-show ‹$\neg\boldsymbol{\mathcal{A}}[E!]b$› **using** *b-prop* &*E*(*2*) **by** *blast*
next
  AOT-show ‹$\Diamond[E!]b$› **using** *b-prop* &*E*(*1*) **by** *blast*
qed
AOT-have *not-act-concrete-a*: ‹$\neg\boldsymbol{\mathcal{A}}[E!]a$›
proof (*rule raa−cor*:*2*)
  AOT-assume ‹$\boldsymbol{\mathcal{A}}[E!]a$›
  AOT-hence *1*: ‹$\Diamond[E!]a$› **by** (*metis Act−Sub*:*3* →*E*)
  AOT-have ‹$[A!]a$› **by** (*simp add*: *a-prop*)
  AOT-hence ‹$[\lambda x\ \neg\Diamond[E!]x]a$›
    by (*rule* $=_{df}E(2)$[*OF AOT-abstract*, *rotated*]) *cqt*:*2*
  AOT-hence ‹$\neg\Diamond[E!]a$› **using** $\beta{\rightarrow}C(1)$ **by** *blast*
  AOT-thus ‹$\Diamond[E!]a$ & $\neg\Diamond[E!]a$› **using** *1* &*I* **by** *blast*
qed
AOT-have *not-delta-concrete-a*: ‹$\neg\boldsymbol{\Delta}[E!]a$›
proof (*rule raa−cor*:*2*)
  AOT-assume ‹$\boldsymbol{\Delta}[E!]a$›
  AOT-hence *1*: ‹$\Diamond[E!]a$› **by** (*metis Delta-pos vdash−properties*:*10*)
  AOT-have ‹$[A!]a$› **by** (*simp add*: *a-prop*)
  AOT-hence ‹$[\lambda x\ \neg\Diamond[E!]x]a$›
    by (*rule* $=_{df}E(2)$[*OF AOT-abstract*, *rotated*]) *cqt*:*2*[*lambda*]
  AOT-hence ‹$\neg\Diamond[E!]a$› **using** $\beta{\rightarrow}C(1)$ **by** *blast*
  AOT-thus ‹$\Diamond[E!]a$ & $\neg\Diamond[E!]a$› **using** *1* &*I* **by** *blast*
qed

AOT-have *not-act-q-zero*: ‹$\neg\boldsymbol{\mathcal{A}}q_0$›
  by (*meson log−prop−prop*:*2 pos−not−pna*:*1*
      q_0-*def reductio−aa*:*1 rule−id−df*:*2*:*a*[*zero*])
AOT-have *delta-q-zero*: ‹$\boldsymbol{\Delta}q_0$›
proof(*rule* $\equiv_{df}I$[*OF necessary-or-contingently-false*];
    rule ∨*I*(*2*); *rule* &*I*)
  AOT-show ‹$\neg\boldsymbol{\mathcal{A}}q_0$› **using** *not-act-q-zero*.
  AOT-show ‹$\Diamond q_0$› **by** (*meson* &*E*(*1*) q_0-*prop*)
qed
AOT-have *act-not-q-zero*: ‹$\boldsymbol{\mathcal{A}}\neg q_0$›
  using *Act−Basic*:*1* ∨*E*(*2*) *not-act-q-zero* **by** *blast*
AOT-have *not-delta-not-q-zero*: ‹$\neg\boldsymbol{\Delta}\neg q_0$›
  using $\equiv_{df}E$ *conventions*:*5 Act−Basic*:*1 act-and-not-nec-not-delta*
    &*E*(*1*) ∨*E*(*2*) *not-act-q-zero* q_0-*prop* **by** *blast*

AOT-have ‹$[L^-]{\downarrow}$› **by** (*simp add*: *rel−neg−T*:*3*)
moreover AOT-have ‹$\neg\boldsymbol{\mathcal{A}}[L^-]b$ & $\neg\boldsymbol{\Delta}[L^-]b$ & $\neg\boldsymbol{\mathcal{A}}[L^-]a$ & $\neg\boldsymbol{\Delta}[L^-]a$›
proof (*safe intro*!: &*I*)
  AOT-show ‹$\neg\boldsymbol{\mathcal{A}}[L^-]b$›

```
    by (meson ≡E(1) logic−actual−nec:1[axiom-inst] nec−imp−act
          nec-not-L-neg →E)
  AOT-show ‹¬Δ[L⁻]b›
    by (meson Delta-pos KBasic2:1 ≡E(1)
          modus−tollens:1 nec-not-L-neg)
  AOT-show ‹¬𝓐[L⁻]a›
    by (meson ≡E(1) logic−actual−nec:1[axiom-inst]
          nec−imp−act nec-not-L-neg →E)
  AOT-show ‹¬Δ[L⁻]a›
    using Delta-pos KBasic2:1 ≡E(1) modus−tollens:1
        nec-not-L-neg by blast
qed
ultimately AOT-obtain F_0 where ‹¬𝓐[F_0]b & ¬Δ[F_0]b & ¬𝓐[F_0]a & ¬Δ[F_0]a›
  using ∃I(1)[rotated, THEN ∃E[rotated]] by fastforce
AOT-hence ‹¬𝓐[F_0]b› and ‹¬Δ[F_0]b› and ‹¬𝓐[F_0]a› and ‹¬Δ[F_0]a›
  using &E by blast+
note props = this

let ?Π = «[λy [A!]y & q_0]»
AOT-modally-strict {
  AOT-have ‹[«?Π»]↓› by cqt:2[lambda]
} note 1 = this
moreover AOT-have ‹¬𝓐[«?Π»]b & ¬Δ[«?Π»]b & ¬𝓐[«?Π»]a & Δ[«?Π»]a›
proof (safe intro!: &I; AOT-subst ‹[λy A!y & q_0]x› ‹A!x & q_0› for: x)
  AOT-show ‹¬𝓐([A!]b & q_0)›
    using Act−Basic:2 &E(1) ≡E(1) not-act-abs-b raa−cor:3 by blast
next AOT-show ‹¬Δ([A!]b & q_0)›
    by (metis Delta-pos KBasic2:3 &E(1) ≡E(4) not-act-abs-b
          oa−facts:4 oa−facts:8 raa−cor:3 →E)
next AOT-show ‹¬𝓐([A!]a & q_0)›
    using Act−Basic:2 &E(2) ≡E(1) not-act-q-zero
        raa−cor:3 by blast
next AOT-show ‹Δ([A!]a & q_0)›
  proof (rule not-act-and-pos-delta)
    AOT-show ‹¬𝓐([A!]a & q_0)›
      using Act−Basic:2 &E(2) ≡E(4) not-act-q-zero
          raa−cor:3 by blast
  next AOT-show ‹◇([A!]a & q_0)›
      by (metis &I →E Delta-pos KBasic:16 &E(1) delta-abs-a
            ≡E(1) oa−facts:6 q_0-prop)
  qed
qed(auto simp: beta−C−meta[THEN →E, OF 1])
ultimately AOT-obtain F_1 where ‹¬𝓐[F_1]b & ¬Δ[F_1]b & ¬𝓐[F_1]a & Δ[F_1]a›
  using ∃I(1)[rotated, THEN ∃E[rotated]] by fastforce
AOT-hence ‹¬𝓐[F_1]b› and ‹¬Δ[F_1]b› and ‹¬𝓐[F_1]a› and ‹Δ[F_1]a›
  using &E by blast+
note props = props this

let ?Π = «[λy [A!]y & ¬q_0]»
AOT-modally-strict {
  AOT-have ‹[«?Π»]↓› by cqt:2[lambda]
} note 1 = this
moreover AOT-have ‹¬𝓐[«?Π»]b & ¬Δ[«?Π»]b & 𝓐[«?Π»]a & ¬Δ[«?Π»]a›
proof (safe intro!: &I; AOT-subst ‹[λy A!y & ¬q_0]x› ‹A!x & ¬q_0› for: x)
  AOT-show ‹¬𝓐([A!]b & ¬q_0)›
    using Act−Basic:2 &E(1) ≡E(1) not-act-abs-b raa−cor:3 by blast
next AOT-show ‹¬Δ([A!]b & ¬q_0)›
    by (meson RM◇ Delta-pos Conjunction Simplification(1) ≡E(4)
          modus−tollens:1 not-act-abs-b oa−facts:4 oa−facts:8)
next AOT-show ‹𝓐([A!]a & ¬q_0)›
    by (metis Act−Basic:1 Act−Basic:2 act-abs-a &I ∨E(2)
          ≡E(3) not-act-q-zero raa−cor:3)
next AOT-show ‹¬Δ([A!]a & ¬q_0)›
```

proof (*rule act-and-not-nec-not-delta*)
AOT-show ‹$\mathcal{A}$([A!]a & ¬q_0)›
by (*metis Act−Basic:1 Act−Basic:2 act-abs-a &I ∨E(2)*
≡E(3) not-act-q-zero raa−cor:3)
next
AOT-show ‹¬□([A!]a & ¬q_0)›
by (*metis KBasic2:1 KBasic:3 &E(1) &E(2) ≡E(4)*
q_0-prop raa−cor:3)
qed
qed(*auto simp: beta−C−meta[THEN →E, OF 1]*)
ultimately AOT-obtain F_2 **where** ‹¬$\mathcal{A}$[F_2]b & ¬**Δ**[F_2]b & $\mathcal{A}$[F_2]a & ¬**Δ**[F_2]a›
using ∃ *I(1)[rotated, THEN* ∃ *E[rotated]]* **by** *fastforce*
AOT-hence ‹¬$\mathcal{A}$[F_2]b› **and** ‹¬**Δ**[F_2]b› **and** ‹$\mathcal{A}$[F_2]a› **and** ‹¬**Δ**[F_2]a›
using &E **by** *blast+*
note *props = props this*

AOT-have *abstract-prop*: ‹¬$\mathcal{A}$[A!]b & ¬**Δ**[A!]b & $\mathcal{A}$[A!]a & **Δ**[A!]a›
using *act-abs-a &I delta-abs-a not-act-abs-b not-delta-abs-b*
by *presburger*
then AOT-obtain F_3 **where** ‹¬$\mathcal{A}$[F_3]b & ¬**Δ**[F_3]b & $\mathcal{A}$[F_3]a & **Δ**[F_3]a›
using ∃ *I(1)[rotated, THEN* ∃ *E[rotated]] oa−exist:2* **by** *fastforce*
AOT-hence ‹¬$\mathcal{A}$[F_3]b› **and** ‹¬**Δ**[F_3]b› **and** ‹$\mathcal{A}$[F_3]a› **and** ‹**Δ**[F_3]a›
using &E **by** *blast+*
note *props = props this*

AOT-have ‹¬$\mathcal{A}$[E!]b & **Δ**[E!]b & ¬$\mathcal{A}$[E!]a & ¬**Δ**[E!]a›
by (*meson &I delta-concrete-b not-act-concrete-a*
not-act-concrete-b not-delta-concrete-a)
then AOT-obtain F_4 **where** ‹¬$\mathcal{A}$[F_4]b & **Δ**[F_4]b & ¬$\mathcal{A}$[F_4]a & ¬**Δ**[F_4]a›
using ∃ *I(1)[rotated, THEN* ∃ *E[rotated]]*
by *fastforce*
AOT-hence ‹¬$\mathcal{A}$[F_4]b› **and** ‹**Δ**[F_4]b› **and** ‹¬$\mathcal{A}$[F_4]a› **and** ‹¬**Δ**[F_4]a›
using &E **by** *blast+*
note *props = props this*

AOT-modally-strict {
AOT-have ‹[λy q_0]↓› **by** *cqt:2[lambda]*
} note *1 = this*
moreover AOT-have ‹¬$\mathcal{A}$[λy q_0]b & **Δ**[λy q_0]b & ¬$\mathcal{A}$[λy q_0]a & **Δ**[λy q_0]a›
by (*safe intro!: &I; AOT-subst* ‹[λy q_0]b› ‹q_0› **for**: *b*)
(*auto simp: not-act-q-zero delta-q-zero beta−C−meta[THEN →E, OF 1]*)
ultimately AOT-obtain F_5 **where** ‹¬$\mathcal{A}$[F_5]b & **Δ**[F_5]b & ¬$\mathcal{A}$[F_5]a & **Δ**[F_5]a›
using ∃ *I(1)[rotated, THEN* ∃ *E[rotated]]*
by *fastforce*
AOT-hence ‹¬$\mathcal{A}$[F_5]b› **and** ‹**Δ**[F_5]b› **and** ‹¬$\mathcal{A}$[F_5]a› **and** ‹**Δ**[F_5]a›
using &E **by** *blast+*
note *props = props this*

let ?Π = «[λy [E!]y ∨ ([A!]y & ¬q_0)]»
AOT-modally-strict {
AOT-have ‹[«?Π»]↓› **by** *cqt:2[lambda]*
} note *1 = this*
moreover AOT-have ‹¬$\mathcal{A}$[«?Π»]b & **Δ**[«?Π»]b & $\mathcal{A}$[«?Π»]a & ¬**Δ**[«?Π»]a›
proof(*safe intro!: &I;*
AOT-subst ‹[λy E!y ∨ (A!y & ¬q_0)]x› ‹E!x ∨ (A!x & ¬q_0)› **for**: *x*)
AOT-have ‹$\mathcal{A}$¬([A!]b & ¬q_0)›
by (*metis Act−Basic:1 Act−Basic:2 abstract-prop &E(1) ∨E(2)*
≡E(1) raa−cor:3)
moreover AOT-have ‹¬$\mathcal{A}$[E!]b›
using *b-prop &E(2)* **by** *blast*
ultimately AOT-have *2*: ‹$\mathcal{A}$(¬[E!]b & ¬([A!]b & ¬q_0))›
by (*metis Act−Basic:2 Act−Sub:1 &I ≡E(3) raa−cor:1*)
AOT-have ‹$\mathcal{A}$¬([E!]b ∨ ([A!]b & ¬q_0))›

```
      by (AOT-subst ‹¬([E!]b ∨ ([A!]b & ¬q_0))› ‹¬[E!]b & ¬([A!]b & ¬q_0)›)
         (auto simp: oth−class−taut:5:d 2)
    AOT-thus ‹¬𝓐([E!]b ∨ ([A!]b & ¬q_0))›
      by (metis ¬¬I Act−Sub:1 ≡E(4))
  next
    AOT-show ‹𝚫([E!]b ∨ ([A!]b & ¬q_0))›
    proof (rule not-act-and-pos-delta)
      AOT-show ‹¬𝓐([E!]b ∨ ([A!]b & ¬q_0))›
        by (metis Act−Basic:2 Act−Basic:9 ∨E(2) raa−cor:3
                  Conjunction Simplification(1) ≡E(4)
                  modus−tollens:1 not-act-abs-b not-act-concrete-b)
    next
      AOT-show ‹◇([E!]b ∨ ([A!]b & ¬q_0))›
        using KBasic2:2 b-prop &E(1) ∨I(1) ≡E(3) raa−cor:3 by blast
    qed
  next AOT-show ‹𝓐([E!]a ∨ ([A!]a & ¬q_0))›
      by (metis Act−Basic:1 Act−Basic:2 Act−Basic:9 act-abs-a &I
              ∨I(2) ∨E(2) ≡E(3) not-act-q-zero raa−cor:1)
  next AOT-show ‹¬𝚫([E!]a ∨ ([A!]a & ¬q_0))›
    proof (rule act-and-not-nec-not-delta)
      AOT-show ‹𝓐([E!]a ∨ ([A!]a & ¬q_0))›
        by (metis Act−Basic:1 Act−Basic:2 Act−Basic:9 act-abs-a &I
                ∨I(2) ∨E(2) ≡E(3) not-act-q-zero raa−cor:1)
    next
      AOT-have ‹□¬[E!]a›
        by (metis ≡_df I conventions:5 &I ∨I(2)
                necessary-or-contingently-false
                not-act-concrete-a not-delta-concrete-a raa−cor:3)
      moreover AOT-have ‹◇¬([A!]a & ¬q_0)›
        by (metis KBasic2:1 KBasic:11 KBasic:3
                &E(1,2) ≡E(1) q_0-prop raa−cor:3)
      ultimately AOT-have ‹◇(¬[E!]a & ¬([A!]a & ¬q_0))›
        by (metis KBasic:16 &I vdash−properties:10)
      AOT-hence ‹◇¬([E!]a ∨ ([A!]a & ¬q_0))›
        by (metis RE◇ ≡E(2) oth−class−taut:5:d)
      AOT-thus ‹¬□([E!]a ∨ ([A!]a & ¬q_0))›
        by (metis KBasic:12 ≡E(1) raa−cor:3)
    qed
 qed(auto simp: beta−C−meta[THEN →E, OF 1])
  ultimately AOT-obtain F_6 where ‹¬𝓐[F_6]b & 𝚫[F_6]b & 𝓐[F_6]a & ¬𝚫[F_6]a›
    using ∃I(1)[rotated, THEN ∃E[rotated]] by fastforce
  AOT-hence ‹¬𝓐[F_6]b› and ‹𝚫[F_6]b› and ‹𝓐[F_6]a› and ‹¬𝚫[F_6]a›
    using &E by blast+
  note props = props this

  let ?Π = «[λy [A!]y ∨ [E!]y]»
  AOT-modally-strict {
    AOT-have ‹[«?Π»]↓› by cqt:2[lambda]
  } note 1 = this
  moreover AOT-have ‹¬𝓐[«?Π»]b & 𝚫[«?Π»]b & 𝓐[«?Π»]a & 𝚫[«?Π»]a›
  proof(safe intro!: &I; AOT-subst ‹[λy A!y ∨ E!y]x› ‹A!x ∨ E!x› for: x)
    AOT-show ‹¬𝓐([A!]b ∨ [E!]b)›
      using Act−Basic:9 ∨E(2) ≡E(4) not-act-abs-b
           not-act-concrete-b raa−cor:3 by blast
  next AOT-show ‹𝚫([A!]b ∨ [E!]b)›
    proof (rule not-act-and-pos-delta)
      AOT-show ‹¬𝓐([A!]b ∨ [E!]b)›
        using Act−Basic:9 ∨E(2) ≡E(4) not-act-abs-b
             not-act-concrete-b raa−cor:3 by blast
    next AOT-show ‹◇([A!]b ∨ [E!]b)›
        using KBasic2:2 b-prop &E(1) ∨I(2) ≡E(2) by blast
    qed
  next AOT-show ‹𝓐([A!]a ∨ [E!]a)›
```

```
    by (meson Act-Basic:9 act-abs-a ∨I(1) ≡E(2))
next AOT-show ‹Δ([A!]a ∨ [E!]a) ›
  proof (rule nec-delta)
    AOT-show ‹□([A!]a ∨ [E!]a)›
      by (metis KBasic:15 act-abs-a act-and-not-nec-not-delta
             Disjunction Addition(1) delta-abs-a raa-cor:3 →E)
  qed
qed(auto simp: beta-C-meta[THEN →E, OF 1])
ultimately AOT-obtain F₇ where ‹¬𝓐[F₇]b & Δ[F₇]b & 𝓐[F₇]a & Δ[F₇]a›
  using ∃I(1)[rotated, THEN ∃E[rotated]] by fastforce
AOT-hence ‹¬𝓐[F₇]b› and ‹Δ[F₇]b› and ‹𝓐[F₇]a› and ‹Δ[F₇]a›
  using &E by blast+
note props = props this

let ?Π = «[λy [O!]y & ¬[E!]y]»
AOT-modally-strict {
  AOT-have ‹[«?Π»]↓› by cqt:2[lambda]
} note 1 = this
moreover AOT-have ‹𝓐[«?Π»]b & ¬Δ[«?Π»]b & ¬𝓐[«?Π»]a & ¬Δ[«?Π»]a›
proof(safe intro!: &I; AOT-subst ‹[λy O!y & ¬E!y]x› ‹O!x & ¬E!x› for: x)
  AOT-show ‹𝓐([O!]b & ¬[E!]b)›
    by (metis Act-Basic:1 Act-Basic:2 act-ord-b &I ∨E(2)
           ≡E(3) not-act-concrete-b raa-cor:3)
next AOT-show ‹¬Δ([O!]b & ¬[E!]b)›
    by (metis (no-types, opaque-lifting) conventions:5 Act-Sub:1 RM:1
           act-and-not-nec-not-delta act-conj-act:3
           act-ord-b b-prop &I &E(1) Conjunction Simplification(2)
           df-rules-formulas[3]
           ≡E(3) raa-cor:1 →E)
next AOT-show ‹¬𝓐([O!]a & ¬[E!]a)›
    using Act-Basic:2 &E(1) ≡E(1) not-act-ord-a raa-cor:3 by blast
next AOT-have ‹¬◇([O!]a & ¬[E!]a)›
    by (metis KBasic2:3 &E(1) ≡E(4) not-act-ord-a oa-facts:3
           oa-facts:7 raa-cor:3 vdash-properties:10)
  AOT-thus ‹¬Δ([O!]a & ¬[E!]a)›
    by (rule impossible-delta)
qed(auto simp: beta-C-meta[THEN →E, OF 1])
ultimately AOT-obtain F₈ where ‹𝓐[F₈]b & ¬Δ[F₈]b & ¬𝓐[F₈]a & ¬Δ[F₈]a›
  using ∃I(1)[rotated, THEN ∃E[rotated]] by fastforce
AOT-hence ‹𝓐[F₈]b› and ‹¬Δ[F₈]b› and ‹¬𝓐[F₈]a› and ‹¬Δ[F₈]a›
  using &E by blast+
note props = props this

let ?Π = «[λy ¬[E!]y & ([O!]y ∨ q₀)]»
AOT-modally-strict {
  AOT-have ‹[«?Π»]↓› by cqt:2[lambda]
} note 1 = this
moreover AOT-have ‹𝓐[«?Π»]b & ¬Δ[«?Π»]b & ¬𝓐[«?Π»]a & Δ[«?Π»]a›
proof(safe intro!: &I;
      AOT-subst ‹[λy ¬E!y & (O!y ∨ q₀)]x› ‹¬E!x & (O!x ∨ q₀)› for: x)
  AOT-show ‹𝓐(¬[E!]b & ([O!]b ∨ q₀))›
    by (metis Act-Basic:1 Act-Basic:2 Act-Basic:9 act-ord-b &I
           ∨I(1) ∨E(2) ≡E(3) not-act-concrete-b raa-cor:1)
next AOT-show ‹¬Δ(¬[E!]b & ([O!]b ∨ q₀))›
  proof (rule act-and-pos-not-not-delta)
    AOT-show ‹𝓐(¬[E!]b & ([O!]b ∨ q₀))›
      by (metis Act-Basic:1 Act-Basic:2 Act-Basic:9 act-ord-b &I
             ∨I(1) ∨E(2) ≡E(3) not-act-concrete-b raa-cor:1)
  next
    AOT-show ‹◇¬(¬[E!]b & ([O!]b ∨ q₀))›
    proof (AOT-subst ‹¬(¬[E!]b & ([O!]b ∨ q₀))› ‹[E!]b ∨ ¬([O!]b ∨ q₀)›)
      AOT-modally-strict {
        AOT-show ‹¬(¬[E!]b & ([O!]b ∨ q₀)) ≡ [E!]b ∨ ¬([O!]b ∨ q₀)›
```

by (*metis* &*I* &*E*(*1*,*2*) ∨*I*(*1*,*2*) ∨*E*(*2*)
→*I* ≡*I* *reductio*−*aa*:*1*)
}
next
AOT-show ‹$\Diamond$([*E*!]*b* ∨ ¬([*O*!]*b* ∨ q_0))›
using *KBasic2*:*2* *b-prop* &*E*(*1*) ∨*I*(*1*) ≡*E*(*3*)
raa−*cor*:*3* **by** *blast*
qed
qed
next
AOT-show ‹¬𝓐(¬[*E*!]*a* & ([*O*!]*a* ∨ q_0))›
using *Act*−*Basic*:*2* *Act*−*Basic*:*9* &*E*(*2*) ∨*E*(*3*) ≡*E*(*1*)
not-act-ord-a *not-act-q-zero* *reductio*−*aa*:*2* **by** *blast*
next
AOT-show ‹**Δ**(¬[*E*!]*a* & ([*O*!]*a* ∨ q_0))›
proof (*rule* *not-act-and-pos-delta*)
AOT-show ‹¬𝓐(¬[*E*!]*a* & ([*O*!]*a* ∨ q_0))›
by (*metis* *Act*−*Basic*:*2* *Act*−*Basic*:*9* &*E*(*2*) ∨*E*(*3*) ≡*E*(*1*)
not-act-ord-a *not-act-q-zero* *reductio*−*aa*:*2*)
next
AOT-have ‹□¬[*E*!]*a*›
using *KBasic2*:*1* ≡*E*(*2*) *not-act-and-pos-delta* *not-act-concrete-a*
not-delta-concrete-a *raa*−*cor*:*5* **by** *blast*
moreover AOT-have ‹$\Diamond$([*O*!]*a* ∨ q_0)›
by (*metis* *KBasic2*:*2* &*E*(*1*) ∨*I*(*2*) ≡*E*(*3*) q_0-*prop* *raa*−*cor*:*3*)
ultimately AOT-show ‹$\Diamond$(¬[*E*!]*a* & ([*O*!]*a* ∨ q_0))›
by (*metis* *KBasic*:*16* &*I* *vdash*−*properties*:*10*)
qed
qed(*auto simp*: *beta*−*C*−*meta*[*THEN* →*E*, *OF* *1*])
ultimately AOT-obtain F_9 **where** ‹𝓐[F_9]*b* & ¬**Δ**[F_9]*b* & ¬𝓐[F_9]*a* & **Δ**[F_9]*a*›
using ∃*I*(*1*)[*rotated*, *THEN* ∃*E*[*rotated*]] **by** *fastforce*
AOT-hence ‹𝓐[F_9]*b*› **and** ‹¬**Δ**[F_9]*b*› **and** ‹¬𝓐[F_9]*a*› **and** ‹**Δ**[F_9]*a*›
using &*E* **by** *blast*+
note *props* = *props* *this*

AOT-modally-strict {
AOT-have ‹[λ*y* ¬q_0]↓› **by** *cqt*:*2*[*lambda*]
} note *1* = *this*
moreover AOT-have ‹𝓐[λ*y* ¬q_0]*b* & ¬**Δ**[λ*y* ¬q_0]*b* & 𝓐[λ*y* ¬q_0]*a* & ¬**Δ**[λ*y* ¬q_0]*a*›
by (*safe intro*!: &*I*; *AOT-subst* ‹[λ*y* ¬q_0]*x*› ‹¬q_0› **for**: *x*)
(*auto simp*: *act-not-q-zero* *not-delta-not-q-zero*
beta−*C*−*meta*[*THEN* →*E*, *OF* *1*])
ultimately AOT-obtain F_{10} **where** ‹𝓐[F_{10}]*b* & ¬**Δ**[F_{10}]*b* & 𝓐[F_{10}]*a* & ¬**Δ**[F_{10}]*a*›
using ∃*I*(*1*)[*rotated*, *THEN* ∃*E*[*rotated*]] **by** *fastforce*
AOT-hence ‹𝓐[F_{10}]*b*› **and** ‹¬**Δ**[F_{10}]*b*› **and** ‹𝓐[F_{10}]*a*› **and** ‹¬**Δ**[F_{10}]*a*›
using &*E* **by** *blast*+
note *props* = *props* *this*

AOT-modally-strict {
AOT-have ‹[λ*y* ¬[*E*!]*y*]↓› **by** *cqt*:*2*[*lambda*]
} note *1* = *this*
moreover AOT-have ‹𝓐[λ*y* ¬[*E*!]*y*]*b* & ¬**Δ**[λ*y* ¬[*E*!]*y*]*b* &
𝓐[λ*y* ¬[*E*!]*y*]*a* & **Δ**[λ*y* ¬[*E*!]*y*]*a*›
proof (*safe intro*!: &*I*; *AOT-subst* ‹[λ*y* ¬[*E*!]*y*]*x*› ‹¬[*E*!]*x*› **for**: *x*)
AOT-show ‹𝓐¬[*E*!]*b*›
using *Act*−*Basic*:*1* ∨*E*(*2*) *not-act-concrete-b* **by** *blast*
next AOT-show ‹¬**Δ**¬[*E*!]*b*›
using $\equiv_{df}$*E* *conventions*:*5* *Act*−*Basic*:*1* *act-and-not-nec-not-delta*
b-prop &*E*(*1*) ∨*E*(*2*) *not-act-concrete-b* **by** *blast*
next AOT-show ‹𝓐¬[*E*!]*a*›
using *Act*−*Basic*:*1* ∨*E*(*2*) *not-act-concrete-a* **by** *blast*
next AOT-show ‹**Δ**¬[*E*!]*a*›
using *KBasic2*:*1* ≡*E*(*2*) *nec-delta* *not-act-and-pos-delta*

```
        not-act-concrete-a not-delta-concrete-a reductio−aa:1
    by blast
qed(auto simp: beta−C−meta[THEN →E, OF 1])
ultimately AOT-obtain F11 where ‹𝓐[F11]b & ¬Δ[F11]b & 𝓐[F11]a & Δ[F11]a›
  using ∃I(1)[rotated, THEN ∃E[rotated]] by fastforce
AOT-hence ‹𝓐[F11]b› and ‹¬Δ[F11]b› and ‹𝓐[F11]a› and ‹Δ[F11]a›
  using &E by blast+
note props = props this

AOT-have ‹𝓐[O!]b & Δ[O!]b & ¬𝓐[O!]a & ¬Δ[O!]a›
  by (simp add: act-ord-b &I delta-ord-b not-act-ord-a not-delta-ord-a)
then AOT-obtain F12 where ‹𝓐[F12]b & Δ[F12]b & ¬𝓐[F12]a & ¬Δ[F12]a›
  using oa−exist:1 ∃I(1)[rotated, THEN ∃E[rotated]] by fastforce
AOT-hence ‹𝓐[F12]b› and ‹Δ[F12]b› and ‹¬𝓐[F12]a› and ‹¬Δ[F12]a›
  using &E by blast+
note props = props this

let ?Π = «[λy [O!]y ∨ q0]»
AOT-modally-strict {
  AOT-have ‹[«?Π»]↓› by cqt:2[lambda]
} note 1 = this
moreover AOT-have ‹𝓐[«?Π»]b & Δ[«?Π»]b & ¬𝓐[«?Π»]a & Δ[«?Π»]a›
proof (safe intro!: &I; AOT-subst ‹[λy O!y ∨ q0]x› ‹O!x ∨ q0› for: x)
  AOT-show ‹𝓐([O!]b ∨ q0)›
    by (meson Act−Basic:9 act-ord-b ∨I(1) ≡E(2))
next AOT-show ‹Δ([O!]b ∨ q0)›
    by (meson KBasic:15 b-ord ∨I(1) nec-delta oa−facts:1 →E)
next AOT-show ‹¬𝓐([O!]a ∨ q0)›
    using Act−Basic:9 ∨E(2) ≡E(4) not-act-ord-a
        not-act-q-zero raa−cor:3 by blast
next AOT-show ‹Δ([O!]a ∨ q0)›
  proof (rule not-act-and-pos-delta)
    AOT-show ‹¬𝓐([O!]a ∨ q0)›
      using Act−Basic:9 ∨E(2) ≡E(4) not-act-ord-a
          not-act-q-zero raa−cor:3 by blast
  next AOT-show ‹◇([O!]a ∨ q0)›
      using KBasic2:2 &E(1) ∨I(2) ≡E(2) q0-prop by blast
  qed
qed(auto simp: beta−C−meta[THEN →E, OF 1])
ultimately AOT-obtain F13 where ‹𝓐[F13]b & Δ[F13]b & ¬𝓐[F13]a & Δ[F13]a›
  using ∃I(1)[rotated, THEN ∃E[rotated]] by fastforce
AOT-hence ‹𝓐[F13]b› and ‹Δ[F13]b› and ‹¬𝓐[F13]a› and ‹Δ[F13]a›
  using &E by blast+
note props = props this

let ?Π = «[λy [O!]y ∨ ¬q0]»
AOT-modally-strict {
   AOT-have ‹[«?Π»]↓› by cqt:2[lambda]
} note 1 = this
moreover AOT-have ‹𝓐[«?Π»]b & Δ[«?Π»]b & 𝓐[«?Π»]a & ¬Δ[«?Π»]a›
proof (safe intro!: &I; AOT-subst ‹[λy O!y ∨ ¬q0]x› ‹O!x ∨ ¬q0› for: x)
  AOT-show ‹𝓐([O!]b ∨ ¬q0)›
    by (meson Act−Basic:9 act-not-q-zero ∨I(2) ≡E(2))
next AOT-show ‹Δ([O!]b ∨ ¬q0)›
    by (meson KBasic:15 b-ord ∨I(1) nec-delta oa−facts:1 →E)
next AOT-show ‹𝓐([O!]a ∨ ¬q0)›
    by (meson Act−Basic:9 act-not-q-zero ∨I(2) ≡E(2))
next AOT-show ‹¬Δ([O!]a ∨ ¬q0)›
  proof(rule act-and-pos-not-not-delta)
    AOT-show ‹𝓐([O!]a ∨ ¬q0)›
      by (meson Act−Basic:9 act-not-q-zero ∨I(2) ≡E(2))
  next
    AOT-have ‹□¬[O!]a›
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          using KBasic2:1 ≡E(2) not-act-and-pos-delta
                not-act-ord-a not-delta-ord-a raa−cor:6 by blast
        moreover AOT-have ‹◇q₀›
          by (meson &E(1) q₀-prop)
        ultimately AOT-have 2: ‹◇(¬[O!]a & q₀)›
           by (metis KBasic:16 &I vdash−properties:10)
        AOT-show ‹◇¬([O!]a ∨ ¬q₀)›
        proof (AOT-subst (reverse) ‹¬([O!]a ∨ ¬q₀)› ‹¬[O!]a & q₀›)
          AOT-modally-strict {
            AOT-show ‹¬[O!]a & q₀ ≡ ¬([O!]a ∨ ¬q₀)›
              by (metis &I &E(1) &E(2) ∨I(1) ∨I(2)
                       ∨E(3) deduction−theorem ≡I raa−cor:3)
          }
        next
          AOT-show ‹◇(¬[O!]a & q₀)›
            using 2 by blast
        qed
      qed
    qed(auto simp: beta−C−meta[THEN →E, OF 1])
    ultimately AOT-obtain F₁₄ where ‹𝓐[F₁₄]b & Δ[F₁₄]b & 𝓐[F₁₄]a & ¬Δ[F₁₄]a›
      using ∃I(1)[rotated, THEN ∃E[rotated]] by fastforce
    AOT-hence ‹𝓐[F₁₄]b› and ‹Δ[F₁₄]b› and ‹𝓐[F₁₄]a› and ‹¬Δ[F₁₄]a›
      using &E by blast+
    note props = props this

    AOT-have ‹[L]↓›
      by (rule =df I(2)[OF L-def]) cqt:2[lambda]+
    moreover AOT-have ‹𝓐[L]b & Δ[L]b & 𝓐[L]a & Δ[L]a›
    proof (safe intro!: &I)
      AOT-show ‹𝓐[L]b›
        by (meson nec-L nec−imp−act vdash−properties:10)
      next AOT-show ‹Δ[L]b› using nec-L nec-delta by blast
      next AOT-show ‹𝓐[L]a› by (meson nec-L nec−imp−act →E)
      next AOT-show ‹Δ[L]a› using nec-L nec-delta by blast
    qed
    ultimately AOT-obtain F₁₅ where ‹𝓐[F₁₅]b & Δ[F₁₅]b & 𝓐[F₁₅]a & Δ[F₁₅]a›
      using ∃I(1)[rotated, THEN ∃E[rotated]] by fastforce
    AOT-hence ‹𝓐[F₁₅]b› and ‹Δ[F₁₅]b› and ‹𝓐[F₁₅]a› and ‹Δ[F₁₅]a›
      using &E by blast+
    note props = props this

    show ?thesis
      by (rule ∃I(2)[where β=F₀]; rule ∃I(2)[where β=F₁];
          rule ∃I(2)[where β=F₂]; rule ∃I(2)[where β=F₃];
          rule ∃I(2)[where β=F₄]; rule ∃I(2)[where β=F₅];
          rule ∃I(2)[where β=F₆]; rule ∃I(2)[where β=F₇];
          rule ∃I(2)[where β=F₈]; rule ∃I(2)[where β=F₉];
          rule ∃I(2)[where β=F₁₀]; rule ∃I(2)[where β=F₁₁];
          rule ∃I(2)[where β=F₁₂]; rule ∃I(2)[where β=F₁₃];
          rule ∃I(2)[where β=F₁₄]; rule ∃I(2)[where β=F₁₅];
          safe intro!: &I)
         (match conclusion in [?v ⊨ [F] ≠ [G]] for F G ⇒ ‹
          match props in A: [?v ⊨ ¬φ{F}] for φ ⇒ ‹
          match (φ) in λa . ?p ⇒ ‹fail› | λa . a ⇒ ‹fail› | - ⇒ ‹
          match props in B: [?v ⊨ φ{G}] ⇒ ‹
          fact pos−not−equiv−ne:4[where F=F and G=G and φ=φ, THEN →E,
                                OF oth−class−taut:4:h[THEN ≡E(2)],
                                OF Disjunction Addition(2)[THEN →E],
                                OF &I, OF A, OF B]››››)+
  qed
```

8.11 The Theory of Objects

```
AOT-theorem o−objects−exist:1: ‹□∃ x O!x›
proof(rule RN)
  AOT-modally-strict {
    AOT-obtain a where ‹◇(E!a & ¬𝓐[E!]a)›
      using ∃ E[rotated, OF qml:4[axiom-inst, THEN BF◇[THEN →E]]]
      by blast
    AOT-hence 1: ‹◇E!a› by (metis KBasic2:3 &E(1) →E)
    AOT-have ‹[λx ◇[E!]x]a›
    proof (rule β←C(1); cqt:2[lambda]?)
      AOT-show ‹a↓› using cqt:2[const-var][axiom-inst] by blast
    next
      AOT-show ‹◇E!a› by (fact 1)
    qed
    AOT-hence ‹O!a› by (rule =df I(2)[OF AOT-ordinary, rotated]) cqt:2
    AOT-thus ‹∃ x [O!]x› by (rule ∃ I)
  }
qed

AOT-theorem o−objects−exist:2: ‹□∃ x A!x›
proof (rule RN)
  AOT-modally-strict {
    AOT-obtain a where ‹[A!]a›
      using A−objects[axiom-inst] ∃ E[rotated] &E by blast
    AOT-thus ‹∃ x A!x› using ∃ I by blast
  }
qed

AOT-theorem o−objects−exist:3: ‹□¬∀ x O!x›
  by (rule RN)
     (metis (no-types, opaque-lifting) ∃ E cqt−orig:1[const-var]
       ≡E(4) modus−tollens:1 o−objects−exist:2 oa−contingent:2
       qml:2[axiom-inst] reductio−aa:2)

AOT-theorem o−objects−exist:4: ‹□¬∀ x A!x›
  by (rule RN)
     (metis (mono-tags, opaque-lifting) ∃ E cqt−orig:1[const-var]
       ≡E(1) modus−tollens:1 o−objects−exist:1 oa−contingent:2
       qml:2[axiom-inst] →E)

AOT-theorem o−objects−exist:5: ‹□¬∀ x E!x›
proof (rule RN; rule raa−cor:2)
  AOT-modally-strict {
    AOT-assume ‹∀ x E!x›
    moreover AOT-obtain a where abs: ‹A!a›
      using o−objects−exist:2[THEN qml:2[axiom-inst, THEN →E]]
            ∃ E[rotated] by blast
    ultimately AOT-have ‹E!a› using ∀ E by blast
    AOT-hence 1: ‹◇E!a› by (metis T◇ →E)
    AOT-have ‹[λy ◇E!y]a›
    proof (rule β←C(1); cqt:2[lambda]?)
      AOT-show ‹a↓› using cqt:2[const-var][axiom-inst].
    next
      AOT-show ‹◇E!a› by (fact 1)
    qed
    AOT-hence ‹O!a›
      by (rule =df I(2)[OF AOT-ordinary, rotated]) cqt:2[lambda]
    AOT-hence ‹¬A!a› by (metis ≡E(1) oa−contingent:2)
    AOT-thus ‹p & ¬p› for p using abs by (metis raa−cor:3)
  }
qed
```

AOT-theorem *partition*: ‹¬∃ *x* (*O*!*x* & *A*!*x*)›
proof(*rule raa−cor*:*2*)
  AOT-assume ‹∃ *x* (*O*!*x* & *A*!*x*)›
  then AOT-obtain *a* **where** ‹*O*!*a* & *A*!*a*›
    using ∃ *E*[*rotated*] **by** *blast*
  AOT-thus ‹*p* & ¬*p*› **for** *p*
    by (*metis* &*E*(*1*) *Conjunction Simplification*(*2*) ≡*E*(*1*)
        modus−tollens:*1 oa−contingent*:*2 raa−cor*:*3*)
qed

AOT-define *eq-E* :: ‹Π› (‹′(=$_E$′)›)
  =*E*: ‹(=$_E$) =$_{df}$ [λ*xy O*!*x* & *O*!*y* & □∀ *F* ([*F*]*x* ≡ [*F*]*y*)]›

syntax *-AOT-eq-E-infix* :: ‹τ ⇒ τ ⇒ φ› (**infixl** ‹=$_E$› *50*)
translations
  -AOT-eq-E-infix κ κ′ == *CONST AOT-exe* (*CONST eq-E*) (*CONST Pair* κ κ′)
print-translation‹
AOT-syntax-print-translations
[(***const-syntax***‹*AOT-exe*›, *fn ctxt* => *fn* [
  Const (***const-name***‹*eq-E*›, -),
  Const (***const-syntax***‹*Pair*›, -) $ *lhs* $ *rhs*
] => *Const* (***syntax-const***‹*-AOT-eq-E-infix*›, *dummyT*) $ *lhs* $ *rhs*)]›

Note: Not explicitly mentioned as theorem in PLM.

AOT-theorem =*E*[*denotes*]: ‹[(=$_E$)]↓›
  by (*rule* =$_{df}$*I*(*2*)[*OF* =*E*]) *cqt*:*2*[*lambda*]+

AOT-theorem =*E−simple*:*1*: ‹*x* =$_E$ *y* ≡ (*O*!*x* & *O*!*y* & □∀ *F* ([*F*]*x* ≡ [*F*]*y*))›
proof −
  AOT-have *1*: ‹[λ*xy* [*O*!]*x* & [*O*!]*y* & □∀ *F* ([*F*]*x* ≡ [*F*]*y*)]↓› **by** *cqt*:*2*
  show *?thesis*
    apply (*rule* =$_{df}$*I*(*2*)[*OF* =*E*]; *cqt*:*2*[*lambda*]*?*)
    using *beta−C−meta*[*THEN* →*E*, *OF 1*, *unvarify* ν$_1$ν$_n$, *of* (-,-),
            OF tuple-denotes[*THEN* ≡$_{df}$*I*], *OF* &*I*,
            OF cqt:*2*[*const-var*][*axiom-inst*],
            OF cqt:*2*[*const-var*][*axiom-inst*]]
    by *fast*
qed

AOT-theorem =*E−simple*:*2*: ‹*x* =$_E$ *y* → *x* = *y*›
proof (*rule* →*I*)
  AOT-assume ‹*x* =$_E$ *y*›
  AOT-hence ‹*O*!*x* & *O*!*y* & □∀ *F* ([*F*]*x* ≡ [*F*]*y*)›
    using =*E−simple*:*1*[*THEN* ≡*E*(*1*)] **by** *blast*
  AOT-thus ‹*x* = *y*›
    using ≡$_{df}$*I*[*OF identity*:*1*] ∨*I* **by** *blast*
qed

AOT-theorem *id−nec3*:*1*: ‹*x* =$_E$ *y* ≡ □(*x* =$_E$ *y*)›
proof (*rule* ≡*I*; *rule* →*I*)
  AOT-assume ‹*x* =$_E$ *y*›
  AOT-hence ‹*O*!*x* & *O*!*y* & □∀ *F* ([*F*]*x* ≡ [*F*]*y*)›
    using =*E−simple*:*1* ≡*E* **by** *blast*
  AOT-hence ‹□*O*!*x* & □*O*!*y* & □□∀ *F* ([*F*]*x* ≡ [*F*]*y*)›
    by (*metis S5Basic*:*6* &*I* &*E*(*1*) &*E*(*2*) ≡*E*(*4*)
        oa−facts:*1 raa−cor*:*3 vdash−properties*:*10*)
  AOT-hence ‹□(*O*!*x* & *O*!*y* & □∀ *F* ([*F*]*x* ≡ [*F*]*y*))›
    by (*metis* &*E*(*1*) &*E*(*2*) ≡*E*(*2*) *KBasic*:*3* &*I*)
  AOT-thus ‹□(*x* =$_E$ *y*)›
    using =*E−simple*:*1*
    by (*AOT-subst* ‹*x* =$_E$ *y*› ‹*O*!*x* & *O*!*y* & □∀ *F* ([*F*]*x* ≡ [*F*]*y*)›) *auto*
next
  AOT-assume ‹□(*x* =$_E$ *y*)›

AOT-thus ‹$x =_E y$› **using** *qml:2[axiom-inst, THEN →E]* **by** *blast*
qed

AOT-theorem *id−nec3:2*: ‹$\Diamond(x =_E y) \equiv x =_E y$›
by (*meson RE◇ S5Basic:2 id−nec3:1 ≡E(1,5) Commutativity of ≡*)

AOT-theorem *id−nec3:3*: ‹$\Diamond(x =_E y) \equiv \Box(x =_E y)$›
by (*meson id−nec3:1 id−nec3:2 ≡E(5)*)

syntax *-AOT-non-eq-E* :: ‹Π› (‹'(≠$_E$')›)
translations
(Π) (≠$_E$) == (Π) (=$_E$)$^-$
syntax *-AOT-non-eq-E-infix* :: ‹$\tau \Rightarrow \tau \Rightarrow \varphi$› (**infixl** ‹≠$_E$› *50*)
translations
-AOT-non-eq-E-infix κ κ' ==
CONST AOT-exe (CONST relation-negation (CONST eq-E)) (CONST Pair κ κ')
print-translation‹
AOT-syntax-print-translations
[(***const-syntax***‹*AOT-exe*›, *fn ctxt => fn* [
Const (***const-syntax***‹*relation-negation*›, -) $ *Const* (***const-name***‹*eq-E*›, -),
Const (***const-syntax***‹*Pair*›, -) $ *lhs* $ *rhs*
] => *Const* (***syntax-const***‹*-AOT-non-eq-E-infix*›, *dummyT*) $ *lhs* $ *rhs*)]›
AOT-theorem *thm−neg=E*: ‹$x \neq_E y \equiv \neg(x =_E y)$›
proof −
AOT-have ϑ: ‹$[\lambda x_1...x_2\ \neg(=_E)x_1...x_2]\downarrow$› **by** *cqt:2*
AOT-have ‹$x \neq_E y \equiv [\lambda x_1...x_2\ \neg(=_E)x_1...x_2]xy$›
by (*rule* $=_{df}I(1)$*[OF df−relation−negation, OF ϑ]*)
(*meson oth−class−taut:3:a*)
also AOT-have ‹$\ldots \equiv \neg(=_E)xy$›
by (*safe intro*!: *beta−C−meta[THEN →E, unvarify* $\nu_1\nu_n$*] cqt:2*
tuple-denotes[THEN $\equiv_{df}I$*] &I*)
finally show *?thesis*.
qed

AOT-theorem *id−nec4:1*: ‹$x \neq_E y \equiv \Box(x \neq_E y)$›
proof −
AOT-have ‹$x \neq_E y \equiv \neg(x =_E y)$› **using** *thm−neg=E*.
also AOT-have ‹$\ldots \equiv \neg\Diamond(x =_E y)$›
by (*meson id−nec3:2 ≡E(1) Commutativity of ≡ oth−class−taut:4:b*)
also AOT-have ‹$\ldots \equiv \Box\neg(x =_E y)$›
by (*meson KBasic2:1 ≡E(2) Commutativity of ≡*)
also AOT-have ‹$\ldots \equiv \Box(x \neq_E y)$›
by (*AOT-subst (reverse)* ‹$\neg(x =_E y)$› ‹$x \neq_E y$›)
(*auto simp*: *thm−neg=E oth−class−taut:3:a*)
finally show *?thesis*.
qed

AOT-theorem *id−nec4:2*: ‹$\Diamond(x \neq_E y) \equiv (x \neq_E y)$›
by (*meson RE◇ S5Basic:2 id−nec4:1 ≡E(2,5) Commutativity of ≡*)

AOT-theorem *id−nec4:3*: ‹$\Diamond(x \neq_E y) \equiv \Box(x \neq_E y)$›
by (*meson id−nec4:1 id−nec4:2 ≡E(5)*)

AOT-theorem *id−act2:1*: ‹$x =_E y \equiv \mathcal{A}x =_E y$›
by (*meson Act−Basic:5 Act−Sub:2 RA[2] id−nec3:2 ≡E(1,6)*)
AOT-theorem *id−act2:2*: ‹$x \neq_E y \equiv \mathcal{A}x \neq_E y$›
by (*meson Act−Basic:5 Act−Sub:2 RA[2] id−nec4:2 ≡E(1,6)*)

AOT-theorem *ord=Eequiv:1*: ‹$O!x \rightarrow x =_E x$›
proof (*rule →I*)
AOT-assume *1*: ‹$O!x$›
AOT-show ‹$x =_E x$›
apply (*rule* $=_{df}I(2)$*[OF =E]*) **apply** *cqt:2[lambda]*

```
    apply (rule β←C(1))
      apply cqt:2[lambda]
     apply (simp add: &I cqt:2[const-var][axiom-inst] prod-denotesI)
    by (simp add: 1 RN &I oth−class−taut:3:a universal−cor)
qed

AOT-theorem ord=Eequiv:2: ‹x =E y → y =E x›
proof(rule CP)
  AOT-assume 1: ‹x =E y›
  AOT-hence 2: ‹x = y› by (metis =E−simple:2 vdash−properties:10)
  AOT-have ‹O!x› using 1 by (meson &E(1) =E−simple:1 ≡E(1))
  AOT-hence ‹x =E x› using ord=Eequiv:1 →E by blast
  AOT-thus ‹y =E x› using rule=E[rotated, OF 2] by fast
qed

AOT-theorem ord=Eequiv:3: ‹(x =E y & y =E z) → x =E z›
proof (rule CP)
  AOT-assume 1: ‹x =E y & y =E z›
  AOT-hence ‹x = y & y = z›
    by (metis &I &E(1) &E(2) =E−simple:2 vdash−properties:6)
  AOT-hence ‹x = z› by (metis id−eq:3 vdash−properties:6)
  moreover AOT-have ‹x =E x›
    using 1[THEN &E(1)] &E(1) =E−simple:1 ≡E(1)
          ord=Eequiv:1 →E by blast
  ultimately AOT-show ‹x =E z›
    using rule=E by fast
qed

AOT-theorem ord−=E=:1: ‹(O!x ∨ O!y) → □(x = y ≡ x =E y)›
proof(rule CP)
  AOT-assume ‹O!x ∨ O!y›
  moreover {
    AOT-assume ‹O!x›
    AOT-hence ‹□O!x› by (metis oa−facts:1 vdash−properties:10)
    moreover {
      AOT-modally-strict {
        AOT-have ‹O!x → (x = y ≡ x =E y)›
        proof (rule →I; rule ≡I; rule →I)
          AOT-assume ‹O!x›
          AOT-hence ‹x =E x› by (metis ord=Eequiv:1 →E)
          moreover AOT-assume ‹x = y›
          ultimately AOT-show ‹x =E y› using rule=E by fast
        next
          AOT-assume ‹x =E y›
          AOT-thus ‹x = y› by (metis =E−simple:2 →E)
        qed
      }
      AOT-hence ‹□O!x → □(x = y ≡ x =E y)› by (metis RM:1)
    }
    ultimately AOT-have ‹□(x = y ≡ x =E y)› using →E by blast
  }
  moreover {
    AOT-assume ‹O!y›
    AOT-hence ‹□O!y› by (metis oa−facts:1 vdash−properties:10)
    moreover {
      AOT-modally-strict {
        AOT-have ‹O!y → (x = y ≡ x =E y)›
        proof (rule →I; rule ≡I; rule →I)
          AOT-assume ‹O!y›
          AOT-hence ‹y =E y› by (metis ord=Eequiv:1 →E)
          moreover AOT-assume ‹x = y›
          ultimately AOT-show ‹x =E y› using rule=E id-sym by fast
        next
```

```
        AOT-assume ‹x =E y›
        AOT-thus ‹x = y› by (metis =E−simple:2 →E)
      qed
    }
    AOT-hence ‹□O!y → □(x = y ≡ x =E y)› by (metis RM:1)
  }
  ultimately AOT-have ‹□(x = y ≡ x =E y)› using →E by blast
 }
 ultimately AOT-show ‹□(x = y ≡ x =E y)› by (metis ∨E(3) raa−cor:1)
qed

AOT-theorem ord−=E=:2: ‹O!y → [λx x = y]↓›
proof (rule →I; rule safe−ext[axiom-inst, THEN →E]; rule &I)
  AOT-show ‹[λx x =E y]↓› by cqt:2[lambda]
next
  AOT-assume ‹O!y›
  AOT-hence 1: ‹□(x = y ≡ x =E y)› for x
    using ord−=E=:1 →E ∨I by blast
  AOT-have ‹□(x =E y ≡ x = y)› for x
    by (AOT-subst ‹x =E y ≡ x = y› ‹x = y ≡ x =E y›)
      (auto simp add: Commutativity of ≡ 1)
  AOT-hence ‹∀x □(x =E y ≡ x = y)› by (rule GEN)
  AOT-thus ‹□∀x (x =E y ≡ x = y)› by (rule BF[THEN →E])
qed


AOT-theorem ord−=E=:3: ‹[λxy O!x & O!y & x = y]↓›
proof (rule safe−ext[2][axiom-inst, THEN →E]; rule &I)
  AOT-show ‹[λxy O!x & O!y & x =E y]↓› by cqt:2[lambda]
next
  AOT-show ‹□∀x∀y ([O!]x & [O!]y & x =E y ≡ [O!]x & [O!]y & x = y)›
  proof (rule RN; rule GEN; rule GEN; rule ≡I; rule →I)
    AOT-modally-strict {
      AOT-show ‹[O!]x & [O!]y & x = y› if ‹[O!]x & [O!]y & x =E y› for x y
        by (metis &I &E(1) Conjunction Simplification(2) =E−simple:2
              modus−tollens:1 raa−cor:1 that)
    }
  next
    AOT-modally-strict {
      AOT-show ‹[O!]x & [O!]y & x =E y› if ‹[O!]x & [O!]y & x = y› for x y
        apply(safe intro!: &I)
          apply (metis that[THEN &E(1), THEN &E(1)])
         apply (metis that[THEN &E(1), THEN &E(2)])
        using rule=E[rotated, OF that[THEN &E(2)]]
            ord=Eequiv:1[THEN →E, OF that[THEN &E(1), THEN &E(1)]]
        by fast
    }
  qed
qed

AOT-theorem ind−nec: ‹∀F ([F]x ≡ [F]y) → □∀F ([F]x ≡ [F]y)›
proof(rule →I)
  AOT-assume ‹∀F ([F]x ≡ [F]y)›
  moreover AOT-have ‹[λx □∀F ([F]x ≡ [F]y)]↓› by cqt:2[lambda]
  ultimately AOT-have ‹[λx □∀F ([F]x ≡ [F]y)]x ≡ [λx □∀F ([F]x ≡ [F]y)]y›
    using ∀E by blast
  moreover AOT-have ‹[λx □∀F ([F]x ≡ [F]y)]y›
    apply (rule β←C(1))
      apply cqt:2[lambda]
     apply (fact cqt:2[const-var][axiom-inst])
    by (simp add: RN GEN oth−class−taut:3:a)
  ultimately AOT-have ‹[λx □∀F ([F]x ≡ [F]y)]x› using ≡E by blast
  AOT-thus ‹□∀F ([F]x ≡ [F]y)›
```

using $\beta{\rightarrow}C(1)$ **by** *blast*
qed

AOT-theorem *ord=E:1*: ‹$(O!x\ \&\ O!y) \rightarrow (\forall F\ ([F]x \equiv [F]y) \rightarrow x =_E y)$›
proof (*rule* $\rightarrow I$; *rule* $\rightarrow I$)
AOT-assume ‹$\forall F\ ([F]x \equiv [F]y)$›
AOT-hence ‹$\Box\forall F\ ([F]x \equiv [F]y)$›
using *ind−nec*[*THEN* $\rightarrow E$] **by** *blast*
moreover AOT-assume ‹$O!x\ \&\ O!y$›
ultimately AOT-have ‹$O!x\ \&\ O!y\ \&\ \Box\forall F\ ([F]x \equiv [F]y)$›
using &*I* **by** *blast*
AOT-thus ‹$x =_E y$› **using** *=E−simple:1*[*THEN* $\equiv E(2)$] **by** *blast*
qed

AOT-theorem *ord=E:2*: ‹$(O!x\ \&\ O!y) \rightarrow (\forall F\ ([F]x \equiv [F]y) \rightarrow x = y)$›
proof (*rule* $\rightarrow I$; *rule* $\rightarrow I$)
AOT-assume ‹$O!x\ \&\ O!y$›
moreover AOT-assume ‹$\forall F\ ([F]x \equiv [F]y)$›
ultimately AOT-have ‹$x =_E y$›
using *ord=E:1* $\rightarrow E$ **by** *blast*
AOT-thus ‹$x = y$› **using** *=E−simple:2*[*THEN* $\rightarrow E$] **by** *blast*
qed

AOT-theorem *ord=E2:1*:
‹$(O!x\ \&\ O!y) \rightarrow (x \neq y \equiv [\lambda z\ z =_E x] \neq [\lambda z\ z =_E y])$›
proof (*rule* $\rightarrow I$; *rule* $\equiv I$; *rule* $\rightarrow I$;
rule $\equiv_{df} I$[*OF* *=−infix*]; *rule* *raa−cor:2*)
AOT-assume *0*: ‹$O!x\ \&\ O!y$›
AOT-assume ‹$x \neq y$›
AOT-hence *1*: ‹$\neg(x = y)$› **using** $\equiv_{df} E$[*OF* *=−infix*] **by** *blast*
AOT-assume ‹$[\lambda z\ z =_E x] = [\lambda z\ z =_E y]$›
moreover AOT-have ‹$[\lambda z\ z =_E x]x$›
apply (*rule* $\beta{\leftarrow}C(1)$)
apply *cqt:2*[*lambda*]
apply (*fact* *cqt:2*[*const-var*][*axiom-inst*])
using *ord=Eequiv:1*[*THEN* $\rightarrow E$, *OF* *0*[*THEN* &*E*(*1*)]]**.**
ultimately AOT-have ‹$[\lambda z\ z =_E y]x$› **using** *rule=E* **by** *fast*
AOT-hence ‹$x =_E y$› **using** $\beta{\rightarrow}C(1)$ **by** *blast*
AOT-hence ‹$x = y$› **by** (*metis* *=E−simple:2* *vdash−properties:6*)
AOT-thus ‹$x = y\ \&\ \neg(x = y)$› **using** *1* &*I* **by** *blast*
next
AOT-assume ‹$[\lambda z\ z =_E x] \neq [\lambda z\ z =_E y]$›
AOT-hence *0*: ‹$\neg([\lambda z\ z =_E x] = [\lambda z\ z =_E y])$›
using $\equiv_{df} E$[*OF* *=−infix*] **by** *blast*
AOT-have ‹$[\lambda z\ z =_E x]{\downarrow}$› **by** *cqt:2*[*lambda*]
AOT-hence ‹$[\lambda z\ z =_E x] = [\lambda z\ z =_E x]$›
by (*metis* *rule=I:1*)
moreover AOT-assume ‹$x = y$›
ultimately AOT-have ‹$[\lambda z\ z =_E x] = [\lambda z\ z =_E y]$›
using *rule=E* **by** *fast*
AOT-thus ‹$[\lambda z\ z =_E x] = [\lambda z\ z =_E y]\ \&\ \neg([\lambda z\ z =_E x] = [\lambda z\ z =_E y])$›
using *0* &*I* **by** *blast*
qed

AOT-theorem *ord=E2:2*:
‹$(O!x\ \&\ O!y) \rightarrow (x \neq y \equiv [\lambda z\ z = x] \neq [\lambda z\ z = y])$›
proof (*rule* $\rightarrow I$; *rule* $\equiv I$; *rule* $\rightarrow I$;
rule $\equiv_{df} I$[*OF* *=−infix*]; *rule* *raa−cor:2*)
AOT-assume *0*: ‹$O!x\ \&\ O!y$›
AOT-assume ‹$x \neq y$›
AOT-hence *1*: ‹$\neg(x = y)$› **using** $\equiv_{df} E$[*OF* *=−infix*] **by** *blast*
AOT-assume ‹$[\lambda z\ z = x] = [\lambda z\ z = y]$›
moreover AOT-have ‹$[\lambda z\ z = x]x$›

```
    apply (rule β←C(1))
    apply (fact ord−=E=:2[THEN →E, OF 0[THEN &E(1)]])
     apply (fact cqt:2[const-var][axiom-inst])
    by (simp add: id−eq:1)
  ultimately AOT-have ‹[λz z = y]x› using rule=E by fast
  AOT-hence ‹x = y› using β→C(1) by blast
  AOT-thus ‹x = y & ¬(x = y)› using 1 &I by blast
next
  AOT-assume 0: ‹O!x & O!y›
  AOT-assume ‹[λz z = x] ≠ [λz z = y]›
  AOT-hence 1: ‹¬([λz z = x] = [λz z = y])›
    using ≡df E[OF =−infix] by blast
  AOT-have ‹[λz z = x]↓›
    by (fact ord−=E=:2[THEN →E, OF 0[THEN &E(1)]])
  AOT-hence ‹[λz z = x] = [λz z = x]›
    by (metis rule=I:1)
  moreover AOT-assume ‹x = y›
  ultimately AOT-have ‹[λz z = x] = [λz z = y]›
    using rule=E by fast
  AOT-thus ‹[λz z = x] = [λz z = y] & ¬([λz z = x] = [λz z = y])›
    using 1 &I by blast
qed

AOT-theorem ordnecfail: ‹O!x → □¬∃F x[F]›
  by (meson RM:1 →I nocoder[axiom-inst] oa−facts:1 →E)

AOT-theorem ab−obey:1: ‹(A!x & A!y) → (∀F (x[F] ≡ y[F]) → x = y)›
proof (rule →I; rule →I)
  AOT-assume 1: ‹A!x & A!y›
  AOT-assume ‹∀F (x[F] ≡ y[F])›
  AOT-hence ‹x[F] ≡ y[F]› for F using ∀E by blast
  AOT-hence ‹□(x[F] ≡ y[F])› for F by (metis en−eq:6[1] ≡E(1))
  AOT-hence ‹∀F □(x[F] ≡ y[F])› by (rule GEN)
  AOT-hence ‹□∀F (x[F] ≡ y[F])› by (rule BF[THEN →E])
  AOT-thus ‹x = y›
    using ≡df I[OF identity:1, OF ∨I(2)] 1 &I by blast
qed

AOT-theorem ab−obey:2:
  ‹(∃F (x[F] & ¬y[F]) ∨ ∃F (y[F] & ¬x[F])) → x ≠ y›
proof (rule →I; rule ≡df I[OF =−infix]; rule raa−cor:2)
  AOT-assume 1: ‹x = y›
  AOT-assume ‹∃F (x[F] & ¬y[F]) ∨ ∃F (y[F] & ¬x[F])›
  moreover {
    AOT-assume ‹∃F (x[F] & ¬y[F])›
    then AOT-obtain F where ‹x[F] & ¬y[F]›
      using ∃E[rotated] by blast
    moreover AOT-have ‹y[F]›
      using calculation[THEN &E(1)] 1 rule=E by fast
    ultimately AOT-have ‹p & ¬p› for p
      by (metis Conjunction Simplification(2) modus−tollens:2 raa−cor:3)
  }
  moreover {
    AOT-assume ‹∃F (y[F] & ¬x[F])›
    then AOT-obtain F where ‹y[F] & ¬x[F]›
      using ∃E[rotated] by blast
    moreover AOT-have ‹¬y[F]›
      using calculation[THEN &E(2)] 1 rule=E by fast
    ultimately AOT-have ‹p & ¬p› for p
      by (metis Conjunction Simplification(1) modus−tollens:1 raa−cor:3)
  }
  ultimately AOT-show ‹p & ¬p› for p
    by (metis ∨E(3) raa−cor:1)
```

qed

AOT-theorem *encoders−are−abstract*: ‹$\exists F\ x[F] \rightarrow A!x$›
by (*meson deduction−theorem* $\equiv E(2)$ *modus−tollens*:*2* *nocoder*
oa−contingent:*3* *vdash−properties*:*1*[*2*])

AOT-theorem *denote=*:*1*: ‹$\forall H \exists x\ x[H]$›
by (*rule GEN*; *rule existence*:*2*[*1*][*THEN* $\equiv_{df}E$]; *cqt*:*2*)

AOT-theorem *denote=*:*2*: ‹$\forall G \exists x_1 ... \exists x_n\ x_1...x_n[H]$›
by (*rule GEN*; *rule existence*:*2*[*THEN* $\equiv_{df}E$]; *cqt*:*2*)

AOT-theorem *denote=*:*2*[*2*]: ‹$\forall G \exists x_1 \exists x_2\ x_1x_2[H]$›
by (*rule GEN*; *rule existence*:*2*[*2*][*THEN* $\equiv_{df}E$]; *cqt*:*2*)

AOT-theorem *denote=*:*2*[*3*]: ‹$\forall G \exists x_1 \exists x_2 \exists x_3\ x_1x_2x_3[H]$›
by (*rule GEN*; *rule existence*:*2*[*3*][*THEN* $\equiv_{df}E$]; *cqt*:*2*)

AOT-theorem *denote=*:*2*[*4*]: ‹$\forall G \exists x_1 \exists x_2 \exists x_3 \exists x_4\ x_1x_2x_3x_4[H]$›
by (*rule GEN*; *rule existence*:*2*[*4*][*THEN* $\equiv_{df}E$]; *cqt*:*2*)

AOT-theorem *denote=*:*3*: ‹$\exists x\ x[\Pi] \equiv \exists H\ (H = \Pi)$›
using *existence*:*2*[*1*] *free−thms*:*1* $\equiv E(2,5)$
Commutativity of $\equiv$ $\equiv Df$ **by** *blast*

AOT-theorem *denote=*:*4*: ‹$(\exists x_1 ... \exists x_n\ x_1...x_n[\Pi]) \equiv \exists H\ (H = \Pi)$›
using *existence*:*2* *free−thms*:*1* $\equiv E(6)$ $\equiv Df$ **by** *blast*

AOT-theorem *denote=*:*4*[*2*]: ‹$(\exists x_1 \exists x_2\ x_1x_2[\Pi]) \equiv \exists H\ (H = \Pi)$›
using *existence*:*2*[*2*] *free−thms*:*1* $\equiv E(6)$ $\equiv Df$ **by** *blast*

AOT-theorem *denote=*:*4*[*3*]: ‹$(\exists x_1 \exists x_2 \exists x_3\ x_1x_2x_3[\Pi]) \equiv \exists H\ (H = \Pi)$›
using *existence*:*2*[*3*] *free−thms*:*1* $\equiv E(6)$ $\equiv Df$ **by** *blast*

AOT-theorem *denote=*:*4*[*4*]: ‹$(\exists x_1 \exists x_2 \exists x_3 \exists x_4\ x_1x_2x_3x_4[\Pi]) \equiv \exists H\ (H = \Pi)$›
using *existence*:*2*[*4*] *free−thms*:*1* $\equiv E(6)$ $\equiv Df$ **by** *blast*

AOT-theorem *A−objects*!: ‹$\exists !x\ (A!x\ \&\ \forall F\ (x[F] \equiv \varphi\{F\}))$›
proof (*rule uniqueness*:*1*[*THEN* $\equiv_{df}I$])
AOT-obtain *a* **where** *a-prop*: ‹$A!a\ \&\ \forall F\ (a[F] \equiv \varphi\{F\})$›
using *A−objects*[*axiom-inst*] $\exists E$[*rotated*] **by** *blast*
AOT-have ‹$(A!\beta\ \&\ \forall F\ (\beta[F] \equiv \varphi\{F\})) \rightarrow \beta = a$› **for** β
proof (*rule* $\rightarrow I$)
AOT-assume *β-prop*: ‹$[A!]\beta\ \&\ \forall F\ (\beta[F] \equiv \varphi\{F\})$›
AOT-hence ‹$\beta[F] \equiv \varphi\{F\}$› **for** *F*
using $\forall E$ $\&E$ **by** *blast*
AOT-hence ‹$\beta[F] \equiv a[F]$› **for** *F*
using *a-prop*[*THEN* $\&E(2)$] $\forall E$ $\equiv E(2,5)$
Commutativity of $\equiv$ **by** *fast*
AOT-hence ‹$\forall F\ (\beta[F] \equiv a[F])$› **by** (*rule GEN*)
AOT-thus ‹$\beta = a$›
using *ab−obey*:*1*[*THEN* $\rightarrow E$,
OF $\&I$[*OF β-prop*[*THEN* $\&E(1)$], *OF a-prop*[*THEN* $\&E(1)$]],
THEN $\rightarrow E$] **by** *blast*
qed
AOT-hence ‹$\forall \beta\ ((A!\beta\ \&\ \forall F\ (\beta[F] \equiv \varphi\{F\})) \rightarrow \beta = a)$› **by** (*rule GEN*)
AOT-thus ‹$\exists \alpha\ ([A!]\alpha\ \&\ \forall F\ (\alpha[F] \equiv \varphi\{F\})\ \&$
$\forall \beta\ ([A!]\beta\ \&\ \forall F\ (\beta[F] \equiv \varphi\{F\}) \rightarrow \beta = \alpha))$›
using $\exists I$ **using** *a-prop* $\&I$ **by** *fast*
qed

AOT-theorem *obj−oth*:*1*: ‹$\exists !x\ (A!x\ \&\ \forall F\ (x[F] \equiv [F]y))$›
using *A−objects*! **by** *fast*

AOT-theorem *obj−oth:2*: ‹∃!x (A!x & ∀F (x[F] ≡ [F]y & [F]z))›
  using *A−objects!* **by** *fast*

AOT-theorem *obj−oth:3*: ‹∃!x (A!x & ∀F (x[F] ≡ [F]y ∨ [F]z))›
  using *A−objects!* **by** *fast*

AOT-theorem *obj−oth:4*: ‹∃!x (A!x & ∀F (x[F] ≡ □[F]y))›
  using *A−objects!* **by** *fast*

AOT-theorem *obj−oth:5*: ‹∃!x (A!x & ∀F (x[F] ≡ F = G))›
  using *A−objects!* **by** *fast*

AOT-theorem *obj−oth:6*: ‹∃!x (A!x & ∀F (x[F] ≡ □∀y([G]y → [F]y)))›
  using *A−objects!* **by** *fast*

AOT-theorem *A−descriptions*: ‹ιx (A!x & ∀F (x[F] ≡ φ{F}))↓›
  by (*rule A−Exists:2[THEN ≡E(2)]*; *rule RA[2]*; *rule A−objects!*)

AOT-act-theorem *thm−can−terms2*:
  ‹y = ιx(A!x & ∀F (x[F] ≡ φ{F})) → (A!y & ∀F (y[F] ≡ φ{F}))›
  using *y−in:2* **by** *blast*

AOT-theorem *can−ab2*: ‹y = ιx(A!x & ∀F (x[F] ≡ φ{F})) → A!y›
proof(*rule →I*)
  AOT-assume ‹y = ιx(A!x & ∀F (x[F] ≡ φ{F}))›
  AOT-hence ‹𝓐(A!y & ∀F (y[F] ≡ φ{F}))›
    using *actual−desc:2[THEN →E]* **by** *blast*
  AOT-hence ‹𝓐A!y› **by** (*metis Act−Basic:2 &E(1) ≡E(1)*)
  AOT-thus ‹A!y› **by** (*metis ≡E(2) oa−facts:8*)
qed

AOT-act-theorem *desc−encode:1*: ‹ιx(A!x & ∀F (x[F] ≡ φ{F}))[F] ≡ φ{F}›
proof −
  AOT-have ‹ιx(A!x & ∀F (x[F] ≡ φ{F}))↓›
    by (*simp add: A−descriptions*)
  AOT-hence ‹A!ιx(A!x & ∀F (x[F] ≡ φ{F})) &
        ∀F(ιx(A!x & ∀F (x[F] ≡ φ{F}))[F] ≡ φ{F})›
    using *y−in:3[THEN →E]* **by** *blast*
  AOT-thus ‹ιx(A!x & ∀F (x[F] ≡ φ{F}))[F] ≡ φ{F}›
    using *&E ∀E* **by** *blast*
qed

AOT-act-theorem *desc−encode:2*: ‹ιx(A!x & ∀F (x[F] ≡ φ{F}))[G] ≡ φ{G}›
  using *desc−encode:1*.

AOT-theorem *desc−nec−encode:1*:
  ‹ιx (A!x & ∀F (x[F] ≡ φ{F}))[F] ≡ 𝓐φ{F}›
proof −
  AOT-have *0*: ‹ιx(A!x & ∀F (x[F] ≡ φ{F}))↓›
    by (*simp add: A−descriptions*)
  AOT-hence ‹𝓐(A!ιx(A!x & ∀F (x[F] ≡ φ{F})) &
        ∀F(ιx(A!x & ∀F (x[F] ≡ φ{F}))[F] ≡ φ{F}))›
    using *actual−desc:4[THEN →E]* **by** *blast*
  AOT-hence ‹𝓐∀F (ιx(A!x & ∀F (x[F] ≡ φ{F}))[F] ≡ φ{F})›
    using *Act−Basic:2 &E(2) ≡E(1)* **by** *blast*
  AOT-hence ‹∀F 𝓐(ιx(A!x & ∀F (x[F] ≡ φ{F}))[F] ≡ φ{F})›
    using *≡E(1) logic−actual−nec:3 vdash−properties:1[2]* **by** *blast*
  AOT-hence ‹𝓐(ιx(A!x & ∀F (x[F] ≡ φ{F}))[F] ≡ φ{F})›
    using *∀E* **by** *blast*
  AOT-hence ‹𝓐ιx(A!x & ∀F (x[F] ≡ φ{F}))[F] ≡ 𝓐φ{F}›
    using *Act−Basic:5 ≡E(1)* **by** *blast*
  AOT-thus ‹ιx(A!x & ∀F (x[F] ≡ φ{F}))[F] ≡ 𝓐φ{F}›

using *en−eq:10[1][unvarify* x_1, *OF 0]* ≡*E(6)* **by** *blast*
qed

AOT-theorem *desc−nec−encode:2*:
‹ιx (A!x & ∀ F (x[F] ≡ φ{F}))[G] ≡ 𝓐φ{G}›
using *desc−nec−encode:1.*

AOT-theorem *Box−desc−encode:1*: ‹□φ{G} → ιx(A!x & ∀ F (x[F] ≡ φ{G}))[G]›
by (*rule* →*I*; *rule desc−nec−encode:2[THEN* ≡*E(2)]*)
(*meson nec−imp−act vdash−properties:10*)

AOT-theorem *Box−desc−encode:2*:
‹□φ{G} → □(ιx(A!x & ∀ F (x[F] ≡ φ{G}))[G] ≡ φ{G})›
proof(*rule CP*)
AOT-assume ‹□φ{G}›
AOT-hence ‹□□φ{G}› **by** (*metis S5Basic:6* ≡*E(1)*)
moreover AOT-have ‹□□φ{G} → □(ιx(A!x & ∀ F (x[F] ≡ φ{G}))[G] ≡ φ{G})›
proof (*rule RM*; *rule* →*I*)
AOT-modally-strict {
AOT-assume *1*: ‹□φ{G}›
AOT-hence ‹ιx(A!x & ∀ F (x[F] ≡ φ{G}))[G]›
using *Box−desc−encode:1* →*E* **by** *blast*
moreover AOT-have ‹φ{G}›
using *1* **by** (*meson qml:2[axiom-inst]* →*E*)
ultimately AOT-show ‹ιx(A!x & ∀ F (x[F] ≡ φ{G}))[G] ≡ φ{G}›
using →*I* ≡*I* **by** *simp*
}
qed
ultimately AOT-show ‹□(ιx(A!x & ∀ F (x[F] ≡ φ{G}))[G] ≡ φ{G})›
using →*E* **by** *blast*
qed

definition *rigid-condition* **where**
‹*rigid-condition* φ ≡ ∀ v . [v ⊨ ∀ α (φ{α} → □φ{α})]›
syntax *rigid-condition* :: ‹*id-position* ⇒ *AOT-prop*› (‹*RIGID′-CONDITION′(-′)*›)

AOT-theorem *strict−can:1[E]*:
assumes ‹*RIGID-CONDITION*(φ)›
shows ‹∀ α (φ{α} → □φ{α})›
using *assms[unfolded rigid-condition-def]* **by** *auto*

AOT-theorem *strict−can:1[I]*:
assumes ‹⊢□ ∀ α (φ{α} → □φ{α})›
shows ‹*RIGID-CONDITION*(φ)›
using *assms rigid-condition-def* **by** *auto*

AOT-theorem *box−phi−a:1*:
assumes ‹*RIGID-CONDITION*(φ)›
shows ‹(A!x & ∀ F (x[F] ≡ φ{F})) → □(A!x & ∀ F (x[F] ≡ φ{F}))›
proof (*rule* →*I*)
AOT-assume *a*: ‹A!x & ∀ F (x[F] ≡ φ{F})›
AOT-hence *b*: ‹□A!x›
by (*metis Conjunction Simplification(1) oa−facts:2* →*E*)
AOT-have ‹x[F] ≡ φ{F}› **for** *F*
using *a[THEN &E(2)]* ∀ *E* **by** *blast*
moreover AOT-have ‹□(x[F] → □x[F])› **for** *F*
by (*meson pre−en−eq:1[1] RN*)
moreover AOT-have ‹□(φ{F} → □φ{F})› **for** *F*
using *RN strict−can:1[E][OF assms]* ∀ *E* **by** *blast*
ultimately AOT-have ‹□(x[F] ≡ φ{F})› **for** *F*
using *sc−eq−box−box:5 qml:2[axiom-inst, THEN* →*E]* →*E* &*I* **by** *metis*
AOT-hence ‹∀ F □(x[F] ≡ φ{F})› **by** (*rule GEN*)
AOT-hence ‹□∀ F (x[F] ≡ φ{F})› **by** (*rule BF[THEN* →*E]*)

```
  AOT-thus ‹□([A!]x & ∀ F (x[F] ≡ φ{F}))›
    using b KBasic:3 ≡S(1) ≡E(2) by blast
qed

AOT-theorem box−phi−a:2:
  assumes ‹RIGID-CONDITION(φ)›
  shows ‹y = ιx(A!x & ∀ F (x[F] ≡ φ{F})) → (A!y & ∀ F (y[F] ≡ φ{F}))›
proof(rule →I)
  AOT-assume ‹y = ιx(A!x & ∀ F (x[F] ≡ φ{F}))›
  AOT-hence ‹𝓐(A!y & ∀ F (y[F] ≡ φ{F}))›
    using actual−desc:2[THEN →E] by fast
  AOT-hence abs: ‹𝓐A!y› and ‹𝓐∀ F (y[F] ≡ φ{F})›
    using Act−Basic:2 &E ≡E(1) by blast+
  AOT-hence ‹∀ F 𝓐(y[F] ≡ φ{F})›
    by (metis ≡E(1) logic−actual−nec:3 vdash−properties:1[2])
  AOT-hence ‹𝓐(y[F] ≡ φ{F})› for F
    using ∀ E by blast
  AOT-hence ‹𝓐y[F] ≡ 𝓐φ{F}› for F
    by (metis Act−Basic:5 ≡E(1))
  AOT-hence ‹y[F] ≡ φ{F}› for F
    using sc−eq−fur:2[THEN →E,
            OF strict−can:1[E][OF assms,
              THEN ∀ E(2)[where β=F], THEN RN]]
    by (metis en−eq:10[1] ≡E(6))
  AOT-hence ‹∀ F (y[F] ≡ φ{F})› by (rule GEN)
  AOT-thus ‹[A!]y & ∀ F (y[F] ≡ φ{F})›
    using abs &I ≡E(2) oa−facts:8 by blast
qed

AOT-theorem box−phi−a:3:
  assumes ‹RIGID-CONDITION(φ)›
  shows ‹ιx(A!x & ∀ F (x[F] ≡ φ{F}))[F] ≡ φ{F}›
  using desc−nec−encode:2
    sc−eq−fur:2[THEN →E,
        OF strict−can:1[E][OF assms,
          THEN ∀ E(2)[where β=F], THEN RN]]
    ≡E(5) by blast

AOT-define Null :: ‹τ ⇒ φ› (‹Null′(-′)›)
  df−null−uni:1: ‹Null(x) ≡df A!x & ¬∃ F x[F]›

AOT-define Universal :: ‹τ ⇒ φ› (‹Universal′(-′)›)
  df−null−uni:2: ‹Universal(x) ≡df A!x & ∀ F x[F]›

AOT-theorem null−uni−uniq:1: ‹∃ !x Null(x)›
proof (rule uniqueness:1[THEN ≡df I])
  AOT-obtain a where a-prop: ‹A!a & ∀ F (a[F] ≡ ¬(F = F))›
    using A−objects[axiom-inst] ∃ E[rotated] by fast
  AOT-have a-null: ‹¬a[F]› for F
  proof (rule raa−cor:2)
    AOT-assume ‹a[F]›
    AOT-hence ‹¬(F = F)› using a-prop[THEN &E(2)] ∀ E ≡E by blast
    AOT-hence ‹F = F & ¬(F = F)› by (metis id−eq:1 raa−cor:3)
    AOT-thus ‹p & ¬p› for p  by (metis raa−cor:1)
  qed
  AOT-have ‹Null(a) & ∀ β (Null(β) → β = a)›
  proof (rule &I)
    AOT-have ‹¬∃ F a[F]›
      using a-null by (metis instantiation reductio−aa:1)
    AOT-thus ‹Null(a)›
      using df−null−uni:1[THEN ≡df I] a-prop[THEN &E(1)] &I by metis
  next
    AOT-show ‹∀ β (Null(β) → β = a)›
```

```
    proof (rule GEN; rule →I)
      fix β
      AOT-assume a: ‹Null(β)›
      AOT-hence ‹¬∃ F β[F]›
        using df−null−uni:1[THEN ≡df E] &E by blast
      AOT-hence β-null: ‹¬β[F]› for F
        by (metis existential:2[const-var] reductio−aa:1)
      AOT-have ‹∀ F (β[F] ≡ a[F])›
        apply (rule GEN; rule ≡I; rule CP)
        using raa−cor:3 β-null a-null by blast+
      moreover AOT-have ‹A!β›
        using a df−null−uni:1[THEN ≡df E] &E by blast
      ultimately AOT-show ‹β = a›
        using a-prop[THEN &E(1)] ab−obey:1[THEN →E, THEN →E]
              &I by blast
    qed
  qed
  AOT-thus ‹∃ α (Null(α) & ∀ β (Null(β) → β = α))›
    using ∃ I(2) by fast
qed

AOT-theorem null−uni−uniq:2: ‹∃ !x Universal(x)›
proof (rule uniqueness:1[THEN ≡df I])
  AOT-obtain a where a-prop: ‹A!a & ∀ F (a[F] ≡ F = F)›
    using A−objects[axiom-inst] ∃ E[rotated] by fast
  AOT-hence aF: ‹a[F]› for F using &E ∀ E ≡E id−eq:1 by fast
  AOT-hence ‹Universal(a)›
    using df−null−uni:2[THEN ≡df I] &I a-prop[THEN &E(1)] GEN by blast
  moreover AOT-have ‹∀ β (Universal(β) → β = a)›
  proof (rule GEN; rule →I)
    fix β
    AOT-assume ‹Universal(β)›
    AOT-hence abs-β: ‹A!β› and ‹β[F]› for F
      using df−null−uni:2[THEN ≡df E] &E ∀ E by blast+
    AOT-hence ‹β[F] ≡ a[F]› for F
      using aF by (metis deduction−theorem ≡I)
    AOT-hence ‹∀ F (β[F] ≡ a[F])› by (rule GEN)
    AOT-thus ‹β = a›
      using a-prop[THEN &E(1)] ab−obey:1[THEN →E, THEN →E]
            &I abs-β by blast
  qed
  ultimately AOT-show ‹∃ α (Universal(α) & ∀ β (Universal(β) → β = α))›
    using &I ∃ I by fast
qed

AOT-theorem null−uni−uniq:3: ‹ιx Null(x)↓›
  using A−Exists:2 RA[2] ≡E(2) null−uni−uniq:1 by blast

AOT-theorem null−uni−uniq:4: ‹ιx Universal(x)↓›
  using A−Exists:2 RA[2] ≡E(2) null−uni−uniq:2 by blast

AOT-define Null-object :: ‹κs› (‹a∅›)
  df−null−uni−terms:1: ‹a∅ =df ιx Null(x)›

AOT-define Universal-object :: ‹κs› (‹aV›)
  df−null−uni−terms:2: ‹aV =df ιx Universal(x)›

AOT-theorem null−uni−facts:1: ‹Null(x) → □Null(x)›
proof (rule →I)
  AOT-assume ‹Null(x)›
  AOT-hence x-abs: ‹A!x› and x-null: ‹¬∃ F x[F]›
    using df−null−uni:1[THEN ≡df E] &E by blast+
  AOT-have ‹¬x[F]› for F using x-null
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```
    using existential:2[const-var] reductio−aa:1
    by metis
  AOT-hence ‹□¬x[F]› for F by (metis en−eq:7[1] ≡E(1))
  AOT-hence ‹∀ F □¬x[F]› by (rule GEN)
  AOT-hence ‹□∀ F ¬x[F]› by (rule BF[THEN →E])
  moreover AOT-have ‹□∀ F ¬x[F] → □¬∃ F x[F]›
    apply (rule RM)
    by (metis (full-types) instantiation cqt:2[const-var][axiom-inst]
                          →I reductio−aa:1 rule−ui:1)
  ultimately AOT-have ‹□¬∃ F x[F]›
    by (metis →E)
  moreover AOT-have ‹□A!x› using x-abs
    using oa−facts:2 vdash−properties:10 by blast
  ultimately AOT-have r: ‹□(A!x & ¬∃ F x[F])›
    by (metis KBasic:3 &I ≡E(3) raa−cor:3)
  AOT-show ‹□Null(x)›
    by (AOT-subst ‹Null(x)› ‹A!x & ¬∃ F x[F]›)
       (auto simp: df−null−uni:1 ≡Df r)
qed

AOT-theorem null−uni−facts:2: ‹Universal(x) → □Universal(x)›
proof (rule →I)
  AOT-assume ‹Universal(x)›
  AOT-hence x-abs: ‹A!x› and x-univ: ‹∀ F x[F]›
    using df−null−uni:2[THEN ≡dfE] &E by blast+
  AOT-have ‹x[F]› for F using x-univ ∀ E by blast
  AOT-hence ‹□x[F]› for F by (metis en−eq:2[1] ≡E(1))
  AOT-hence ‹∀ F □x[F]› by (rule GEN)
  AOT-hence ‹□∀ F x[F]› by (rule BF[THEN →E])
  moreover AOT-have ‹□A!x› using x-abs
    using oa−facts:2 vdash−properties:10 by blast
  ultimately AOT-have r: ‹□(A!x & ∀ F x[F])›
    by (metis KBasic:3 &I ≡E(3) raa−cor:3)
  AOT-show ‹□Universal(x)›
    by (AOT-subst ‹Universal(x)› ‹A!x & ∀ F x[F]›)
       (auto simp add: df−null−uni:2 ≡Df r)
qed

AOT-theorem null−uni−facts:3: ‹Null(a∅)›
  apply (rule =dfI(2)[OF df−null−uni−terms:1])
   apply (simp add: null−uni−uniq:3)
  using actual−desc:4[THEN →E, OF null−uni−uniq:3]
    sc−eq−fur:2[THEN →E,
        OF null−uni−facts:1[unvarify x, THEN RN, OF null−uni−uniq:3],
        THEN ≡E(1)]
  by blast

AOT-theorem null−uni−facts:4: ‹Universal(aV)›
  apply (rule =dfI(2)[OF df−null−uni−terms:2])
   apply (simp add: null−uni−uniq:4)
  using actual−desc:4[THEN →E, OF null−uni−uniq:4]
    sc−eq−fur:2[THEN →E,
        OF null−uni−facts:2[unvarify x, THEN RN, OF null−uni−uniq:4],
        THEN ≡E(1)]
  by blast

AOT-theorem null−uni−facts:5: ‹a∅ ≠ aV›
proof (rule =dfI(2)[OF df−null−uni−terms:1, OF null−uni−uniq:3];
    rule =dfI(2)[OF df−null−uni−terms:2, OF null−uni−uniq:4];
    rule ≡dfI[OF =−infix];
    rule raa−cor:2)
  AOT-obtain x where nullx: ‹Null(x)›
    by (metis instantiation df−null−uni−terms:1 existential:1
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```
        null−uni−facts:3 null−uni−uniq:3 rule−id−df:2:b[zero])
  AOT-hence act-null: ‹𝓐Null(x)›
   by (metis nec−imp−act null−uni−facts:1 →E)
  AOT-assume ‹ιx Null(x) = ιx Universal(x)›
  AOT-hence ‹𝓐∀x(Null(x) ≡ Universal(x))›
   using actual−desc:5[THEN →E] by blast
  AOT-hence ‹∀x 𝓐(Null(x) ≡ Universal(x))›
   by (metis ≡E(1) logic−actual−nec:3 vdash−properties:1[2])
  AOT-hence ‹𝓐Null(x) ≡ 𝓐Universal(x)›
   using Act−Basic:5 ≡E(1) rule−ui:3 by blast
  AOT-hence ‹𝓐Universal(x)› using act-null ≡E by blast
  AOT-hence ‹Universal(x)›
   by (metis RN ≡E(1) null−uni−facts:2 sc−eq−fur:2 →E)
  AOT-hence ‹∀F x[F]› using ≡df E[OF df−null−uni:2] &E by metis
  moreover AOT-have ‹¬∃F x[F]›
   using nullx ≡df E[OF df−null−uni:1] &E by metis
  ultimately AOT-show ‹p & ¬p› for p
   by (metis cqt−further:1 raa−cor:3 →E)
qed

AOT-theorem null−uni−facts:6: ‹a∅ = ιx(A!x & ∀F (x[F] ≡ F ≠ F))›
proof (rule ab−obey:1[unvarify x y, THEN →E, THEN →E])
  AOT-show ‹ιx([A!]x & ∀F (x[F] ≡ F ≠ F))↓›
   by (simp add: A−descriptions)
next
  AOT-show ‹a∅↓›
   by (rule =df I(2)[OF df−null−uni−terms:1, OF null−uni−uniq:3])
     (simp add: null−uni−uniq:3)
next
  AOT-have ‹ιx([A!]x & ∀F (x[F] ≡ F ≠ F))↓›
   by (simp add: A−descriptions)
  AOT-hence 1: ‹ιx([A!]x & ∀F (x[F] ≡ F ≠ F)) = ιx([A!]x & ∀F (x[F] ≡ F ≠ F))›
   using rule=I:1 by blast
  AOT-show ‹[A!]a∅ & [A!]ιx([A!]x & ∀F (x[F] ≡ F ≠ F))›
   apply (rule =df I(2)[OF df−null−uni−terms:1, OF null−uni−uniq:3];
        rule &I)
    apply (meson ≡df E Conjunction Simplification(1)
             df−null−uni:1 df−null−uni−terms:1 null−uni−facts:3
             null−uni−uniq:3 rule−id−df:2:a[zero] →E)
   using can−ab2[unvarify y, OF A−descriptions, THEN →E, OF 1].
next
  AOT-show ‹∀F (a∅[F] ≡ ιx([A!]x & ∀F (x[F] ≡ F ≠ F))[F])›
  proof (rule GEN)
   fix F
   AOT-have ‹¬a∅[F]›
    by (rule =df I(2)[OF df−null−uni−terms:1, OF null−uni−uniq:3])
      (metis (no-types, lifting) ≡df E &E(2) ∨I(2) ∨E(3) ∃I(2)
          df−null−uni:1 df−null−uni−terms:1 null−uni−facts:3
          raa−cor:2 rule−id−df:2:a[zero]
          russell−axiom[enc,1].ψ-denotes-asm)
   moreover AOT-have ‹¬ιx([A!]x & ∀F (x[F] ≡ F ≠ F))[F]›
   proof(rule raa−cor:2)
    AOT-assume 0: ‹ιx([A!]x & ∀F (x[F] ≡ F ≠ F))[F]›
    AOT-hence ‹𝓐(F ≠ F)›
     using desc−nec−encode:2[THEN ≡E(1), OF 0] by blast
    moreover AOT-have ‹¬𝓐(F ≠ F)›
     using ≡df E id−act:2 id−eq:1 ≡E(2)
        =−infix raa−cor:3 by blast
    ultimately AOT-show ‹𝓐(F ≠ F) & ¬𝓐(F ≠ F)› by (rule &I)
   qed
   ultimately AOT-show ‹a∅[F] ≡ ιx([A!]x & ∀F (x[F] ≡ F ≠ F))[F]›
    using deduction−theorem ≡I raa−cor:4 by blast
  qed
```

qed

AOT-theorem *null−uni−facts:7*: ‹a_V = $\iota x(A!x$ & $\forall F$ $(x[F] \equiv F = F))$›
proof (*rule ab−obey:1*[*unvarify x y*, *THEN* →*E*, *THEN* →*E*])
  AOT-show ‹$\iota x([A!]x$ & $\forall F$ $(x[F] \equiv F = F))\downarrow$›
    by (*simp add*: *A−descriptions*)
next
  AOT-show ‹$a_V\downarrow$›
    by (*rule* $=_{df}I(2)$[*OF df−null−uni−terms:2*, *OF null−uni−uniq:4*])
       (*simp add*: *null−uni−uniq:4*)
next
  AOT-have ‹$\iota x([A!]x$ & $\forall F$ $(x[F] \equiv F = F))\downarrow$›
    by (*simp add*: *A−descriptions*)
  AOT-hence *1*: ‹$\iota x([A!]x$ & $\forall F$ $(x[F] \equiv F = F)) = \iota x([A!]x$ & $\forall F$ $(x[F] \equiv F = F))$›
    using *rule=I:1* **by** *blast*
  AOT-show ‹$[A!]a_V$ & $[A!]\iota x([A!]x$ & $\forall F$ $(x[F] \equiv F = F))$›
    apply (*rule* $=_{df}I(2)$[*OF df−null−uni−terms:2*, *OF null−uni−uniq:4*];
          rule &*I*)
     apply (*meson* $\equiv_{df}E$ *Conjunction Simplification*(*1*) *df−null−uni:2*
                df−null−uni−terms:2 null−uni−facts:4 null−uni−uniq:4
                rule−id−df:2:a[*zero*] →*E*)
    using *can−ab2*[*unvarify y*, *OF A−descriptions*, *THEN* →*E*, *OF 1*]**.**
next
  AOT-show ‹$\forall F$ $(a_V[F] \equiv \iota x([A!]x$ & $\forall F$ $(x[F] \equiv F = F))[F])$›
  proof (*rule GEN*)
    fix *F*
    AOT-have ‹$a_V[F]$›
      apply (*rule* $=_{df}I(2)$[*OF df−null−uni−terms:2*, *OF null−uni−uniq:4*])
      using $\equiv_{df}E$ &*E*(*2*) *df−null−uni:2 df−null−uni−terms:2*
            null−uni−facts:4 null−uni−uniq:4 rule−id−df:2:a[*zero*]
            rule−ui:3 **by** *blast*
    moreover AOT-have ‹$\iota x([A!]x$ & $\forall F$ $(x[F] \equiv F = F))[F]$›
      using *RA*[*2*] *desc−nec−encode:2 id−eq:1* $\equiv E(2)$ **by** *fastforce*
    ultimately AOT-show ‹$a_V[F] \equiv \iota x([A!]x$ & $\forall F$ $(x[F] \equiv F = F))[F]$›
      using *deduction−theorem* $\equiv I$ **by** *simp*
  qed
qed

AOT-theorem *aclassical:1*:
‹$\forall R \exists x \exists y(A!x$ & $A!y$ & $x \neq y$ & $[\lambda z\ [R]zx] = [\lambda z\ [R]zy])$›
proof(*rule GEN*)
  fix *R*
  AOT-obtain *a* **where** *a-prop*:
    ‹$A!a$ & $\forall F$ $(a[F] \equiv \exists y(A!y$ & $F = [\lambda z\ [R]zy]$ & $\neg y[F]))$›
    using *A−objects*[*axiom-inst*] $\exists E$[*rotated*] **by** *fast*
  AOT-have *a-enc*: ‹$a[\lambda z\ [R]za]$›
  proof (*rule raa−cor:1*)
    AOT-assume *0*: ‹$\neg a[\lambda z\ [R]za]$›
    AOT-hence ‹$\neg\exists y(A!y$ & $[\lambda z\ [R]za] = [\lambda z\ [R]zy]$ & $\neg y[\lambda z\ [R]za])$›
      by (*rule a-prop*[*THEN* &*E*(*2*), *THEN* $\forall E(1)$[**where** τ=«$[\lambda z\ [R]za]$»],
              THEN oth−class−taut:4:b[*THEN* $\equiv E(1)$],
              THEN $\equiv E(1)$, *rotated*])
        cqt:2[*lambda*]
    AOT-hence ‹$\forall y$ $\neg(A!y$ & $[\lambda z\ [R]za] = [\lambda z\ [R]zy]$ & $\neg y[\lambda z\ [R]za])$›
      using *cqt−further:4 vdash−properties:10* **by** *blast*
    AOT-hence ‹$\neg(A!a$ & $[\lambda z\ [R]za] = [\lambda z\ [R]za]$ & $\neg a[\lambda z\ [R]za])$›
      using $\forall E$ **by** *blast*
    AOT-hence ‹$(A!a$ & $[\lambda z\ [R]za] = [\lambda z\ [R]za]) \rightarrow a[\lambda z\ [R]za]$›
      by (*metis* &*I deduction−theorem raa−cor:3*)
    moreover AOT-have ‹$[\lambda z\ [R]za] = [\lambda z\ [R]za]$›
      by (*rule* =*I*) *cqt:2*[*lambda*]
    ultimately AOT-have ‹$a[\lambda z\ [R]za]$›
      using *a-prop*[*THEN* &*E*(*1*)] →*E* &*I* **by** *blast*

AOT-thus ‹a[λz [R]za] & ¬a[λz [R]za]›
using *0 &I* **by** *blast*
qed
AOT-hence ‹∃ y(A!y & [λz [R]za] = [λz [R]zy] & ¬y[λz [R]za])›
by (*rule a-prop*[*THEN &E*(*2*), *THEN* ∀ *E*(*1*), *THEN* ≡*E*(*1*), *rotated*])
cqt:*2*
then AOT-obtain *b* **where** *b-prop*:
‹A!b & [λz [R]za] = [λz [R]zb] & ¬b[λz [R]za]›
using ∃ *E*[*rotated*] **by** *blast*
AOT-have ‹a ≠ b›
apply (*rule* $\equiv_{df}$*I*[*OF* =−*infix*])
using *a-enc b-prop*[*THEN &E*(*2*)]
using ¬¬*I rule*=*E id-sym* ≡*E*(*4*) *oth*−*class*−*taut*:*3*:*a*
raa−*cor*:*3 reductio*−*aa*:*1* **by** *fast*
AOT-hence ‹A!a & A!b & a ≠ b & [λz [R]za] = [λz [R]zb]›
using *b-prop &E a-prop &I* **by** *meson*
AOT-hence ‹∃ y (A!a & A!y & a ≠ y & [λz [R]za] = [λz [R]zy])› **by** (*rule* ∃ *I*)
AOT-thus ‹∃ x∃ y (A!x & A!y & x ≠ y & [λz [R]zx] = [λz [R]zy])› **by** (*rule* ∃ *I*)
qed

AOT-theorem *aclassical*:*2*:
‹∀ R∃ x∃ y(A!x & A!y & x ≠ y & [λz [R]xz] = [λz [R]yz])›
proof(*rule GEN*)
fix *R*
AOT-obtain *a* **where** *a-prop*:
‹A!a & ∀ F (a[F] ≡ ∃ y(A!y & F = [λz [R]yz] & ¬y[F]))›
using *A*−*objects*[*axiom-inst*] ∃ *E*[*rotated*] **by** *fast*
AOT-have *a-enc*: ‹a[λz [R]az]›
proof (*rule raa*−*cor*:*1*)
AOT-assume *0*: ‹¬a[λz [R]az]›
AOT-hence ‹¬∃ y(A!y & [λz [R]az] = [λz [R]yz] & ¬y[λz [R]az])›
by (*rule a-prop*[*THEN &E*(*2*), *THEN* ∀ *E*(*1*)[**where** τ=«[λz [R]az]»],
THEN oth−*class*−*taut*:*4*:*b*[*THEN* ≡*E*(*1*)],
THEN ≡*E*(*1*), *rotated*])
cqt:*2*[*lambda*]
AOT-hence ‹∀ y ¬(A!y & [λz [R]az] = [λz [R]yz] & ¬y[λz [R]az])›
using *cqt*−*further*:*4 vdash*−*properties*:*10* **by** *blast*
AOT-hence ‹¬(A!a & [λz [R]az] = [λz [R]az] & ¬a[λz [R]az])›
using ∀ *E* **by** *blast*
AOT-hence ‹(A!a & [λz [R]az] = [λz [R]az]) → a[λz [R]az]›
by (*metis &I deduction*−*theorem raa*−*cor*:*3*)
moreover AOT-have ‹[λz [R]az] = [λz [R]az]›
by (*rule* =*I*) *cqt*:*2*[*lambda*]
ultimately AOT-have ‹a[λz [R]az]›
using *a-prop*[*THEN &E*(*1*)] →*E &I* **by** *blast*
AOT-thus ‹a[λz [R]az] & ¬a[λz [R]az]›
using *0 &I* **by** *blast*
qed
AOT-hence ‹∃ y(A!y & [λz [R]az] = [λz [R]yz] & ¬y[λz [R]az])›
by (*rule a-prop*[*THEN &E*(*2*), *THEN* ∀ *E*(*1*), *THEN* ≡*E*(*1*), *rotated*])
cqt:*2*
then AOT-obtain *b* **where** *b-prop*:
‹A!b & [λz [R]az] = [λz [R]bz] & ¬b[λz [R]az]›
using ∃ *E*[*rotated*] **by** *blast*
AOT-have ‹a ≠ b›
apply (*rule* $\equiv_{df}$*I*[*OF* =−*infix*])
using *a-enc b-prop*[*THEN &E*(*2*)]
using ¬¬*I rule*=*E id-sym* ≡*E*(*4*) *oth*−*class*−*taut*:*3*:*a*
raa−*cor*:*3 reductio*−*aa*:*1* **by** *fast*
AOT-hence ‹A!a & A!b & a ≠ b & [λz [R]az] = [λz [R]bz]›
using *b-prop &E a-prop &I* **by** *meson*
AOT-hence ‹∃ y (A!a & A!y & a ≠ y & [λz [R]az] = [λz [R]yz])› **by** (*rule* ∃ *I*)
AOT-thus ‹∃ x∃ y (A!x & A!y & x ≠ y & [λz [R]xz] = [λz [R]yz])› **by** (*rule* ∃ *I*)

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qed

AOT-theorem aclassical:3:
  ‹∀ F∃ x∃ y(A!x & A!y & x ≠ y & [λ [F]x] = [λ [F]y])›
proof(rule GEN)
  fix R
  AOT-obtain a where a-prop:
    ‹A!a & ∀ F (a[F] ≡ ∃ y(A!y & F = [λz [R]y] & ¬y[F]))›
    using A−objects[axiom-inst] ∃ E[rotated] by fast
  AOT-have den: ‹[λz [R]a]↓› by cqt:2[lambda]
  AOT-have a-enc: ‹a[λz [R]a]›
  proof (rule raa−cor:1)
    AOT-assume 0: ‹¬a[λz [R]a]›
    AOT-hence ‹¬∃ y(A!y & [λz [R]a] = [λz [R]y] & ¬y[λz [R]a])›
      by (safe intro!: a-prop[THEN &E(2), THEN ∀ E(1)[where τ=‹«[λz [R]a]»›],
                THEN oth−class−taut:4:b[THEN ≡E(1)],
                THEN ≡E(1), rotated] cqt:2)
    AOT-hence ‹∀ y ¬(A!y & [λz [R]a] = [λz [R]y] & ¬y[λz [R]a])›
      using cqt−further:4 →E by blast
    AOT-hence ‹¬(A!a & [λz [R]a] = [λz [R]a] & ¬a[λz [R]a])› using ∀ E by blast
    AOT-hence ‹(A!a & [λz [R]a] = [λz [R]a]) → a[λz [R]a]›
      by (metis &I deduction−theorem raa−cor:3)
    AOT-hence ‹a[λz [R]a]›
      using a-prop[THEN &E(1)] →E &I
      by (metis rule=I:1 den)
    AOT-thus ‹a[λz [R]a] & ¬a[λz [R]a]›  by (metis 0 raa−cor:3)
  qed
  AOT-hence ‹∃ y(A!y & [λz [R]a] = [λz [R]y] & ¬y[λz [R]a])›
    by (rule a-prop[THEN &E(2), THEN ∀ E(1), OF den, THEN ≡E(1), rotated])
  then AOT-obtain b where b-prop: ‹A!b & [λz [R]a] = [λz [R]b] & ¬b[λz [R]a]›
    using ∃ E[rotated] by blast
  AOT-have 1: ‹a ≠ b›
    apply (rule ≡df I[OF =−infix])
    using a-enc b-prop[THEN &E(2)]
    using ¬¬I rule=E id-sym ≡E(4) oth−class−taut:3:a
          raa−cor:3 reductio−aa:1 by fast
  AOT-have a: ‹[λ [R]a] = ([R]a)›
    apply (rule lambda−predicates:3[zero][axiom-inst, unvarify p])
    by (meson log−prop−prop:2)
  AOT-have b: ‹[λ [R]b] = ([R]b)›
    apply (rule lambda−predicates:3[zero][axiom-inst, unvarify p])
    by (meson log−prop−prop:2)
  AOT-have ‹[λ [R]a] = [λ [R]b]›
    apply (rule rule=E[rotated, OF a[THEN id-sym]])
    apply (rule rule=E[rotated, OF b[THEN id-sym]])
    apply (rule identity:4[THEN ≡df I, OF &I, rotated])
    using b-prop &E apply blast
    apply (safe intro!: &I)
    by (simp add: log−prop−prop:2)+
  AOT-hence ‹A!a & A!b & a ≠ b & [λ [R]a] = [λ [R]b]›
    using 1 a-prop[THEN &E(1)] b-prop[THEN &E(1), THEN &E(1)]
          &I by auto
  AOT-hence ‹∃ y (A!a & A!y & a ≠ y & [λ [R]a] = [λ [R]y])› by (rule ∃ I)
  AOT-thus ‹∃ x∃ y (A!x & A!y & x ≠ y & [λ [R]x] = [λ [R]y])› by (rule ∃ I)
qed

AOT-theorem aclassical2: ‹∃ x∃ y (A!x & A!y & x ≠ y & ∀ F ([F]x ≡ [F]y))›
proof −
  AOT-have ‹∃ x ∃ y ([A!]x & [A!]y & x ≠ y &
              [λz [λxy ∀ F ([F]x ≡ [F]y)]zx] =
              [λz [λxy ∀ F ([F]x ≡ [F]y)]zy])›
    by (rule aclassical:1[THEN ∀ E(1)[where τ=«[λxy ∀ F ([F]x ≡ [F]y)]»]])
       cqt:2
```

```
  then AOT-obtain x where ‹∃ y ([A!]x & [A!]y & x ≠ y &
              [λz [λxy ∀ F ([F]x ≡ [F]y)]zx] =
              [λz [λxy ∀ F ([F]x ≡ [F]y)]zy])›
    using ∃ E[rotated] by blast
  then AOT-obtain y where 0: ‹([A!]x & [A!]y & x ≠ y &
              [λz [λxy ∀ F ([F]x ≡ [F]y)]zx] =
              [λz [λxy ∀ F ([F]x ≡ [F]y)]zy])›
    using ∃ E[rotated] by blast
  AOT-have ‹[λz [λxy ∀ F ([F]x ≡ [F]y)]zx]x›
    by (auto intro!: β←C(1) cqt:2
            simp: &I ex:1:a prod-denotesI rule−ui:3
                 oth−class−taut:3:a universal−cor)
  AOT-hence ‹[λz [λxy ∀ F ([F]x ≡ [F]y)]zy]x›
    by (rule rule=E[rotated, OF 0[THEN &E(2)]])
  AOT-hence ‹[λxy ∀ F ([F]x ≡ [F]y)]xy›
    by (rule β→C(1))
  AOT-hence ‹∀ F ([F]x ≡ [F]y)›
    using β→C(1) old.prod.case by fast
  AOT-hence ‹[A!]x & [A!]y & x ≠ y & ∀ F ([F]x ≡ [F]y)›
    using 0 &E &I by blast
  AOT-hence ‹∃ y ([A!]x & [A!]y & x ≠ y & ∀ F ([F]x ≡ [F]y))› by (rule ∃ I)
  AOT-thus ‹∃ x∃ y ([A!]x & [A!]y & x ≠ y & ∀ F ([F]x ≡ [F]y))› by (rule ∃ I(2))
qed

AOT-theorem kirchner−thm:1:
  ‹[λx φ{x}]↓ ≡ □∀ x∀ y(∀ F([F]x ≡ [F]y) → (φ{x} ≡ φ{y}))›
proof(rule ≡I; rule →I)
  AOT-assume ‹[λx φ{x}]↓›
  AOT-hence ‹□[λx φ{x}]↓› by (metis exist−nec vdash−properties:10)
  moreover AOT-have ‹□[λx φ{x}]↓ → □∀ x∀ y(∀ F([F]x ≡ [F]y) → (φ{x} ≡ φ{y}))›
  proof (rule RM:1; rule →I; rule GEN; rule GEN; rule →I)
    AOT-modally-strict {
      fix x y
      AOT-assume 0: ‹[λx φ{x}]↓›
      moreover AOT-assume ‹∀ F([F]x ≡ [F]y)›
      ultimately AOT-have ‹[λx φ{x}]x ≡ [λx φ{x}]y›
        using ∀ E by blast
      AOT-thus ‹(φ{x} ≡ φ{y})›
        using beta−C−meta[THEN →E, OF 0] ≡E(6) by meson
    }
  qed
  ultimately AOT-show ‹□∀ x∀ y(∀ F([F]x ≡ [F]y) → (φ{x} ≡ φ{y}))›
    using →E by blast
next
  AOT-have ‹□∀ x∀ y(∀ F([F]x ≡ [F]y) → (φ{x} ≡ φ{y})) →
          □∀ y(∃ x(∀ F([F]x ≡ [F]y) & φ{x}) ≡ φ{y})›
  proof(rule RM:1; rule →I; rule GEN)
    AOT-modally-strict {
      AOT-assume ‹∀ x∀ y(∀ F([F]x ≡ [F]y) → (φ{x} ≡ φ{y}))›
      AOT-hence indisc: ‹φ{x} ≡ φ{y}› if ‹∀ F([F]x ≡ [F]y)› for x y
        using ∀ E(2) →E that by blast
      AOT-show ‹(∃ x(∀ F([F]x ≡ [F]y) & φ{x}) ≡ φ{y})› for y
      proof (rule raa−cor:1)
        AOT-assume ‹¬(∃ x(∀ F([F]x ≡ [F]y) & φ{x}) ≡ φ{y})›
        AOT-hence ‹(∃ x(∀ F([F]x ≡ [F]y) & φ{x}) & ¬φ{y}) ∨
                (¬(∃ x(∀ F([F]x ≡ [F]y) & φ{x})) & φ{y})›
          using ≡E(1) oth−class−taut:4:h by blast
        moreover {
          AOT-assume 0: ‹∃ x(∀ F([F]x ≡ [F]y) & φ{x}) & ¬φ{y}›
          AOT-obtain a where ‹∀ F([F]a ≡ [F]y) & φ{a}›
            using ∃ E[rotated, OF 0[THEN &E(1)]]  by blast
          AOT-hence ‹φ{y}›
            using indisc[THEN ≡E(1)] &E by blast
```

```
        AOT-hence ‹p & ¬p› for p
          using 0[THEN &E(2)] &I raa−cor:3 by blast
      }
      moreover {
        AOT-assume 0: ‹(¬(∃x(∀F([F]x ≡ [F]y) & φ{x})) & φ{y})›
        AOT-hence ‹∀x ¬(∀F([F]x ≡ [F]y) & φ{x})›
          using &E(1) cqt−further:4 →E by blast
        AOT-hence ‹¬(∀F([F]y ≡ [F]y) & φ{y})›
          using ∀E by blast
        AOT-hence ‹¬∀F([F]y ≡ [F]y) ∨ ¬φ{y}›
          using ≡E(1) oth−class−taut:5:c by blast
        moreover AOT-have ‹∀F([F]y ≡ [F]y)›
          by (simp add: oth−class−taut:3:a universal−cor)
        ultimately AOT-have ‹¬φ{y}› by (metis ¬¬I ∨E(2))
        AOT-hence ‹p & ¬p› for p
          using 0[THEN &E(2)] &I raa−cor:3 by blast
      }
      ultimately AOT-show ‹p & ¬p› for p
        using ∨E(3) raa−cor:1 by blast
    qed
  }
qed
moreover AOT-assume ‹□∀x∀y(∀F([F]x ≡ [F]y) → (φ{x} ≡ φ{y}))›
ultimately AOT-have ‹□∀y(∃x(∀F([F]x ≡ [F]y) & φ{x}) ≡ φ{y})›
  using →E by blast
AOT-thus ‹[λx φ{x}]↓›
  by (rule safe−ext[axiom-inst, THEN →E, OF &I, rotated]) cqt:2
qed

AOT-theorem kirchner−thm:2:
‹[λx1...xn φ{x1...xn}]↓ ≡ □∀x1...∀xn∀y1...∀yn
  (∀F([F]x1...xn ≡ [F]y1...yn) → (φ{x1...xn} ≡ φ{y1...yn}))›
proof(rule ≡I; rule →I)
  AOT-assume ‹[λx1...xn φ{x1...xn}]↓›
  AOT-hence ‹□[λx1...xn φ{x1...xn}]↓› by (metis exist−nec →E)
  moreover AOT-have ‹□[λx1...xn φ{x1...xn}]↓ → □∀x1...∀xn∀y1...∀yn
    (∀F([F]x1...xn ≡ [F]y1...yn) → (φ{x1...xn} ≡ φ{y1...yn}))›
  proof (rule RM:1; rule →I; rule GEN; rule GEN; rule →I)
    AOT-modally-strict {
      fix x1xn y1yn :: ‹'a AOT-var›
      AOT-assume 0: ‹[λx1...xn φ{x1...xn}]↓›
      moreover AOT-assume ‹∀F([F]x1...xn ≡ [F]y1...yn)›
      ultimately AOT-have ‹[λx1...xn φ{x1...xn}]x1...xn ≡
                          [λx1...xn φ{x1...xn}]y1...yn›
        using ∀E by blast
      AOT-thus ‹(φ{x1...xn} ≡ φ{y1...yn})›
        using beta−C−meta[THEN →E, OF 0] ≡E(6) by meson
    }
  qed
  ultimately AOT-show ‹□∀x1...∀xn∀y1...∀yn(
    ∀F([F]x1...xn ≡ [F]y1...yn) → (φ{x1...xn} ≡ φ{y1...yn})
  )›
    using →E by blast
next
  AOT-have ‹
    □(∀x1...∀xn∀y1...∀yn
      (∀F([F]x1...xn ≡ [F]y1...yn) → (φ{x1...xn} ≡ φ{y1...yn})))
    → □∀y1...∀yn
       ((∃x1...∃xn(∀F([F]x1...xn ≡ [F]y1...yn) & φ{x1...xn})) ≡
        φ{y1...yn})›
  proof(rule RM:1; rule →I; rule GEN)
    AOT-modally-strict {
      AOT-assume ‹∀x1...∀xn∀y1...∀yn
```

$(\forall F([F]x_1...x_n \equiv [F]y_1...y_n) \rightarrow (\varphi\{x_1...x_n\} \equiv \varphi\{y_1...y_n\}))$›
AOT-hence *indisc*: ‹$\varphi\{x_1...x_n\} \equiv \varphi\{y_1...y_n\}$›
if ‹$\forall F([F]x_1...x_n \equiv [F]y_1...y_n)$› **for** x_1x_n y_1y_n
using $\forall E(2)$ $\rightarrow E$ *that* **by** *blast*
AOT-show ‹$(\exists x_1...\exists x_n(\forall F([F]x_1...x_n \equiv [F]y_1...y_n)$ & $\varphi\{x_1...x_n\})) \equiv$
$\varphi\{y_1...y_n\}$› **for** y_1y_n
proof (*rule raa−cor*:*1*)
AOT-assume ‹$\neg((\exists x_1...\exists x_n(\forall F([F]x_1...x_n \equiv [F]y_1...y_n)$ & $\varphi\{x_1...x_n\})) \equiv$
$\varphi\{y_1...y_n\})$›
AOT-hence ‹$((\exists x_1...\exists x_n(\forall F([F]x_1...x_n \equiv [F]y_1...y_n)$
& $\varphi\{x_1...x_n\}))$
& $\neg\varphi\{y_1...y_n\}) \vee$
$(\neg(\exists x_1...\exists x_n(\forall F([F]x_1...x_n \equiv [F]y_1...y_n)$ & $\varphi\{x_1...x_n\}))$
& $\varphi\{y_1...y_n\})$›
using $\equiv E(1)$ *oth−class−taut*:*4*:*h* **by** *blast*
moreover {
AOT-assume *0*: ‹$(\exists x_1...\exists x_n(\forall F([F]x_1...x_n \equiv [F]y_1...y_n)$ & $\varphi\{x_1...x_n\}))$
& $\neg\varphi\{y_1...y_n\}$›
AOT-obtain a_1a_n **where** ‹$\forall F([F]a_1...a_n \equiv [F]y_1...y_n)$ & $\varphi\{a_1...a_n\}$›
using $\exists E$[*rotated*, *OF 0*[*THEN* &*E*(*1*)]] **by** *blast*
AOT-hence ‹$\varphi\{y_1...y_n\}$›
using *indisc*[*THEN* $\equiv E(1)$] &*E* **by** *blast*
AOT-hence ‹p & $\neg p$› **for** p
using *0*[*THEN* &*E*(*2*)] &*I* *raa−cor*:*3* **by** *blast*
}
moreover {
AOT-assume *0*: ‹$\neg(\exists x_1...\exists x_n(\forall F([F]x_1...x_n \equiv [F]y_1...y_n)$ & $\varphi\{x_1...x_n\}))$
& $\varphi\{y_1...y_n\}$›
AOT-hence ‹$\forall x_1...\forall x_n$ $\neg(\forall F([F]x_1...x_n \equiv [F]y_1...y_n)$ & $\varphi\{x_1...x_n\})$›
using &*E*(*1*) *cqt−further*:*4* $\rightarrow E$ **by** *blast*
AOT-hence ‹$\neg(\forall F([F]y_1...y_n \equiv [F]y_1...y_n)$ & $\varphi\{y_1...y_n\})$›
using $\forall E$ **by** *blast*
AOT-hence ‹$\neg\forall F([F]y_1...y_n \equiv [F]y_1...y_n) \vee \neg\varphi\{y_1...y_n\}$›
using $\equiv E(1)$ *oth−class−taut*:*5*:*c* **by** *blast*
moreover AOT-have ‹$\forall F([F]y_1...y_n \equiv [F]y_1...y_n)$›
by (*simp add*: *oth−class−taut*:*3*:*a universal−cor*)
ultimately AOT-have ‹$\neg\varphi\{y_1...y_n\}$›
by (*metis* $\neg\neg I$ $\vee E(2)$)
AOT-hence ‹p & $\neg p$› **for** p
using *0*[*THEN* &*E*(*2*)] &*I* *raa−cor*:*3* **by** *blast*
}
ultimately AOT-show ‹p & $\neg p$› **for** p
using $\vee E(3)$ *raa−cor*:*1* **by** *blast*
qed
}
qed
moreover AOT-assume ‹$\Box\forall x_1...\forall x_n\forall y_1...\forall y_n$
$(\forall F([F]x_1...x_n \equiv [F]y_1...y_n) \rightarrow (\varphi\{x_1...x_n\} \equiv \varphi\{y_1...y_n\}))$›
ultimately AOT-have ‹$\Box\forall y_1...\forall y_n$
$((\exists x_1...\exists x_n(\forall F([F]x_1...x_n \equiv [F]y_1...y_n)$ & $\varphi\{x_1...x_n\})) \equiv$
$\varphi\{y_1...y_n\})$›
using $\rightarrow E$ **by** *blast*
AOT-thus ‹$[\lambda x_1...x_n\ \varphi\{x_1...x_n\}]\downarrow$›
by (*rule safe−ext*[*axiom-inst*, *THEN* $\rightarrow E$, *OF* &*I*, *rotated*]) *cqt*:*2*
qed

AOT-theorem *kirchner−thm−cor*:*1*:
‹$[\lambda x\ \varphi\{x\}]\downarrow \rightarrow \forall x\forall y(\forall F([F]x \equiv [F]y) \rightarrow \Box(\varphi\{x\} \equiv \varphi\{y\}))$›
proof(*rule* $\rightarrow I$; *rule GEN*; *rule GEN*; *rule* $\rightarrow I$)
fix x y
AOT-assume ‹$[\lambda x\ \varphi\{x\}]\downarrow$›
AOT-hence ‹$\Box\forall x\forall y\ (\forall F\ ([F]x \equiv [F]y) \rightarrow (\varphi\{x\} \equiv \varphi\{y\}))$›
by (*rule kirchner−thm*:*1*[*THEN* $\equiv E(1)$])

AOT-hence ‹$\forall x \Box \forall y\ (\forall F\ ([F]x \equiv [F]y) \rightarrow (\varphi\{x\} \equiv \varphi\{y\}))$›
  using *CBF*[*THEN* →*E*] **by** *blast*
AOT-hence ‹$\Box \forall y\ (\forall F\ ([F]x \equiv [F]y) \rightarrow (\varphi\{x\} \equiv \varphi\{y\}))$›
  using ∀*E* **by** *blast*
AOT-hence ‹$\forall y\ \Box(\forall F\ ([F]x \equiv [F]y) \rightarrow (\varphi\{x\} \equiv \varphi\{y\}))$›
  using *CBF*[*THEN* →*E*] **by** *blast*
AOT-hence ‹$\Box(\forall F\ ([F]x \equiv [F]y) \rightarrow (\varphi\{x\} \equiv \varphi\{y\}))$›
  using ∀*E* **by** *blast*
AOT-hence ‹$\Box \forall F\ ([F]x \equiv [F]y) \rightarrow \Box(\varphi\{x\} \equiv \varphi\{y\})$›
  using *qml*:*1*[*axiom-inst*] *vdash−properties*:*6* **by** *blast*
moreover AOT-assume ‹$\forall F([F]x \equiv [F]y)$›
ultimately AOT-show ‹$\Box(\varphi\{x\} \equiv \varphi\{y\})$› **using** →*E* *ind−nec* **by** *blast*
qed

AOT-theorem *kirchner−thm−cor*:*2*:
‹$[\lambda x_1...x_n\ \varphi\{x_1...x_n\}]\downarrow \rightarrow \forall x_1...\forall x_n \forall y_1...\forall y_n$
$(\forall F([F]x_1...x_n \equiv [F]y_1...y_n) \rightarrow \Box(\varphi\{x_1...x_n\} \equiv \varphi\{y_1...y_n\}))$›
proof(*rule* →*I*; *rule GEN*; *rule GEN*; *rule* →*I*)
  fix $x_1x_n\ y_1y_n$
  AOT-assume ‹$[\lambda x_1...x_n\ \varphi\{x_1...x_n\}]\downarrow$›
  AOT-hence *0*: ‹$\Box \forall x_1...\forall x_n \forall y_1...\forall y_n$
  $(\forall F\ ([F]x_1...x_n \equiv [F]y_1...y_n) \rightarrow (\varphi\{x_1...x_n\} \equiv \varphi\{y_1...y_n\}))$›
    by (*rule kirchner−thm*:*2*[*THEN* ≡*E*(*1*)])
  AOT-have ‹$\forall x_1...\forall x_n \forall y_1...\forall y_n$
  $\Box(\forall F\ ([F]x_1...x_n \equiv [F]y_1...y_n) \rightarrow (\varphi\{x_1...x_n\} \equiv \varphi\{y_1...y_n\}))$›
  proof(*rule GEN*; *rule GEN*)
    fix $x_1x_n\ y_1y_n$
    AOT-show ‹$\Box(\forall F\ ([F]x_1...x_n \equiv [F]y_1...y_n) \rightarrow (\varphi\{x_1...x_n\} \equiv \varphi\{y_1...y_n\}))$›
      apply (*rule RM*:*1*[*THEN* →*E*, *rotated*, *OF 0*]; *rule* →*I*)
      using ∀*E* **by** *blast*
  qed
  AOT-hence ‹$\forall y_1...\forall y_n\ \Box(\forall F\ ([F]x_1...x_n \equiv [F]y_1...y_n) \rightarrow$
  $(\varphi\{x_1...x_n\} \equiv \varphi\{y_1...y_n\}))$›
    using ∀*E* **by** *blast*
  AOT-hence ‹$\Box(\forall F\ ([F]x_1...x_n \equiv [F]y_1...y_n) \rightarrow (\varphi\{x_1...x_n\} \equiv \varphi\{y_1...y_n\}))$›
    using ∀*E* **by** *blast*
  AOT-hence ‹$\Box(\forall F\ ([F]x_1...x_n \equiv [F]y_1...y_n) \rightarrow (\varphi\{x_1...x_n\} \equiv \varphi\{y_1...y_n\}))$›
    using ∀*E* **by** *blast*
  AOT-hence *0*: ‹$\Box \forall F\ ([F]x_1...x_n \equiv [F]y_1...y_n) \rightarrow \Box(\varphi\{x_1...x_n\} \equiv \varphi\{y_1...y_n\})$›
    using *qml*:*1*[*axiom-inst*] *vdash−properties*:*6* **by** *blast*
  moreover AOT-assume ‹$\forall F([F]x_1...x_n \equiv [F]y_1...y_n)$›
  moreover AOT-have ‹$[\lambda x_1...x_n\ \Box \forall F\ ([F]x_1...x_n \equiv [F]y_1...y_n)]\downarrow$› **by** *cqt*:*2*
  ultimately AOT-have ‹$[\lambda x_1...x_n\ \Box \forall F\ ([F]x_1...x_n \equiv [F]y_1...y_n)]x_1...x_n \equiv$
  $[\lambda x_1...x_n\ \Box \forall F\ ([F]x_1...x_n \equiv [F]y_1...y_n)]y_1...y_n$›
    using ∀*E* **by** *blast*
  moreover AOT-have ‹$[\lambda x_1...x_n\ \Box \forall F\ ([F]x_1...x_n \equiv [F]y_1...y_n)]y_1...y_n$›
    apply (*rule* β←*C*(*1*))
      apply *cqt*:*2*[*lambda*]
     apply (*fact cqt*:*2*[*const-var*][*axiom-inst*])
    by (*simp add*: *RN GEN oth−class−taut*:*3*:*a*)
  ultimately AOT-have ‹$[\lambda x_1...x_n\ \Box \forall F\ ([F]x_1...x_n \equiv [F]y_1...y_n)]x_1...x_n$›
    using ≡*E*(*2*) **by** *blast*
  AOT-hence ‹$\Box \forall F\ ([F]x_1...x_n \equiv [F]y_1...y_n)$›
    using β→*C*(*1*) **by** *blast*
  AOT-thus ‹$\Box(\varphi\{x_1...x_n\} \equiv \varphi\{y_1...y_n\})$› **using** →*E* *0* **by** *blast*
qed

8.12 Propositional Properties

AOT-define *propositional* :: ‹$\Pi \Rightarrow \varphi$› (‹*Propositional*′(-′)›)
  prop−prop1: ‹$Propositional([F]) \equiv_{df} \exists p(F = [\lambda y\ p])$›

AOT-theorem *prop−prop2*:*1*: ‹$\forall p\ [\lambda y\ p]\downarrow$›

by (*rule GEN*) *cqt*:*2*[*lambda*]

AOT-theorem *prop*−*prop2*:*2*: ‹[λν φ]↓›
by *cqt*:*2*[*lambda*]

AOT-theorem *prop*−*prop2*:*3*: ‹*F* = [λ*y p*] → □∀ *x*([*F*]*x* ≡ *p*)›
proof (*rule* →*I*)
AOT-assume *0*: ‹*F* = [λ*y p*]›
AOT-show ‹□∀ *x*([*F*]*x* ≡ *p*)›
by (*rule rule*=*E*[*rotated*, *OF 0*[*symmetric*]];
rule RN; *rule GEN*; *rule beta*−*C*−*meta*[*THEN* →*E*])
cqt:*2*[*lambda*]
qed

AOT-theorem *prop*−*prop2*:*4*: ‹*Propositional*([*F*]) → □*Propositional*([*F*])›
proof(*rule* →*I*)
AOT-assume ‹*Propositional*([*F*])›
AOT-hence ‹∃ *p*(*F* = [λ*y p*])›
using $\equiv_{df}$*E*[*OF prop*−*prop1*] **by** *blast*
then AOT-obtain *p* **where** ‹*F* = [λ*y p*]›
using ∃ *E*[*rotated*] **by** *blast*
AOT-hence ‹□(*F* = [λ*y p*])›
using *id*−*nec*:*2 modus*−*tollens*:*1 raa*−*cor*:*3* **by** *blast*
AOT-hence ‹∃ *p* □(*F* = [λ*y p*])›
using ∃ *I* **by** *fast*
AOT-hence *0*: ‹□∃ *p* (*F* = [λ*y p*])›
by (*metis Buridan vdash*−*properties*:*10*)
AOT-thus ‹□*Propositional*([*F*])›
using *prop*−*prop1*[*THEN* ≡*Df*]
by (*AOT-subst* ‹*Propositional*([*F*])› ‹∃ *p* (*F* = [λ*y p*])›) *auto*
qed

AOT-define *indicriminate* :: ‹Π ⇒ φ› (‹*Indiscriminate*′(-′)›)
prop−*indis*: ‹*Indiscriminate*([*F*]) $\equiv_{df}$ *F*↓ & □(∃ *x* [*F*]*x* → ∀ *x* [*F*]*x*)›

AOT-theorem *prop*−*in*−*thm*: ‹*Propositional*([Π]) → *Indiscriminate*([Π])›
proof(*rule* →*I*)
AOT-assume ‹*Propositional*([Π])›
AOT-hence ‹∃ *p* Π = [λ*y p*]› **using** $\equiv_{df}$*E*[*OF prop*−*prop1*] **by** *blast*
then AOT-obtain *p* **where** Π-*def*: ‹Π = [λ*y p*]› **using** ∃ *E*[*rotated*] **by** *blast*
AOT-show ‹*Indiscriminate*([Π])›
proof (*rule* $\equiv_{df}$*I*[*OF prop*−*indis*]; *rule* &*I*)
AOT-show ‹Π↓›
using Π-*def* **by** (*meson t*=*t*−*proper*:*1 vdash*−*properties*:*6*)
next
AOT-show ‹□(∃ *x* [Π]*x* → ∀ *x* [Π]*x*)›
proof (*rule rule*=*E*[*rotated*, *OF* Π-*def*[*symmetric*]];
rule RN; *rule* →*I*; *rule GEN*)
AOT-modally-strict {
AOT-assume ‹∃ *x* [λ*y p*]*x*›
then AOT-obtain *a* **where** ‹[λ*y p*]*a*› **using** ∃ *E*[*rotated*] **by** *blast*
AOT-hence *0*: ‹*p*› **by** (*metis* β→*C*(*1*))
AOT-show ‹[λ*y p*]*x*› **for** *x*
apply (*rule* β←*C*(*1*))
apply *cqt*:*2*[*lambda*]
apply (*fact cqt*:*2*[*const-var*][*axiom-inst*])
by (*fact 0*)
}
qed
qed
qed

AOT-theorem *prop*−*in*−*f*:*1*: ‹*Necessary*([*F*]) → *Indiscriminate*([*F*])›

```
proof (rule →I)
  AOT-assume ‹Necessary([F])›
  AOT-hence 0: ‹□∀x_1...∀x_n [F]x_1...x_n›
    using ≡_df E[OF contingent−properties:1] by blast
  AOT-show ‹Indiscriminate([F])›
    by (rule ≡_df I[OF prop−indis])
       (metis 0 KBasic:1 &I ex:1:a rule−ui:2[const-var] →E)
qed

AOT-theorem prop−in−f:2: ‹Impossible([F]) → Indiscriminate([F])›
proof (rule →I)
  AOT-modally-strict {
    AOT-have ‹∀x ¬[F]x → (∃x [F]x → ∀x [F]x)›
      by (metis ∃E cqt−orig:3 Hypothetical Syllogism →I raa−cor:3)
  }
  AOT-hence 0: ‹□∀x ¬[F]x → □(∃x [F]x → ∀x [F]x)›
    by (rule RM:1)
  AOT-assume ‹Impossible([F])›
  AOT-hence ‹□∀x ¬[F]x›
    using ≡_df E[OF contingent−properties:2] &E by blast
  AOT-hence 1: ‹□(∃x [F]x → ∀x [F]x)›
    using 0 →E by blast
  AOT-show ‹Indiscriminate([F])›
    by (rule ≡_df I[OF prop−indis]; rule &I)
       (simp add: ex:1:a rule−ui:2[const-var] 1)+
qed

AOT-theorem prop−in−f:3:a: ‹¬Indiscriminate([E!])›
proof(rule raa−cor:2)
  AOT-assume ‹Indiscriminate([E!])›
  AOT-hence 0: ‹□(∃x [E!]x → ∀x [E!]x)›
    using ≡_df E[OF prop−indis] &E by blast
  AOT-hence ‹◇∃x [E!]x → ◇∀x [E!]x›
    using KBasic:13 vdash−properties:10 by blast
  moreover AOT-have ‹◇∃x [E!]x›
    by (simp add: thm−cont−e:3)
  ultimately AOT-have ‹◇∀x [E!]x›
    by (metis vdash−properties:6)
  AOT-thus ‹p & ¬p› for p
    by (metis ≡_df E conventions:5 o−objects−exist:5 reductio−aa:1)
qed

AOT-theorem prop−in−f:3:b: ‹¬Indiscriminate([E!]⁻)›
proof (rule rule=E[rotated, OF rel−neg−T:2[symmetric]];
       rule raa−cor:2)
  AOT-assume ‹Indiscriminate([λx ¬[E!]x])›
  AOT-hence 0: ‹□(∃x [λx ¬[E!]x]x → ∀x [λx ¬[E!]x]x)›
    using ≡_df E[OF prop−indis] &E by blast
  AOT-hence ‹□∃x [λx ¬[E!]x]x → □∀x [λx ¬[E!]x]x›
    using →E qml:1 vdash−properties:1[2] by blast
  moreover AOT-have ‹□∃x [λx ¬[E!]x]x›
    apply (AOT-subst ‹[λx ¬E!x]x› ‹¬E!x› for: x)
    apply (rule beta−C−meta[THEN →E])
     apply cqt:2
    by (metis (full-types) B◇ RN T◇ cqt−further:2
                       o−objects−exist:5 →E)
  ultimately AOT-have 1: ‹□∀x [λx ¬[E!]x]x›
    by (metis vdash−properties:6)
  AOT-hence ‹□∀x ¬[E!]x›
    by (AOT-subst (reverse) ‹¬[E!]x›  ‹[λx ¬[E!]x]x› for: x)
       (auto intro!: cqt:2 beta−C−meta[THEN →E])
  AOT-hence ‹∀x □¬[E!]x› by (metis CBF vdash−properties:10)
  moreover AOT-obtain a where abs-a: ‹O!a›
```

  using $\exists E$ *o−objects−exist:1 qml:2[axiom-inst]* $\rightarrow E$ **by** *blast*
 ultimately AOT-have ‹$\Box\neg[E!]a$› **using** $\forall E$ **by** *blast*
 AOT-hence *2*: ‹$\neg\Diamond[E!]a$› **by** (*metis* $\equiv_{df}E$ *conventions:5 reductio−aa:1*)
 AOT-have ‹$A!a$›
  apply (*rule* $=_{df}I(2)$*[OF AOT-abstract]*)
   apply *cqt:2[lambda]*
  apply (*rule* $\beta{\leftarrow}C(1)$)
   apply *cqt:2[lambda]*
  using *cqt:2[const-var][axiom-inst]* **apply** *blast*
  by (*fact 2*)
 AOT-thus ‹p & $\neg p$› **for** p **using** *abs-a*
  by (*metis* $\equiv E(1)$ *oa−contingent:2 reductio−aa:1*)
qed

AOT-theorem *prop−in−f:3:c*: ‹$\neg$*Indiscriminate*$(O!)$›
proof(*rule raa−cor:2*)
 AOT-assume ‹*Indiscriminate*$(O!)$›
 AOT-hence *0*: ‹$\Box(\exists x\ O!x \rightarrow \forall x\ O!x)$›
  using $\equiv_{df}E$*[OF prop−indis]* &E **by** *blast*
 AOT-hence ‹$\Box\exists x\ O!x \rightarrow \Box\forall x\ O!x$›
  using *qml:1[axiom-inst] vdash−properties:6* **by** *blast*
 moreover AOT-have ‹$\Box\exists x\ O!x$›
  using *o−objects−exist:1* **by** *blast*
 ultimately AOT-have ‹$\Box\forall x\ O!x$›
  by (*metis vdash−properties:6*)
 AOT-thus ‹p & $\neg p$› **for** p
  by (*metis o−objects−exist:3 qml:2[axiom-inst] raa−cor:3* $\rightarrow E$)
qed

AOT-theorem *prop−in−f:3:d*: ‹$\neg$*Indiscriminate*$(A!)$›
proof(*rule raa−cor:2*)
 AOT-assume ‹*Indiscriminate*$(A!)$›
 AOT-hence *0*: ‹$\Box(\exists x\ A!x \rightarrow \forall x\ A!x)$›
  using $\equiv_{df}E$*[OF prop−indis]* &E **by** *blast*
 AOT-hence ‹$\Box\exists x\ A!x \rightarrow \Box\forall x\ A!x$›
  using *qml:1[axiom-inst] vdash−properties:6* **by** *blast*
 moreover AOT-have ‹$\Box\exists x\ A!x$›
  using *o−objects−exist:2* **by** *blast*
 ultimately AOT-have ‹$\Box\forall x\ A!x$›
  by (*metis vdash−properties:6*)
 AOT-thus ‹p & $\neg p$› **for** p
  by (*metis o−objects−exist:4 qml:2[axiom-inst] raa−cor:3* $\rightarrow E$)
qed

AOT-theorem *prop−in−f:4:a*: ‹$\neg$*Propositional*$(E!)$›
 using *modus−tollens:1 prop−in−f:3:a prop−in−thm* **by** *blast*

AOT-theorem *prop−in−f:4:b*: ‹$\neg$*Propositional*$(E!^{-})$›
 using *modus−tollens:1 prop−in−f:3:b prop−in−thm* **by** *blast*

AOT-theorem *prop−in−f:4:c*: ‹$\neg$*Propositional*$(O!)$›
 using *modus−tollens:1 prop−in−f:3:c prop−in−thm* **by** *blast*

AOT-theorem *prop−in−f:4:d*: ‹$\neg$*Propositional*$(A!)$›
 using *modus−tollens:1 prop−in−f:3:d prop−in−thm* **by** *blast*

AOT-theorem *prop−prop−nec:1*: ‹$\Diamond\exists p\ (F = [\lambda y\ p]) \rightarrow \exists p(F = [\lambda y\ p])$›
proof(*rule* $\rightarrow I$)
 AOT-assume ‹$\Diamond\exists p\ (F = [\lambda y\ p])$›
 AOT-hence ‹$\exists p\ \Diamond(F = [\lambda y\ p])$›
  by (*metis* $BF\Diamond$ $\rightarrow E$)
 then AOT-obtain p **where** ‹$\Diamond(F = [\lambda y\ p])$›
  using $\exists E$*[rotated]* **by** *blast*

AOT-hence ‹$F = [\lambda y\ p]$›
  by (*metis derived−S5−rules*:*2* *emptyE* *id−nec*:*2* $\to E$)
AOT-thus ‹$\exists p(F = [\lambda y\ p])$› **by** (*rule* $\exists I$)
qed

AOT-theorem *prop−prop−nec*:*2*: ‹$\forall p\ (F \neq [\lambda y\ p]) \to \Box\forall p(F \neq [\lambda y\ p])$›
proof(*rule* $\to I$)
AOT-assume ‹$\forall p\ (F \neq [\lambda y\ p])$›
AOT-hence ‹$(F \neq [\lambda y\ p])$› **for** *p*
  using $\forall E$ **by** *blast*
AOT-hence ‹$\Box(F \neq [\lambda y\ p])$› **for** *p*
  by (*rule id−nec2*:*2*[*unvarify* β, *THEN* $\to E$, *rotated*]) *cqt*:*2*
AOT-hence ‹$\forall p\ \Box(F \neq [\lambda y\ p])$› **by** (*rule GEN*)
AOT-thus ‹$\Box\forall p\ (F \neq [\lambda y\ p])$› **using** *BF*[*THEN* $\to E$] **by** *fast*
qed

AOT-theorem *prop−prop−nec*:*3*: ‹$\exists p\ (F = [\lambda y\ p]) \to \Box\exists p(F = [\lambda y\ p])$›
proof(*rule* $\to I$)
AOT-assume ‹$\exists p\ (F = [\lambda y\ p])$›
then AOT-obtain *p* **where** ‹$(F = [\lambda y\ p])$› **using** $\exists E$[*rotated*] **by** *blast*
AOT-hence ‹$\Box(F = [\lambda y\ p])$› **by** (*metis id−nec*:*2* $\to E$)
AOT-hence ‹$\exists p\Box(F = [\lambda y\ p])$› **by** (*rule* $\exists I$)
AOT-thus ‹$\Box\exists p(F = [\lambda y\ p])$› **by** (*metis Buridan* $\to E$)
qed

AOT-theorem *prop−prop−nec*:*4*: ‹$\Diamond\forall p\ (F \neq [\lambda y\ p]) \to \forall p(F \neq [\lambda y\ p])$›
proof(*rule* $\to I$)
AOT-assume ‹$\Diamond\forall p\ (F \neq [\lambda y\ p])$›
AOT-hence ‹$\forall p\ \Diamond(F \neq [\lambda y\ p])$› **by** (*metis Buridan*$\Diamond$ $\to E$)
AOT-hence ‹$\Diamond(F \neq [\lambda y\ p])$› **for** *p*
  using $\forall E$ **by** *blast*
AOT-hence ‹$F \neq [\lambda y\ p]$› **for** *p*
  by (*rule id−nec2*:*3*[*unvarify* β, *THEN* $\to E$, *rotated*]) *cqt*:*2*
AOT-thus ‹$\forall p\ (F \neq [\lambda y\ p])$› **by** (*rule GEN*)
qed

AOT-theorem *enc−prop−nec*:*1*:
‹$\Diamond\forall F\ (x[F] \to \exists p(F = [\lambda y\ p])) \to \forall F(x[F] \to \exists p\ (F = [\lambda y\ p]))$›
proof(*rule* $\to I$; *rule GEN*; *rule* $\to I$)
fix *F*
AOT-assume ‹$\Diamond\forall F\ (x[F] \to \exists p(F = [\lambda y\ p]))$›
AOT-hence ‹$\forall F\ \Diamond(x[F] \to \exists p(F = [\lambda y\ p]))$›
  using *Buridan*$\Diamond$ *vdash−properties*:*10* **by** *blast*
AOT-hence *0*: ‹$\Diamond(x[F] \to \exists p(F = [\lambda y\ p]))$› **using** $\forall E$ **by** *blast*
AOT-assume ‹$x[F]$›
AOT-hence ‹$\Box x[F]$› **by** (*metis en−eq*:*2*[*1*] $\equiv E(1)$)
AOT-hence ‹$\Diamond\exists p(F = [\lambda y\ p])$›
  using *0* **by** (*metis KBasic2*:*4* $\equiv E(1)$ *vdash−properties*:*10*)
AOT-thus ‹$\exists p(F = [\lambda y\ p])$›
  using *prop−prop−nec*:*1*[*THEN* $\to E$] **by** *blast*
qed

AOT-theorem *enc−prop−nec*:*2*:
‹$\forall F\ (x[F] \to \exists p(F = [\lambda y\ p])) \to \Box\forall F(x[F] \to \exists p\ (F = [\lambda y\ p]))$›
using *derived−S5−rules*:*1*[**where** Γ={}, *simplified*, *OF enc−prop−nec*:*1*]
by *blast*

9 Basic Logical Objects

AOT-define *TruthValueOf* :: ‹$\tau \Rightarrow \varphi \Rightarrow \varphi$› (‹*TruthValueOf*′(-,-′)›)
tv−p: ‹*TruthValueOf*$(x,p) \equiv_{df} A!x\ \&\ \forall F\ (x[F] \equiv \exists q((q \equiv p)\ \&\ F = [\lambda y\ q]))$›

AOT-theorem *p−has−!tv:1*: ‹∃ *x TruthValueOf*(*x*,*p*)›
using *tv−p*[*THEN* ≡*Df*]
by (*AOT-subst* ‹*TruthValueOf*(*x*,*p*)›
‹*A*!*x* & ∀ *F* (*x*[*F*] ≡ ∃ *q*((*q* ≡ *p*) & *F* = [λ*y q*]))› **for**: *x*)
(*simp add*: *A−objects*[*axiom-inst*])

AOT-theorem *p−has−!tv:2*: ‹∃!*x TruthValueOf*(*x*,*p*)›
using *tv−p*[*THEN* ≡*Df*]
by (*AOT-subst* ‹*TruthValueOf*(*x*,*p*)›
‹*A*!*x* & ∀ *F* (*x*[*F*] ≡ ∃ *q*((*q* ≡ *p*) & *F* = [λ*y q*]))› **for**: *x*)
(*simp add*: *A−objects*!)

AOT-theorem *uni−tv*: ‹ι*x TruthValueOf*(*x*,*p*)↓›
using *A−Exists:2 RA*[*2*] ≡*E*(*2*) *p−has−!tv:2* **by** *blast*

AOT-define *TheTruthValueOf* :: ‹φ ⇒ κ_s› (‹∘-› [*100*] *100*)
the−tv−p: ‹∘*p* $=_{df}$ ι*x TruthValueOf*(*x*,*p*)›

AOT-define *PropEnc* :: ‹τ ⇒ φ ⇒ φ› (**infixl** ‹**Σ**› *40*)
prop−enc: ‹*x***Σ***p* $\equiv_{df}$ *x*↓ & *x*[λ*y p*]›

AOT-theorem *tv−id:1*: ‹∘*p* = ι*x* (*A*!*x* & ∀ *F* (*x*[*F*] ≡ ∃ *q*((*q* ≡ *p*) & *F* = [λ*y q*])))›
proof −
AOT-have ‹□∀ *x*(*TruthValueOf*(*x*,*p*) ≡ *A*!*x* & ∀ *F* (*x*[*F*] ≡ ∃ *q*((*q* ≡ *p*) & *F* = [λ*y q*])))›
by (*rule RN*; *rule GEN*; *rule tv−p*[*THEN* ≡*Df*])
AOT-hence ‹ι*x TruthValueOf*(*x*,*p*) = ι*x* (*A*!*x* & ∀ *F* (*x*[*F*] ≡ ∃ *q*((*q* ≡ *p*) & *F* = [λ*y q*])))›
using *equiv−desc−eq:3*[*THEN* →*E*, *OF* &*I*, *OF uni−tv*] **by** *simp*
thus *?thesis*
using $=_{df}$*I*(*1*)[*OF the−tv−p*, *OF uni−tv*] **by** *fast*
qed

AOT-theorem *tv−id:2*: ‹∘*p***Σ***p*›
proof −
AOT-modally-strict {
AOT-have ‹(*p* ≡ *p*) & [λ*y p*] = [λ*y p*]›
by (*auto simp*: *prop−prop2:2 rule*=*I:1 intro*!: ≡*I* →*I* &*I*)
AOT-hence ‹∃ *q* ((*q* ≡ *p*) & [λ*y p*] = [λ*y q*])›
using ∃ *I* **by** *fast*
}
AOT-hence ‹𝓐∃ *q* ((*q* ≡ *p*) & [λ*y p*] = [λ*y q*])›
using *RA*[*2*] **by** *blast*
AOT-hence ‹ι*x*(*A*!*x* & ∀ *F* (*x*[*F*] ≡ ∃ *q* ((*q* ≡ *p*) & *F* = [λ*y q*])))[λ*y p*]›
by (*safe intro*!: *desc−nec−encode:1*[*unvarify F*, *THEN* ≡*E*(*2*)] *cqt:2*)
AOT-hence ‹ι*x*(*A*!*x* & ∀ *F* (*x*[*F*] ≡ ∃ *q* ((*q* ≡ *p*) & *F* = [λ*y q*])))**Σ***p*›
by (*safe intro*!: *prop−enc*[*THEN* $\equiv_{df}$*I*] &*I A−descriptions*)
AOT-thus ‹∘*p***Σ***p*›
by (*rule rule*=*E*[*rotated*, *OF tv−id:1*[*symmetric*]])
qed

AOT-theorem *TV−lem1:1*:
‹*p* ≡ ∀ *F*(∃ *q* (*q* & *F* = [λ*y q*]) ≡ ∃ *q*((*q* ≡ *p*) & *F* = [λ*y q*]))›
proof(*safe intro*!: ≡*I* →*I GEN*)
fix *F*
AOT-assume ‹∃ *q* (*q* & *F* = [λ*y q*])›
then AOT-obtain *q* **where** ‹*q* & *F* = [λ*y q*]› **using** ∃ *E*[*rotated*] **by** *blast*
moreover AOT-assume *p*
ultimately AOT-have ‹(*q* ≡ *p*) & *F* = [λ*y q*]›
by (*metis* &*I* &*E*(*1*) &*E*(*2*) *deduction−theorem* ≡*I*)
AOT-thus ‹∃ *q* ((*q* ≡ *p*) & *F* = [λ*y q*])› **by** (*rule* ∃ *I*)

next
  fix F
  AOT-assume ‹∃ q ((q ≡ p) & F = [λy q])›
  then AOT-obtain q **where** ‹(q ≡ p) & F = [λy q]› **using** ∃ *E*[*rotated*] **by** *blast*
  moreover AOT-assume p
  ultimately AOT-have ‹q & F = [λy q]›
    by (*metis* &*I* &*E*(*1*) &*E*(*2*) ≡*E*(*2*))
  AOT-thus ‹∃ q (q & F = [λy q])› **by** (*rule* ∃ *I*)
next
  AOT-assume ‹∀ F (∃ q (q & F = [λy q]) ≡ ∃ q ((q ≡ p) & F = [λy q]))›
  AOT-hence ‹∃ q (q & [λy p] = [λy q]) ≡ ∃ q ((q ≡ p) & [λy p] = [λy q])›
    using ∀ *E*(*1*)[*rotated*, *OF prop−prop2*:*2*] **by** *blast*
  moreover AOT-have ‹∃ q ((q ≡ p) & [λy p] = [λy q])›
    by (*rule* ∃ *I*(*2*)[**where** β=p])
      (*simp add*: *rule*=*I*:*1* &*I oth−class−taut*:*3*:*a prop−prop2*:*2*)
  ultimately AOT-have ‹∃ q (q & [λy p] = [λy q])› **using** ≡*E*(*2*) **by** *blast*
  then AOT-obtain q **where** ‹q & [λy p] = [λy q]› **using** ∃ *E*[*rotated*] **by** *blast*
  AOT-thus ‹p›
    using *rule*=*E* &*E*(*1*) &*E*(*2*) *id-sym* ≡*E*(*2*) *p−identity−thm2*:*3* **by** *fast*
qed

AOT-theorem *TV−lem1*:*2*:
  ‹¬p ≡ ∀ F(∃ q (¬q & F = [λy q]) ≡ ∃ q((q ≡ p) & F = [λy q]))›
proof(*safe intro*!: ≡*I* →*I GEN*)
  fix F
  AOT-assume ‹∃ q (¬q & F = [λy q])›
  then AOT-obtain q **where** ‹¬q & F = [λy q]› **using** ∃ *E*[*rotated*] **by** *blast*
  moreover AOT-assume ‹¬p›
  ultimately AOT-have ‹(q ≡ p) & F = [λy q]›
    by (*metis* &*I* &*E*(*1*) &*E*(*2*) *deduction−theorem* ≡*I raa−cor*:*3*)
  AOT-thus ‹∃ q ((q ≡ p) & F = [λy q])› **by** (*rule* ∃ *I*)
next
  fix F
  AOT-assume ‹∃ q ((q ≡ p) & F = [λy q])›
  then AOT-obtain q **where** ‹(q ≡ p) & F = [λy q]› **using** ∃ *E*[*rotated*] **by** *blast*
  moreover AOT-assume ‹¬p›
  ultimately AOT-have ‹¬q & F = [λy q]›
    by (*metis* &*I* &*E*(*1*) &*E*(*2*) ≡*E*(*1*) *raa−cor*:*3*)
  AOT-thus ‹∃ q (¬q & F = [λy q])› **by** (*rule* ∃ *I*)
next
  AOT-assume ‹∀ F (∃ q (¬q & F = [λy q]) ≡ ∃ q ((q ≡ p) & F = [λy q]))›
  AOT-hence ‹∃ q (¬q & [λy p] = [λy q]) ≡ ∃ q ((q ≡ p) & [λy p] = [λy q])›
    using ∀ *E*(*1*)[*rotated*, *OF prop−prop2*:*2*] **by** *blast*
  moreover AOT-have ‹∃ q ((q ≡ p) & [λy p] = [λy q])›
    by (*rule* ∃ *I*(*2*)[**where** β=p])
      (*simp add*: *rule*=*I*:*1* &*I oth−class−taut*:*3*:*a prop−prop2*:*2*)
  ultimately AOT-have ‹∃ q (¬q & [λy p] = [λy q])› **using** ≡*E*(*2*) **by** *blast*
  then AOT-obtain q **where** ‹¬q & [λy p] = [λy q]› **using** ∃ *E*[*rotated*] **by** *blast*
  AOT-thus ‹¬p›
    using *rule*=*E* &*E*(*1*) &*E*(*2*) *id-sym* ≡*E*(*2*) *p−identity−thm2*:*3* **by** *fast*
qed

AOT-define *TruthValue* :: ‹τ ⇒ φ› (‹*TruthValue*′(-′)›)
  T−value: ‹*TruthValue*(x) $\equiv_{df}$ ∃ p (*TruthValueOf*(x,p))›

AOT-act-theorem *T−lem*:*1*: ‹*TruthValueOf*(∘p, p)›
proof −
  AOT-have ϑ: ‹∘p = ɩx *TruthValueOf*(x, p)›
    using *rule−id−df*:*1 the−tv−p uni−tv* **by** *blast*
  moreover AOT-have ‹∘p↓›

```
    using t=t−proper:1 calculation vdash−properties:10 by blast
  ultimately show ?thesis by (metis rule=E id-sym vdash−properties:10 y−in:3)
qed

AOT-act-theorem T−lem:2: ‹∀ F (∘p[F] ≡ ∃ q((q ≡ p) & F = [λy q]))›
  using T−lem:1[THEN tv−p[THEN ≡df E], THEN &E(2)].

AOT-act-theorem T−lem:3: ‹∘pΣr ≡ (r ≡ p)›
proof −
  AOT-have ϑ: ‹∘p[λy r] ≡ ∃ q ((q ≡ p) & [λy r] = [λy q])›
    using T−lem:2[THEN ∀ E(1), OF prop−prop2:2].
  show ?thesis
  proof(rule ≡I; rule →I)
    AOT-assume ‹∘pΣr›
    AOT-hence ‹∘p[λy r]› by (metis ≡df E &E(2) prop−enc)
    AOT-hence ‹∃ q ((q ≡ p) & [λy r] = [λy q])› using ϑ ≡E(1) by blast
    then AOT-obtain q where ‹(q ≡ p) & [λy r] = [λy q]› using ∃ E[rotated] by blast
    moreover AOT-have ‹r = q› using calculation
      using &E(2) ≡E(2) p−identity−thm2:3 by blast
    ultimately AOT-show ‹r ≡ p›
      by (metis rule=E &E(1) ≡E(6) oth−class−taut:3:a)
  next
    AOT-assume ‹r ≡ p›
    moreover AOT-have ‹[λy r] = [λy r]›
      by (simp add: rule=I:1 prop−prop2:2)
    ultimately AOT-have ‹(r ≡ p) & [λy r] = [λy r]› using &I by blast
    AOT-hence ‹∃ q ((q ≡ p) & [λy r] = [λy q])› by (rule ∃ I(2)[where β=r])
    AOT-hence ‹∘p[λy r]› using ϑ ≡E(2) by blast
    AOT-thus ‹∘pΣr›
      by (metis ≡df I &I prop−enc russell−axiom[enc,1].ψ-denotes-asm)
  qed
qed

AOT-act-theorem T−lem:4: ‹TruthValueOf(x, p) ≡ x = ∘p›
proof −
  AOT-have ‹∀ x (x = ιx TruthValueOf(x, p) ≡ ∀ z (TruthValueOf(z, p) ≡ z = x))›
    by (simp add: fund−cont−desc GEN)
  moreover AOT-have ‹∘p↓›
    using ≡df E tv−id:2 &E(1) prop−enc by blast
  ultimately AOT-have
    ‹(∘p = ιx TruthValueOf(x, p)) ≡ ∀ z (TruthValueOf(z, p) ≡ z = ∘p)›
    using ∀ E(1) by blast
  AOT-hence ‹∀ z (TruthValueOf(z, p) ≡ z = ∘p)›
    using ≡E(1) rule−id−df:1 the−tv−p uni−tv by blast
  AOT-thus ‹TruthValueOf(x, p) ≡ x = ∘p› using ∀ E(2) by blast
qed



AOT-theorem TV−lem2:1:
  ‹(A!x & ∀ F (x[F] ≡ ∃ q (q & F = [λy q]))) → TruthValue(x)›
proof(safe intro!: →I T−value[THEN ≡df I] tv−p[THEN ≡df I]
                  ∃ I(1)[rotated, OF log−prop−prop:2])
  AOT-assume ‹[A!]x & ∀ F (x[F] ≡ ∃ q (q & F = [λy q]))›
  AOT-thus ‹[A!]x & ∀ F (x[F] ≡ ∃ q ((q ≡ (∀ p (p → p))) & F = [λy q]))›
    apply (AOT-subst ‹∃ q ((q ≡ (∀ p (p → p))) & F = [λy q])›
                     ‹∃ q (q & F = [λy q])› for: F :: ‹<κ>›)
     apply (AOT-subst ‹q ≡ ∀ p (p →p)› ‹q› for: q)
    apply (metis (no-types, lifting) →I ≡I ≡E(2) GEN)
    by (auto simp: cqt−further:7)
qed
```

AOT-theorem *TV−lem2:2*:
‹(A!x & ∀ F (x[F] ≡ ∃ q (¬q & F = [λy q]))) → TruthValue(x)›
proof(*safe intro*!: →I T−value[THEN $\equiv_{df}$I] tv−p[THEN $\equiv_{df}$I]
∃ I(1)[rotated, OF log−prop−prop:2])
AOT-assume ‹[A!]x & ∀ F (x[F] ≡ ∃ q (¬q & F = [λy q]))›
AOT-thus ‹[A!]x & ∀ F (x[F] ≡ ∃ q ((q ≡ (∃ p (p & ¬p))) & F = [λy q]))›
apply (*AOT-subst* ‹∃ q ((q ≡ (∃ p (p & ¬p))) & F = [λy q])›
‹∃ q (¬q & F = [λy q])› **for**: F :: ‹<κ>›)
apply (*AOT-subst* ‹q ≡ ∃ p (p & ¬p)› ‹¬q› **for**: q)
apply (*metis* (*no-types*, *lifting*)
→I ∃ E ≡E(1) ≡I raa−cor:1 raa−cor:3)
by (*auto simp add*: cqt−further:7)
qed

AOT-define *TheTrue* :: κ_s (‹⊤›)
the−true:1: ‹⊤ $=_{df}$ ιx (A!x & ∀ F (x[F] ≡ ∃ p(p & F = [λy p])))›
AOT-define *TheFalse* :: κ_s (‹⊥›)
the−true:2: ‹⊥ $=_{df}$ ιx (A!x & ∀ F (x[F] ≡ ∃ p(¬p & F = [λy p])))›

AOT-theorem *the−true:3*: ‹⊤ ≠ ⊥›
proof(*safe intro*!: ab−obey:2[unvarify x y, THEN →E, rotated 2, OF ∨I(1)]
∃ I(1)[**where** τ=‹«[λx ∀ q(q → q)]»›] &I prop−prop2:2)
AOT-have *false-def*: ‹⊥ = ιx (A!x & ∀ F (x[F] ≡ ∃ p(¬p & F = [λy p])))›
by (*simp add*: A−descriptions rule−id−df:1[zero] the−true:2)
moreover AOT-show *false-den*: ‹⊥↓›
by (*meson* →E t=t−proper:1 A−descriptions
rule−id−df:1[zero] the−true:2)
ultimately AOT-have *false-prop*: ‹𝓐(A!⊥ & ∀ F (⊥[F] ≡ ∃ p(¬p & F = [λy p])))›
using nec−hintikka−scheme[unvarify x, THEN ≡E(1), THEN &E(1)] **by** *blast*
AOT-hence ‹𝓐∀ F (⊥[F] ≡ ∃ p(¬p & F = [λy p]))›
using Act−Basic:2 &E(2) ≡E(1) **by** *blast*
AOT-hence ‹∀ F 𝓐(⊥[F] ≡ ∃ p(¬p & F = [λy p]))›
using ≡E(1) logic−actual−nec:3[axiom-inst] **by** *blast*
AOT-hence *false-enc-cond*:
‹𝓐(⊥[λx ∀ q(q → q)] ≡ ∃ p(¬p & [λx ∀ q(q → q)] = [λy p]))›
using ∀ E(1)[rotated, OF prop−prop2:2] **by** *blast*

AOT-have *true-def*: ‹⊤ = ιx (A!x & ∀ F (x[F] ≡ ∃ p(p & F = [λy p])))›
by (*simp add*: A−descriptions rule−id−df:1[zero] the−true:1)
moreover AOT-show *true-den*: ‹⊤↓›
by (*meson* t=t−proper:1 A−descriptions rule−id−df:1[zero] the−true:1 →E)
ultimately AOT-have *true-prop*: ‹𝓐(A!⊤ & ∀ F (⊤[F] ≡ ∃ p(p & F = [λy p])))›
using nec−hintikka−scheme[unvarify x, THEN ≡E(1), THEN &E(1)] **by** *blast*
AOT-hence ‹𝓐∀ F (⊤[F] ≡ ∃ p(p & F = [λy p]))›
using Act−Basic:2 &E(2) ≡E(1) **by** *blast*
AOT-hence ‹∀ F 𝓐(⊤[F] ≡ ∃ p(p & F = [λy p]))›
using ≡E(1) logic−actual−nec:3[axiom-inst] **by** *blast*
AOT-hence ‹𝓐(⊤[λx ∀ q(q → q)] ≡ ∃ p(p & [λx ∀ q(q → q)] = [λy p]))›
using ∀ E(1)[rotated, OF prop−prop2:2] **by** *blast*
moreover AOT-have ‹𝓐∃ p(p & [λx ∀ q(q → q)] = [λy p])›
by (*safe intro*!: nec−imp−act[THEN →E] RN ∃ I(1)[**where** τ=«∀ q(q → q)»] &I
GEN →I log−prop−prop:2 rule=I:1 prop−prop2:2)
ultimately AOT-have ‹𝓐(⊤[λx ∀ q(q → q)])›
using Act−Basic:5 ≡E(1,2) **by** *blast*
AOT-thus ‹⊤[λx ∀ q(q → q)]›
using en−eq:10[1][unvarify x_1 F, THEN ≡E(1)] true-den prop−prop2:2 **by** *blast*

AOT-show ‹¬⊥[λx ∀ q(q → q)]›
proof(*rule* raa−cor:2)
AOT-assume ‹⊥[λx ∀ q(q → q)]›
AOT-hence ‹𝓐⊥[λx ∀ q(q → q)]›
using en−eq:10[1][unvarify x_1 F, THEN ≡E(2)]

false-den prop−prop2:*2* **by** *blast*
AOT-hence ‹𝓐∃ *p*(¬*p* & [λ*x* ∀ *q*(*q* → *q*)] = [λ*y p*])›
using *false-enc-cond Act−Basic*:*5* ≡*E*(*1*) **by** *blast*
AOT-hence ‹∃ *p* 𝓐(¬*p* & [λ*x* ∀ *q*(*q* → *q*)] = [λ*y p*])›
using *Act−Basic*:*10* ≡*E*(*1*) **by** *blast*
then AOT-obtain *p* **where** *p-prop*: ‹𝓐(¬*p* & [λ*x* ∀ *q*(*q* → *q*)] = [λ*y p*])›
using ∃ *E*[*rotated*] **by** *blast*
AOT-hence ‹𝓐[λ*x* ∀ *q*(*q* → *q*)] = [λ*y p*]›
by (*metis Act−Basic*:*2* &*E*(*2*) ≡*E*(*1*))
AOT-hence ‹[λ*x* ∀ *q*(*q* → *q*)] = [λ*y p*]›
using *id−act*:*1*[*unvarify* α β, *THEN* ≡*E*(*2*)] *prop−prop2*:*2* **by** *blast*
AOT-hence ‹(∀ *q*(*q* → *q*)) = *p*›
using *p−identity−thm2*:*3*[*unvarify p*, *THEN* ≡*E*(*2*)]
log−prop−prop:*2* **by** *blast*
moreover AOT-have ‹𝓐¬*p*› **using** *p-prop*
using *Act−Basic*:*2* &*E*(*1*) ≡*E*(*1*) **by** *blast*
ultimately AOT-have ‹𝓐¬∀ *q*(*q* → *q*)›
by (*metis Act−Sub*:*1* ≡*E*(*1*,*2*) *raa−cor*:*3 rule*=*E*)
moreover AOT-have ‹¬𝓐¬∀ *q*(*q* → *q*)›
by (*meson Act−Sub*:*1 RA*[*2*] *if−p−then−p* ≡*E*(*1*) *universal−cor*)
ultimately AOT-show ‹𝓐¬∀ *q*(*q* → *q*) & ¬𝓐¬∀ *q*(*q* → *q*)›
using &*I* **by** *blast*
qed
qed

AOT-act-theorem *T−T−value*:*1*: ‹*TruthValue*(⊤)›
proof −
AOT-have *true-def*: ‹⊤ = ι*x* (*A*!*x* & ∀ *F* (*x*[*F*] ≡ ∃ *p*(*p* & *F* = [λ*y p*])))›
by (*simp add*: *A−descriptions rule−id−df*:*1*[*zero*] *the−true*:*1*)
AOT-hence *true-den*: ‹⊤↓›
using *t*=*t−proper*:*1 vdash−properties*:*6* **by** *blast*
AOT-show ‹*TruthValue*(⊤)›
using *y−in*:*2*[*unvarify z*, *OF true-den*, *THEN* →*E*, *OF true-def*]
TV−lem2:*1*[*unvarify x*, *OF true-den*, *THEN* →*E*] **by** *blast*
qed

AOT-act-theorem *T−T−value*:*2*: ‹*TruthValue*(⊥)›
proof −
AOT-have *false-def*: ‹⊥ = ι*x* (*A*!*x* & ∀ *F* (*x*[*F*] ≡ ∃ *p*(¬*p* & *F* = [λ*y p*])))›
by (*simp add*: *A−descriptions rule−id−df*:*1*[*zero*] *the−true*:*2*)
AOT-hence *false-den*: ‹⊥↓›
using *t*=*t−proper*:*1 vdash−properties*:*6* **by** *blast*
AOT-show ‹*TruthValue*(⊥)›
using *y−in*:*2*[*unvarify z*, *OF false-den*, *THEN* →*E*, *OF false-def*]
TV−lem2:*2*[*unvarify x*, *OF false-den*, *THEN* →*E*] **by** *blast*
qed

AOT-theorem *two−T*: ‹∃ *x*∃ *y*(*TruthValue*(*x*) & *TruthValue*(*y*) & *x* ≠ *y* &
∀ *z* (*TruthValue*(*z*) → *z* = *x* ∨ *z* = *y*))›
proof −
AOT-obtain *a* **where** *a-prop*: ‹*A*!*a* & ∀ *F* (*a*[*F*] ≡ ∃ *p* (*p* & *F* = [λ*y p*]))›
using *A−objects*[*axiom-inst*] ∃ *E*[*rotated*] **by** *fast*
AOT-obtain *b* **where** *b-prop*: ‹*A*!*b* & ∀ *F* (*b*[*F*] ≡ ∃ *p* (¬*p* & *F* = [λ*y p*]))›
using *A−objects*[*axiom-inst*] ∃ *E*[*rotated*] **by** *fast*
AOT-obtain *p* **where** *p*: *p*
by (*metis log−prop−prop*:*2 raa−cor*:*3 rule−ui*:*1 universal−cor*)
show *?thesis*
proof(*rule* ∃ *I*(*2*)[**where** β=*a*]; *rule* ∃ *I*(*2*)[**where** β=*b*];
safe intro!: &*I GEN* →*I*)
AOT-show ‹*TruthValue*(*a*)›
using *TV−lem2*:*1 a-prop vdash−properties*:*10* **by** *blast*
next
AOT-show ‹*TruthValue*(*b*)›

```
    using TV−lem2:2 b-prop vdash−properties:10 by blast
next
  AOT-show ‹a ≠ b›
  proof(rule ab−obey:2[THEN →E, OF ∨I(1)])
    AOT-show ‹∃ F (a[F] & ¬b[F])›
    proof(rule ∃ I(1)[where τ=«[λy p]»]; rule &I prop−prop2:2)
      AOT-show ‹a[λy p]›
        by(safe intro!: ∃ I(2)[where β=p] &I p rule=I:1[OF prop−prop2:2]
            a-prop[THEN &E(2), THEN ∀ E(1), THEN ≡E(2), OF prop−prop2:2])
    next
      AOT-show ‹¬b[λy p]›
      proof (rule raa−cor:2)
        AOT-assume ‹b[λy p]›
        AOT-hence ‹∃ q (¬q & [λy p] = [λy q])›
          using ∀ E(1)[rotated, OF prop−prop2:2, THEN ≡E(1)]
                b-prop[THEN &E(2)] by fast
        then AOT-obtain q where ‹¬q & [λy p] = [λy q]›
          using ∃ E[rotated] by blast
        AOT-hence ‹¬p›
          by (metis rule=E &E(1) &E(2) deduction−theorem ≡I
                ≡E(2) p−identity−thm2:3 raa−cor:3)
        AOT-thus ‹p & ¬p› using p &I by blast
      qed
    qed
  qed
next
  fix z
  AOT-assume ‹TruthValue(z)›
  AOT-hence ‹∃ p (TruthValueOf(z, p))›
    by (metis ≡df E T−value)
  then AOT-obtain p where ‹TruthValueOf(z, p)› using ∃ E[rotated] by blast
  AOT-hence z-prop: ‹A!z & ∀ F (z[F] ≡ ∃ q ((q ≡ p) & F = [λy q]))›
    using ≡df E tv−p by blast
  {
    AOT-assume p: ‹p›
    AOT-have ‹z = a›
    proof(rule ab−obey:1[THEN →E, THEN →E, OF &I,
                        OF z-prop[THEN &E(1)], OF a-prop[THEN &E(1)]];
          rule GEN)
      fix G
      AOT-have ‹z[G] ≡ ∃ q ((q ≡ p) & G = [λy q])›
        using z-prop[THEN &E(2)] ∀ E(2) by blast
      also AOT-have ‹∃ q ((q ≡ p) & G = [λy q]) ≡ ∃ q (q & G = [λy q])›
        using TV−lem1:1[THEN ≡E(1), OF p, THEN ∀ E(2)[where β=G], symmetric].
      also AOT-have ‹... ≡ a[G]›
        using a-prop[THEN &E(2), THEN ∀ E(2)[where β=G], symmetric].
      finally AOT-show ‹z[G] ≡ a[G]›.
    qed
    AOT-hence ‹z = a ∨ z = b› by (rule ∨I)
  }
  moreover {
    AOT-assume notp: ‹¬p›
    AOT-have ‹z = b›
    proof(rule ab−obey:1[THEN →E, THEN →E, OF &I,
                        OF z-prop[THEN &E(1)], OF b-prop[THEN &E(1)]];
          rule GEN)
      fix G
      AOT-have ‹z[G] ≡ ∃ q ((q ≡ p) & G = [λy q])›
        using z-prop[THEN &E(2)] ∀ E(2) by blast
      also AOT-have ‹∃ q ((q ≡ p) & G = [λy q]) ≡ ∃ q (¬q & G = [λy q])›
        using TV−lem1:2[THEN ≡E(1), OF notp, THEN ∀ E(2), symmetric].
      also AOT-have ‹... ≡ b[G]›
        using b-prop[THEN &E(2), THEN ∀ E(2), symmetric].
```

finally AOT-show ‹$z[G] \equiv b[G]$›.
qed
AOT-hence ‹$z = a \vee z = b$› **by** (*rule* $\vee I$)
}
ultimately AOT-show ‹$z = a \vee z = b$›
by (*metis reductio−aa*:*1*)
qed
qed

AOT-act-theorem *valueof−facts*:*1*: ‹$TruthValueOf(x, p) \rightarrow (p \equiv x = \top)$›
proof(*safe intro*!: $\rightarrow I$ *dest*!: *tv−p*[*THEN* $\equiv_{df} E$])
AOT-assume ϑ: ‹$[A!]x$ & $\forall F\ (x[F] \equiv \exists q\ ((q \equiv p)$ & $F = [\lambda y\ q]))$›
AOT-have *a*: ‹$A!\top$›
using $\exists E$ *T−T−value*:*1* *T−value* &$E(1)$ $\equiv_{df} E$ *tv−p* **by** *blast*
AOT-have *true-def*: ‹$\top = \iota x\ (A!x$ & $\forall F\ (x[F] \equiv \exists p(p$ & $F = [\lambda y\ p])))$›
by (*simp add*: *A−descriptions rule−id−df*:*1*[*zero*] *the−true*:*1*)
AOT-hence *true-den*: ‹$\top\downarrow$›
using *t=t−proper*:*1* *vdash−properties*:*6* **by** *blast*
AOT-have *b*: ‹$\forall F\ (\top[F] \equiv \exists q\ (q$ & $F = [\lambda y\ q]))$›
using *y−in*:*2*[*unvarify z*, *OF true-den*, *THEN* $\rightarrow E$, *OF true-def*] &E **by** *blast*
AOT-show ‹$p \equiv x = \top$›
proof(*safe intro*!: $\equiv I$ $\rightarrow I$)
AOT-assume *p*
AOT-hence ‹$\forall F\ (\exists q\ (q$ & $F = [\lambda y\ q]) \equiv \exists q\ ((q \equiv p)$ & $F = [\lambda y\ q]))$›
using *TV−lem1*:*1*[*THEN* $\equiv E(1)$] **by** *blast*
AOT-hence ‹$\forall F(\top[F] \equiv \exists q\ ((q \equiv p)$ & $F = [\lambda y\ q]))$›
using *b cqt−basic*:*10*[*THEN* $\rightarrow E$, *OF* &I, *OF b*] **by** *fast*
AOT-hence *c*: ‹$\forall F(\exists q((q \equiv p)$ & $F = [\lambda y\ q]) \equiv \top[F])$›
using *cqt−basic*:*11*[*THEN* $\equiv E(1)$] **by** *fast*
AOT-hence ‹$\forall F\ (x[F] \equiv \top[F])$›
using *cqt−basic*:*10*[*THEN* $\rightarrow E$, *OF* &I, *OF* ϑ[*THEN* &$E(2)$]] **by** *fast*
AOT-thus ‹$x = \top$›
by (*rule ab−obey*:*1*[*unvarify y*, *OF true-den*, *THEN* $\rightarrow E$, *THEN* $\rightarrow E$,
OF &I, *OF* ϑ[*THEN* &$E(1)$], *OF a*])
next
AOT-assume ‹$x = \top$›
AOT-hence *d*: ‹$\forall F\ (\top[F] \equiv \exists q\ ((q \equiv p)$ & $F = [\lambda y\ q]))$›
using *rule=E* ϑ[*THEN* &$E(2)$] **by** *fast*
AOT-have ‹$\forall F\ (\exists q\ (q$ & $F = [\lambda y\ q]) \equiv \exists q\ ((q \equiv p)$ & $F = [\lambda y\ q]))$›
using *cqt−basic*:*10*[*THEN* $\rightarrow E$, *OF* &I,
OF b[*THEN cqt−basic*:*11*[*THEN* $\equiv E(1)$]], *OF d*].
AOT-thus *p* **using** *TV−lem1*:*1*[*THEN* $\equiv E(2)$] **by** *blast*
qed
qed

AOT-act-theorem *valueof−facts*:*2*: ‹$TruthValueOf(x, p) \rightarrow (\neg p \equiv x = \bot)$›
proof(*safe intro*!: $\rightarrow I$ *dest*!: *tv−p*[*THEN* $\equiv_{df} E$])
AOT-assume ϑ: ‹$[A!]x$ & $\forall F\ (x[F] \equiv \exists q\ ((q \equiv p)$ & $F = [\lambda y\ q]))$›
AOT-have *a*: ‹$A!\bot$›
using $\exists E$ *T−T−value*:*2* *T−value* &$E(1)$ $\equiv_{df} E$ *tv−p* **by** *blast*
AOT-have *false-def*: ‹$\bot = \iota x\ (A!x$ & $\forall F\ (x[F] \equiv \exists p(\neg p$ & $F = [\lambda y\ p])))$›
by (*simp add*: *A−descriptions rule−id−df*:*1*[*zero*] *the−true*:*2*)
AOT-hence *false-den*: ‹$\bot\downarrow$›
using *t=t−proper*:*1* *vdash−properties*:*6* **by** *blast*
AOT-have *b*: ‹$\forall F\ (\bot[F] \equiv \exists q\ (\neg q$ & $F = [\lambda y\ q]))$›
using *y−in*:*2*[*unvarify z*, *OF false-den*, *THEN* $\rightarrow E$, *OF false-def*] &E **by** *blast*
AOT-show ‹$\neg p \equiv x = \bot$›
proof(*safe intro*!: $\equiv I$ $\rightarrow I$)
AOT-assume ‹$\neg p$›
AOT-hence ‹$\forall F\ (\exists q\ (\neg q$ & $F = [\lambda y\ q]) \equiv \exists q\ ((q \equiv p)$ & $F = [\lambda y\ q]))$›
using *TV−lem1*:*2*[*THEN* $\equiv E(1)$] **by** *blast*
AOT-hence ‹$\forall F(\bot[F] \equiv \exists q\ ((q \equiv p)$ & $F = [\lambda y\ q]))$›
using *b cqt−basic*:*10*[*THEN* $\rightarrow E$, *OF* &I, *OF b*] **by** *fast*

AOT-hence *c*: ‹$\forall F(\exists q((q \equiv p)\ \&\ F = [\lambda y\ q]) \equiv \bot[F])$›
  using *cqt−basic*:*11*[*THEN* ≡*E*(*1*)] **by** *fast*
AOT-hence ‹$\forall F\ (x[F] \equiv \bot[F])$›
  using *cqt−basic*:*10*[*THEN* →*E*, *OF* &*I*, *OF* ϑ[*THEN* &*E*(*2*)]] **by** *fast*
AOT-thus ‹$x = \bot$›
  by (*rule ab−obey*:*1*[*unvarify y*, *OF false-den*, *THEN* →*E*, *THEN* →*E*,
  OF &*I*, *OF* ϑ[*THEN* &*E*(*1*)], *OF a*])
next
AOT-assume ‹$x = \bot$›
AOT-hence *d*: ‹$\forall F\ (\bot[F] \equiv \exists q\ ((q \equiv p)\ \&\ F = [\lambda y\ q]))$›
  using *rule*=*E* ϑ[*THEN* &*E*(*2*)] **by** *fast*
AOT-have ‹$\forall F\ (\exists q\ (\neg q\ \&\ F = [\lambda y\ q]) \equiv \exists q\ ((q \equiv p)\ \&\ F = [\lambda y\ q]))$›
  using *cqt−basic*:*10*[*THEN* →*E*, *OF* &*I*,
  OF b[*THEN cqt−basic*:*11*[*THEN* ≡*E*(*1*)]], *OF d*]**.**
AOT-thus ‹$\neg p$› **using** *TV−lem1*:*2*[*THEN* ≡*E*(*2*)] **by** *blast*
qed
qed

AOT-act-theorem *q−True*:*1*: ‹$p \equiv (\circ p = \top)$›
apply (*rule valueof−facts*:*1*[*unvarify x*, *THEN* →*E*, *rotated*, *OF T−lem*:*1*])
using $\equiv_{df}E$ *tv−id*:*2* &*E*(*1*) *prop−enc* **by** *blast*

AOT-act-theorem *q−True*:*2*: ‹$\neg p \equiv (\circ p = \bot)$›
apply (*rule valueof−facts*:*2*[*unvarify x*, *THEN* →*E*, *rotated*, *OF T−lem*:*1*])
using $\equiv_{df}E$ *tv−id*:*2* &*E*(*1*) *prop−enc* **by** *blast*

AOT-act-theorem *q−True*:*3*: ‹$p \equiv \top\mathbf{\Sigma}p$›
proof(*safe intro*!: ≡*I* →*I*)
AOT-assume *p*
AOT-hence ‹$\circ p = \top$› **by** (*metis* ≡*E*(*1*) *q−True*:*1*)
moreover AOT-have ‹$\circ p\mathbf{\Sigma}p$›
  by (*simp add*: *tv−id*:*2*)
ultimately AOT-show ‹$\top\mathbf{\Sigma}p$›
  using *rule*=*E T−lem*:*4* **by** *fast*
next
AOT-have *true-def*: ‹$\top = \iota x\ (A!x\ \&\ \forall F\ (x[F] \equiv \exists p(p\ \&\ F = [\lambda y\ p])))$›
  by (*simp add*: *A−descriptions rule−id−df*:*1*[*zero*] *the−true*:*1*)
AOT-hence *true-den*: ‹$\top\downarrow$›
  using *t*=*t−proper*:*1 vdash−properties*:*6* **by** *blast*
AOT-have *b*: ‹$\forall F\ (\top[F] \equiv \exists q\ (q\ \&\ F = [\lambda y\ q]))$›
  using *y−in*:*2*[*unvarify z*, *OF true-den*, *THEN* →*E*, *OF true-def*] &*E* **by** *blast*

AOT-assume ‹$\top\mathbf{\Sigma}p$›
AOT-hence ‹$\top[\lambda y\ p]$› **by** (*metis* $\equiv_{df}E$ &*E*(*2*) *prop−enc*)
AOT-hence ‹$\exists q\ (q\ \&\ [\lambda y\ p] = [\lambda y\ q])$›
  using *b*[*THEN* ∀*E*(*1*), *OF prop−prop2*:*2*, *THEN* ≡*E*(*1*)] **by** *blast*
then AOT-obtain *q* **where** ‹$q\ \&\ [\lambda y\ p] = [\lambda y\ q]$› **using** ∃*E*[*rotated*] **by** *blast*
AOT-thus ‹p›
  using *rule*=*E* &*E*(*1*) &*E*(*2*) *id-sym* ≡*E*(*2*) *p−identity−thm2*:*3* **by** *fast*
qed

AOT-act-theorem *q−True*:*5*: ‹$\neg p \equiv \bot\mathbf{\Sigma}p$›
proof(*safe intro*!: ≡*I* →*I*)
AOT-assume ‹$\neg p$›
AOT-hence ‹$\circ p = \bot$› **by** (*metis* ≡*E*(*1*) *q−True*:*2*)
moreover AOT-have ‹$\circ p\mathbf{\Sigma}p$›
  by (*simp add*: *tv−id*:*2*)
ultimately AOT-show ‹$\bot\mathbf{\Sigma}p$›
  using *rule*=*E T−lem*:*4* **by** *fast*
next
AOT-have *false-def*: ‹$\bot = \iota x\ (A!x\ \&\ \forall F\ (x[F] \equiv \exists p(\neg p\ \&\ F = [\lambda y\ p])))$›
  by (*simp add*: *A−descriptions rule−id−df*:*1*[*zero*] *the−true*:*2*)

```
  AOT-hence false-den: ‹⊥↓›
    using t=t−proper:1 vdash−properties:6 by blast
  AOT-have b: ‹∀ F (⊥[F] ≡ ∃ q (¬q & F = [λy q]))›
    using y−in:2[unvarify z, OF false-den, THEN →E, OF false-def] &E by blast

  AOT-assume ‹⊥Σp›
  AOT-hence ‹⊥[λy p]› by (metis ≡df E &E(2) prop−enc)
  AOT-hence ‹∃ q (¬q & [λy p] = [λy q])›
    using b[THEN ∀ E(1), OF prop−prop2:2, THEN ≡E(1)] by blast
  then AOT-obtain q where ‹¬q & [λy p] = [λy q]› using ∃ E[rotated] by blast
  AOT-thus ‹¬p›
    using rule=E &E(1) &E(2) id-sym ≡E(2) p−identity−thm2:3 by fast
qed

AOT-act-theorem q−True:4: ‹p ≡ ¬(⊥Σp)›
  using q−True:5
  by (metis deduction−theorem ≡I ≡E(2) ≡E(4) raa−cor:3)

AOT-act-theorem q−True:6: ‹¬p ≡ ¬(⊤Σp)›
  using ≡E(1) oth−class−taut:4:b q−True:3 by blast

AOT-define ExtensionOf :: ‹τ ⇒ φ ⇒ φ› (‹ExtensionOf'(-,-')›)
  exten−p: ‹ExtensionOf(x,p) ≡df A!x &
                                 ∀ F (x[F] → Propositional([F])) &
                                 ∀ q ((xΣq) ≡ (q ≡ p))›

AOT-theorem extof−e: ‹ExtensionOf(x, p) ≡ TruthValueOf(x, p)›
proof (safe intro!: ≡I →I tv−p[THEN ≡df I] exten−p[THEN ≡df I]
            dest!: tv−p[THEN ≡df E] exten−p[THEN ≡df E])
  AOT-assume 1: ‹[A!]x & ∀ F (x[F] → Propositional([F])) & ∀ q (x Σ q ≡ (q ≡ p))›
  AOT-hence ϑ: ‹[A!]x & ∀ F (x[F] → ∃ q(F = [λy q])) & ∀ q (x Σ q ≡ (q ≡ p))›
    by (AOT-subst ‹∃ q(F = [λy q])› ‹Propositional([F])› for: F :: ‹<κ>›)
       (auto simp add: df−rules−formulas[3] df−rules−formulas[4]
                       ≡I prop−prop1)
  AOT-show ‹[A!]x & ∀ F (x[F] ≡ ∃ q ((q ≡ p) & F = [λy q]))›
  proof(safe intro!: &I GEN 1[THEN &E(1), THEN &E(1)] ≡I →I)
    fix F
    AOT-assume 0: ‹x[F]›
    AOT-hence ‹∃ q (F = [λy q])›
      using ϑ[THEN &E(1), THEN &E(2)] ∀ E(2) →E by blast
    then AOT-obtain q where q-prop: ‹F = [λy q]› using ∃ E[rotated] by blast
    AOT-hence ‹x[λy q]› using 0 rule=E by blast
    AOT-hence ‹xΣq› by (metis ≡df I &I ex:1:a prop−enc rule−ui:3)
    AOT-hence ‹q ≡ p› using ϑ[THEN &E(2)] ∀ E(2) ≡E(1) by blast
    AOT-hence ‹(q ≡ p) & F = [λy q]› using q-prop &I by blast
    AOT-thus ‹∃ q ((q ≡ p) & F = [λy q])› by (rule ∃ I)
  next
    fix F
    AOT-assume ‹∃ q ((q ≡ p) & F = [λy q])›
    then AOT-obtain q where q-prop: ‹(q ≡ p) & F = [λy q]›
      using ∃ E[rotated] by blast
    AOT-hence ‹xΣq› using ϑ[THEN &E(2)] ∀ E(2) &E ≡E(2) by blast
    AOT-hence ‹x[λy q]› by (metis ≡df E &E(2) prop−enc)
    AOT-thus ‹x[F]› using q-prop[THEN &E(2), symmetric] rule=E by blast
  qed
next
  AOT-assume 0: ‹[A!]x & ∀ F (x[F] ≡ ∃ q ((q ≡ p) & F = [λy q]))›
  AOT-show ‹[A!]x & ∀ F (x[F] → Propositional([F])) & ∀ q (x Σ q ≡ (q ≡ p))›
  proof(safe intro!: &I 0[THEN &E(1)] GEN →I)
    fix F
    AOT-assume ‹x[F]›
    AOT-hence ‹∃ q ((q ≡ p) & F = [λy q])›
      using 0[THEN &E(2)] ∀ E(2) ≡E(1) by blast
```

```
    then AOT-obtain q where ‹(q ≡ p) & F = [λy q]›
      using ∃ E[rotated] by blast
    AOT-hence ‹F = [λy q]› using &E(2) by blast
    AOT-hence ‹∃ q F = [λy q]› by (rule ∃ I)
    AOT-thus ‹Propositional([F])› by (metis ≡df I prop−prop1)
  next
    AOT-show ‹xΣr ≡ (r ≡ p)› for r
    proof(rule ≡I; rule →I)
      AOT-assume ‹xΣr›
      AOT-hence ‹x[λy r]› by (metis ≡df E &E(2) prop−enc)
      AOT-hence ‹∃ q ((q ≡ p) & [λy r] = [λy q])›
        using 0[THEN &E(2), THEN ∀ E(1), OF prop−prop2:2, THEN ≡E(1)] by blast
      then AOT-obtain q where ‹(q ≡ p) & [λy r] = [λy q]›
        using ∃ E[rotated] by blast
      AOT-thus ‹r ≡ p›
        by (metis rule=E &E(1,2) id-sym ≡E(2) Commutativity of ≡
              p−identity−thm2:3)
    next
      AOT-assume ‹r ≡ p›
      AOT-hence ‹(r ≡ p) & [λy r] = [λy r]›
        by (metis rule=I:1 ≡S(1) ≡E(2) Commutativity of & prop−prop2:2)
      AOT-hence ‹∃ q ((q ≡ p) & [λy r] = [λy q])› by (rule ∃ I)
      AOT-hence ‹x[λy r]›
        using 0[THEN &E(2), THEN ∀ E(1), OF prop−prop2:2, THEN ≡E(2)] by blast
      AOT-thus ‹xΣr› by (metis ≡df I &I ex:1:a prop−enc rule−ui:3)
    qed
  qed
qed

AOT-theorem ext−p−tv:1: ‹∃!x ExtensionOf(x, p)›
  by (AOT-subst ‹ExtensionOf(x, p)› ‹TruthValueOf(x, p)› for: x)
     (auto simp: extof−e p−has−!tv:2)

AOT-theorem ext−p−tv:2: ‹ιx(ExtensionOf(x, p))↓›
  using A−Exists:2 RA[2] ext−p−tv:1 ≡E(2) by blast

AOT-theorem ext−p−tv:3: ‹ιx(ExtensionOf(x, p)) = ∘p›
proof −
  AOT-have 0: ‹𝒜∀ x(ExtensionOf(x, p) ≡ TruthValueOf(x,p))›
    by (rule RA[2]; rule GEN; rule extof−e)
  AOT-have 1: ‹∘p = ιx TruthValueOf(x,p)›
    using rule−id−df:1 the−tv−p uni−tv by blast
  show ?thesis
    apply (rule equiv−desc−eq:1[THEN →E, OF 0, THEN ∀ E(1)[where τ=‹«∘p»›],
                                THEN ≡E(2), symmetric])
    using 1 t=t−proper:1 vdash−properties:10 apply blast
    by (fact 1)
qed
```

10 Restricted Variables

```
locale AOT-restriction-condition =
  fixes ψ :: ‹'a::AOT-Term-id-2 ⇒ o›
  assumes res−var:2[AOT]: ‹[v ⊨ ∃ α ψ{α}]›
  assumes res−var:3[AOT]: ‹[v ⊨ ψ{τ} → τ↓]›

ML‹
fun register-restricted-type (name:string, restriction:string) thy =
let
val ctxt = thy
val ctxt = setupStrictWorld ctxt
val trm = Syntax.check-term ctxt (AOT-read-term @{nonterminal φ'} ctxt restriction)
val free = case (Term.add-frees trm []) of [f] => f |
```

```
    - => raise Term.TERM (Expected single free variable., [trm])
val trm = Term.absfree free trm
val localeTerm = Const (const-name ‹AOT-restriction-condition›, dummyT) $ trm
val localeTerm = Syntax.check-term ctxt localeTerm
fun after-qed thms thy = let
val st = Interpretation.global-interpretation
  (([(@{locale AOT-restriction-condition}, ((name, true),
          (Expression.Named [(ψ, trm)], [])))], [])) [] thy
val st = Proof.refine-insert (flat thms) st
val thy = Proof.global-immediate-proof st

val thy = Local-Theory.background-theory
  (AOT-Constraints.map (Symtab.update
    (name, (term-of (snd free), term-of (snd free))))) thy
val thy = Local-Theory.background-theory
  (AOT-Restriction.map (Symtab.update
    (name, (trm, Const (const-name ‹AOT-term-of-var›, dummyT))))) thy

in thy end
in
Proof.theorem NONE after-qed [[(HOLogic.mk-Trueprop localeTerm, [])]] ctxt
end

val - =
  Outer-Syntax.command
    command-keyword ‹AOT-register-restricted-type›
    Register a restricted type.
    (((Parse.short-ident --| Parse.$$$ :) -- Parse.term) >>
    (Toplevel.local-theory-to-proof NONE NONE o register-restricted-type));
›

locale AOT-rigid-restriction-condition = AOT-restriction-condition +
  assumes rigid[AOT]: ‹[v ⊨ ∀ α(ψ{α} → □ψ{α})]›
begin
lemma rigid-condition[AOT]: ‹[v ⊨ □(ψ{α} → □ψ{α})]›
  using rigid[THEN ∀ E(2)] RN by simp
lemma type-set-nonempty[AOT-no-atp, no-atp]: ‹∃ x . x ∈ { α . [w₀ ⊨ ψ{α}]}›
  by (metis instantiation mem-Collect-eq res−var:2)
end

locale AOT-restricted-type = AOT-rigid-restriction-condition +
  fixes Rep and Abs
  assumes AOT-restricted-type-definition[AOT-no-atp]:
    ‹type-definition Rep Abs { α . [w₀ ⊨ ψ{α}]}›
begin

AOT-theorem restricted-var-condition: ‹ψ{«AOT-term-of-var (Rep α)»}›
proof −
  interpret type-definition Rep Abs { α . [w₀ ⊨ ψ{α}]}
    using AOT-restricted-type-definition.
  AOT-actually {
    AOT-have ‹«AOT-term-of-var (Rep α)»↓› and ‹ψ{«AOT-term-of-var (Rep α)»}›
      using AOT-sem-imp Rep res−var:3 by auto
  }
  moreover AOT-actually {
    AOT-have ‹ψ{α} → □ψ{α}› for α
      using AOT-sem-box rigid-condition by presburger
    AOT-hence ‹ψ{τ} → □ψ{τ}› if ‹τ↓› for τ
      by (metis AOT-model.AOT-term-of-var-cases AOT-sem-denotes that)
  }
  ultimately AOT-show ‹ψ{«AOT-term-of-var (Rep α)»}›
    using AOT-sem-box AOT-sem-imp by blast
qed
```

lemmas ψ = *restricted-var-condition*

AOT-theorem *GEN*: **assumes** ‹*for arbitrary* α: $\varphi\{$«*AOT-term-of-var* (*Rep* α)»$\}$›
  shows ‹$\forall\,\alpha$ ($\psi\{\alpha\} \rightarrow \varphi\{\alpha\}$)›
proof(*rule GEN*; *rule* $\rightarrow I$)
  interpret *type-definition Rep Abs* { α . [w_0 $\models$ $\psi\{\alpha\}$]}
    using *AOT-restricted-type-definition***.**
  fix α
  AOT-assume ‹$\psi\{\alpha\}$›
  AOT-hence ‹$\mathcal{A}\psi\{\alpha\}$›
    by (*metis AOT-model-axiom-def AOT-sem-box AOT-sem-imp act-closure rigid-condition*)
  hence *0*: ‹[w_0 $\models$ $\psi\{\alpha\}$]› **by** (*metis AOT-sem-act*)
  {
    fix τ
    assume α-*def*: ‹α = *Rep* τ›
    AOT-have ‹$\varphi\{\alpha\}$›
      unfolding α-*def*
      using *assms* **by** *blast*
  }
  AOT-thus ‹$\varphi\{\alpha\}$›
    using *Rep-cases*[*simplified*, *OF 0*]
    by *blast*
qed
lemmas $\forall\,I$ = *GEN*

end

lemma *AOT-restricted-type-intro*[*AOT-no-atp*, *no-atp*]:
  assumes ‹*type-definition Rep Abs* { α . [w_0 $\models$ $\psi\{\alpha\}$]}›
    and ‹*AOT-rigid-restriction-condition* ψ›
  shows ‹*AOT-restricted-type* ψ *Rep Abs*›
  by (*auto intro*!: *assms AOT-restricted-type-axioms.intro AOT-restricted-type.intro*)

ML‹
fun register-rigid-restricted-type (*name*:*string*, *restriction*:*string*) *thy* =
let
val ctxt = *thy*
val ctxt = *setupStrictWorld ctxt*
val trm = *Syntax.check-term ctxt* (*AOT-read-term* @{*nonterminal* φ'} *ctxt restriction*)
val free = *case* (*Term.add-frees trm* []) *of* [*f*] => *f*
  | - => *raise Term.TERM* (*Expected single free variable.*, [*trm*])
val trm = *Term.absfree free trm*
val localeTerm = *HOLogic.mk-Trueprop*
  (*Const* (***const-name***‹*AOT-rigid-restriction-condition*›, *dummyT*) $ *trm*)
val localeTerm = *Syntax.check-prop ctxt localeTerm*
val int-bnd = *Binding.concealed* (*Binding.qualify true internal* (*Binding.name name*))
val bnds = {*Rep-name* = *Binding.qualify true name* (*Binding.name Rep*),
      Abs-name = *Binding.qualify true Abs int-bnd*,
      type-definition-name = *Binding.qualify true type-definition int-bnd*}

fun after-qed witts thy = *let*
  val thms = (*map* (*Element.conclude-witness ctxt*) (*flat witts*))

  val typeset = *HOLogic.mk-Collect* (α, *dummyT*,
    const‹*AOT-model-valid-in*› $ ***const***‹w_0› $
    (*trm* $ (*Const* (***const-name***‹*AOT-term-of-var*›, *dummyT*) $ *Bound 0*)))
  val typeset = *Syntax.check-term thy typeset*
  val nonempty-thm = *Drule.OF*
    (@{*thm AOT-rigid-restriction-condition.type-set-nonempty*}, *thms*)

```
  val ((-,st),thy) = Typedef.add-typedef {overloaded=true}
    (Binding.name name, [], Mixfix.NoSyn) typeset (SOME bnds)
    (fn ctxt => (Tactic.resolve-tac ctxt ([nonempty-thm]) 1)) thy
  val ({rep-type = -, abs-type = -, Rep-name = Rep-name, Abs-name = Abs-name,
        axiom-name = -},
      {inhabited = -, type-definition = type-definition, Rep = -,
       Rep-inverse = -, Abs-inverse = -, Rep-inject = -, Abs-inject = -,
       Rep-cases = -, Abs-cases = -, Rep-induct = -, Abs-induct = -}) = st

  val locale-thm = Drule.OF (@{thm AOT-restricted-type-intro}, type-definition::thms)

  val st = Interpretation.global-interpretation (([(@{locale AOT-restricted-type},
    ((name, true), (Expression.Named [
      (ψ, trm),
      (Rep, Const (Rep-name, dummyT)),
      (Abs, Const (Abs-name, dummyT))], [])))
    ], [])) [] thy

  val st = Proof.refine-insert [locale-thm] st
  val thy = Proof.global-immediate-proof st

  val thy = Local-Theory.background-theory (AOT-Constraints.map (
    Symtab.update (name, (term-of (snd free), term-of (snd free))))) thy
  val thy = Local-Theory.background-theory (AOT-Restriction.map (
    Symtab.update (name, (trm, Const (Rep-name, dummyT))))) thy

  in thy end
in
Element.witness-proof after-qed [[localeTerm]] thy
end

val - =
  Outer-Syntax.command
    command-keyword‹AOT-register-rigid-restricted-type›
    Register a restricted type.
    (((Parse.short-ident --| Parse.$$$ :) -- Parse.term) >>
    (Toplevel.local-theory-to-proof NONE NONE o register-rigid-restricted-type));
›
```

ML‹

```
fun get-instantiated-allI ctxt varname thm = let
val trm = Thm.concl-of thm
val trm = case trm of (@{const Trueprop} $ (@{const AOT-model-valid-in} $ - $ x)) => x
                      | - => raise Term.TERM (Expected simple theorem., [trm])
fun extractVars (Const (const-name‹AOT-term-of-var›, t) $ (Const rep $ Var v)) =
    (if fst (fst v) = fst varname
     then [Const (const-name‹AOT-term-of-var›, t) $ (Const rep $ Var v)]
     else []) (* TODO: care about the index *)
  | extractVars (t1 $ t2) = extractVars t1 @ extractVars t2
  | extractVars (Abs (-, -, t)) = extractVars t
  | extractVars - = []
val vars = extractVars trm
val vartrm = hd vars
val vars = fold Term.add-vars vars []
val var = hd vars
val trmty = (case vartrm of (Const (-, Type (fun, [-, ty])) $ -) => ty
                         | - => raise Match)
val varty = snd var
val tyname = fst (Term.dest-Type varty)
val b = tyname^.∀I (* TODO: better way to find the theorem *)
val thms = fst (Context.map-proof-result (fn ctxt => (Attrib.eval-thms ctxt
    [(Facts.Named ((b,Position.none),NONE),[])], ctxt)) ctxt)
```

```
val allthm = (case thms of (thm::-) => thm
    | - => raise Fail Unknown restricted type.)
val trm = Abs (Term.string-of-vname (fst var), trmty, Term.abstract-over (vartrm, trm))
val trm = Thm.cterm-of (Context.proof-of ctxt) trm
val phi = hd (Term.add-vars (Thm.prop-of allthm) [])
val allthm = Drule.instantiate-normalize (TVars.empty, Vars.make [(phi,trm)]) allthm
in
allthm
end
›
```

```
attribute-setup unconstrain =
  ‹Scan.lift (Scan.repeat1 Args.var) >> (fn args => Thm.rule-attribute []
  (fn ctxt => fn thm =>
    let
    val thm = fold (fn arg => fn thm => thm RS get-instantiated-allI ctxt arg thm)
                args thm
    val thm = fold (fn - => fn thm => thm RS @{thm ∀E(2)}) args thm
    in
     thm
    end))›
  Generalize a statement about restricted variables to a statement about
   unrestricted variables with explicit restriction condition.
```

```
context AOT-restricted-type
begin

AOT-theorem rule−ui:
  assumes ‹∀α(ψ{α} → φ{α})›
  shows ‹φ{«AOT-term-of-var (Rep α)»}›
proof −
  AOT-have ‹φ{α}› if ‹ψ{α}› for α using assms[THEN ∀E(2), THEN →E] that by blast
  moreover AOT-have ‹ψ{«AOT-term-of-var (Rep α)»}›
    by (auto simp: ψ)
  ultimately show ?thesis by blast
qed
lemmas ∀E = rule−ui

AOT-theorem instantiation:
  assumes ‹for arbitrary β: φ{«AOT-term-of-var (Rep β)»} ⊢ χ› and ‹∃α (ψ{α} & φ{α})›
  shows ‹χ›
proof −
  AOT-have ‹φ{«AOT-term-of-var (Rep α)»} → χ› for α
    using assms(1)
    by (simp add: deduction−theorem)
  AOT-hence 0: ‹∀α (ψ{α} → (φ{α} → χ))›
    using GEN by simp
  moreover AOT-obtain α where ‹ψ{α} & φ{α}› using assms(2) ∃E[rotated] by blast
  ultimately AOT-show ‹χ› using AOT-PLM.∀E(2)[THEN →E, THEN →E] &E by fast
qed
lemmas ∃E = instantiation

AOT-theorem existential: assumes ‹φ{«AOT-term-of-var (Rep β)»}›
  shows ‹∃ α (ψ{α} & φ{α})›
  by (meson AOT-restricted-type.ψ AOT-restricted-type-axioms assms
          &I existential:2[const-var])
lemmas ∃I = existential
end
```

context *AOT-rigid-restriction-condition*
begin

AOT-theorem *res−var−bound−reas*[*1*]:
‹∀α(ψ{α} → ∀β φ{α, β}) ≡ ∀β∀α (ψ{α} → φ{α, β})›
proof(*safe intro*!: ≡*I* →*I GEN*)
fix β α
AOT-assume ‹∀α (ψ{α} → ∀β φ{α, β})›
AOT-hence ‹ψ{α} → ∀β φ{α, β}› **using** ∀*E*(*2*) **by** *blast*
moreover AOT-assume ‹ψ{α}›
ultimately AOT-have ‹∀β φ{α, β}› **using** →*E* **by** *blast*
AOT-thus ‹φ{α, β}› **using** ∀*E*(*2*) **by** *blast*
next
fix α β
AOT-assume ‹∀β∀α(ψ{α} → φ{α, β})›
AOT-hence ‹∀α(ψ{α} → φ{α, β})› **using** ∀*E*(*2*) **by** *blast*
AOT-hence ‹ψ{α} → φ{α, β}› **using** ∀*E*(*2*) **by** *blast*
moreover AOT-assume ‹ψ{α}›
ultimately AOT-show ‹φ{α, β}› **using** →*E* **by** *blast*
qed

AOT-theorem *res−var−bound−reas*[*BF*]:
‹∀α(ψ{α} → □φ{α}) → □∀α(ψ{α} → φ{α})›
proof(*safe intro*!: →*I*)
AOT-assume ‹∀α(ψ{α} → □φ{α})›
AOT-hence ‹ψ{α} → □φ{α}› **for** α
using ∀*E*(*2*) **by** *blast*
AOT-hence ‹□(ψ{α} → φ{α})› **for** α
by (*metis sc−eq−box−box*:*6 rigid-condition vdash−properties*:*6*)
AOT-hence ‹∀α □(ψ{α} → φ{α})›
by (*rule GEN*)
AOT-thus ‹□∀α (ψ{α} → φ{α})›
by (*metis BF vdash−properties*:*6*)
qed

AOT-theorem *res−var−bound−reas*[*CBF*]:
‹□∀α(ψ{α} → φ{α}) → ∀α(ψ{α} → □φ{α})›
proof(*safe intro*!: →*I GEN*)
fix α
AOT-assume ‹□∀α (ψ{α} → φ{α})›
AOT-hence ‹∀α □(ψ{α} → φ{α})›
by (*metis CBF vdash−properties*:*6*)
AOT-hence *1*: ‹□(ψ{α} → φ{α})›
using ∀*E*(*2*) **by** *blast*
AOT-assume ‹ψ{α}›
AOT-hence ‹□ψ{α}›
by (*metis B*◇ *T*◇ *rigid-condition vdash−properties*:*6*)
AOT-thus ‹□φ{α}›
using *1 qml*:*1*[*axiom-inst*, *THEN* →*E*, *THEN* →*E*] **by** *blast*
qed

AOT-theorem *res−var−bound−reas*[*2*]:
‹∀α (ψ{α} → **𝓐**φ{α}) → **𝓐**∀α (ψ{α} → φ{α})›
proof(*safe intro*!: →*I*)
AOT-assume ‹∀α (ψ{α} → **𝓐**φ{α})›
AOT-hence ‹ψ{α} → **𝓐**φ{α}› **for** α
using ∀*E*(*2*) **by** *blast*
AOT-hence ‹**𝓐**(ψ{α} → φ{α})› **for** α
by (*metis sc−eq−box−box*:*7 rigid-condition vdash−properties*:*6*)
AOT-hence ‹∀α **𝓐**(ψ{α} → φ{α})›
by (*rule GEN*)
AOT-thus ‹**𝓐**∀α(ψ{α} → φ{α})›
by (*metis* ≡*E*(*2*) *logic−actual−nec*:*3*[*axiom-inst*])

qed

AOT-theorem *res−var−bound−reas*[*3*]:
‹$\mathcal{A}$∀ α (ψ{α} → φ{α}) → ∀ α (ψ{α} → $\mathcal{A}$φ{α})›
proof(*safe intro*!: →*I GEN*)
fix α
AOT-assume ‹$\mathcal{A}$∀ α (ψ{α} → φ{α})›
AOT-hence ‹∀ α $\mathcal{A}$(ψ{α} → φ{α})›
by (*metis* ≡*E*(*1*) *logic−actual−nec*:*3*[*axiom-inst*])
AOT-hence *1*: ‹$\mathcal{A}$(ψ{α} → φ{α})› **by** (*metis rule−ui*:*3*)
AOT-assume ‹ψ{α}›
AOT-hence ‹$\mathcal{A}$ψ{α}›
by (*metis nec−imp−act qml*:*2*[*axiom-inst*] *rigid-condition* →*E*)
AOT-thus ‹$\mathcal{A}$φ{α}›
using *1* **by** (*metis act−cond* →*E*)
qed

AOT-theorem *res−var−bound−reas*[*Buridan*]:
‹∃ α (ψ{α} & □φ{α}) → □∃ α (ψ{α} & φ{α})›
proof (*rule* →*I*)
AOT-assume ‹∃ α (ψ{α} & □φ{α})›
then AOT-obtain α **where** ‹ψ{α} & □φ{α}›
using ∃ *E*[*rotated*] **by** *blast*
AOT-hence ‹□(ψ{α} & φ{α})›
by (*metis KBasic*:*11 KBasic*:*3 T*◇ &*I* &*E*(*1*) &*E*(*2*)
≡*E*(*2*) *reductio−aa*:*1 rigid-condition vdash−properties*:*6*)
AOT-hence ‹∃ α □(ψ{α} & φ{α})›
by (*rule* ∃ *I*)
AOT-thus ‹□∃ α (ψ{α} & φ{α})›
by (*rule Buridan*[*THEN* →*E*])
qed

AOT-theorem *res−var−bound−reas*[*BF*◇]:
‹◇∃ α (ψ{α} & φ{α}) → ∃ α (ψ{α} & ◇φ{α})›
proof(*rule* →*I*)
AOT-assume ‹◇∃ α (ψ{α} & φ{α})›
AOT-hence ‹∃ α ◇(ψ{α} & φ{α})›
using *BF*◇[*THEN* →*E*] **by** *blast*
then AOT-obtain α **where** ‹◇(ψ{α} & φ{α})›
using ∃ *E*[*rotated*] **by** *blast*
AOT-hence ‹◇ψ{α}› **and** ‹◇φ{α}›
using *KBasic2*:*3* &*E* →*E* **by** *blast*+
moreover AOT-have ‹ψ{α}›
using *calculation rigid-condition* **by** (*metis B*◇ *K*◇ →*E*)
ultimately AOT-have ‹ψ{α} & ◇φ{α}›
using &*I* **by** *blast*
AOT-thus ‹∃ α (ψ{α} & ◇φ{α})›
by (*rule* ∃ *I*)
qed

AOT-theorem *res−var−bound−reas*[*CBF*◇]:
‹∃ α (ψ{α} & ◇φ{α}) → ◇∃ α (ψ{α} & φ{α})›
proof(*rule* →*I*)
AOT-assume ‹∃ α (ψ{α} & ◇φ{α})›
then AOT-obtain α **where** ‹ψ{α} & ◇φ{α}›
using ∃ *E*[*rotated*] **by** *blast*
AOT-hence ‹□ψ{α}› **and** ‹◇φ{α}›
using *rigid-condition*[*THEN qml*:*2*[*axiom-inst*, *THEN* →*E*], *THEN* →*E*] &*E* **by** *blast*+
AOT-hence ‹◇(ψ{α} & φ{α})›
by (*metis KBasic*:*16 con−dis−taut*:*5* →*E*)
AOT-hence ‹∃ α ◇(ψ{α} & φ{α})›
by (*rule* ∃ *I*)

```
  AOT-thus ‹◇∃α (ψ{α} & φ{α})›
    using CBF◇[THEN →E] by fast
qed

AOT-theorem res−var−bound−reas[A−Exists:1]:
  ‹𝓐∃!α (ψ{α} & φ{α}) ≡ ∃!α (ψ{α} & 𝓐φ{α})›
proof(safe intro!: ≡I →I)
  AOT-assume ‹𝓐∃!α (ψ{α} & φ{α})›
  AOT-hence ‹∃!α 𝓐(ψ{α} & φ{α})›
    using A−Exists:1[THEN ≡E(1)] by blast
  AOT-hence ‹∃!α (𝓐ψ{α} & 𝓐φ{α})›
    apply(AOT-subst ‹𝓐ψ{α} & 𝓐φ{α}› ‹𝓐(ψ{α} & φ{α})› for: α)
     apply (meson Act−Basic:2 intro−elim:3:f oth−class−taut:3:a)
    by simp
  AOT-thus ‹∃!α (ψ{α} & 𝓐φ{α})›
    apply (AOT-subst ‹ψ{α}› ‹𝓐ψ{α}› for: α)
    using Commutativity of ≡ intro−elim:3:b sc−eq−fur:2
          →E rigid-condition by blast
next
  AOT-assume ‹∃!α (ψ{α} & 𝓐φ{α})›
  AOT-hence ‹∃!α (𝓐ψ{α} & 𝓐φ{α})›
    apply (AOT-subst ‹𝓐ψ{α}› ‹ψ{α}› for: α)
     apply (meson sc−eq−fur:2 →E rigid-condition)
    by simp
  AOT-hence ‹∃!α 𝓐(ψ{α} & φ{α})›
    apply (AOT-subst ‹𝓐(ψ{α} & φ{α})› ‹𝓐ψ{α} & 𝓐φ{α}› for: α)
     using Act−Basic:2 apply presburger
     by simp
  AOT-thus ‹𝓐∃!α (ψ{α} & φ{α})›
     by (metis A−Exists:1 intro−elim:3:b)
qed

end

theory AOT-ExtendedRelationComprehension
  imports AOT-RestrictedVariables
begin
```

11 Extended Relation Comprehension

This theory depends on choosing extended models.

interpretation *AOT-ExtendedModel* **by** (*standard*; *auto*)

Auxiliary lemma: negations of denoting relations denote.

```
AOT-theorem negation-denotes: ‹[λx φ{x}]↓ → [λx ¬φ{x}]↓›
proof(rule →I)
  AOT-assume 0: ‹[λx φ{x}]↓›
  AOT-show ‹[λx ¬φ{x}]↓›
  proof (rule safe−ext[axiom-inst, THEN →E, OF &I])
    AOT-show ‹[λx ¬[λx φ{x}]x]↓› by cqt:2
  next
    AOT-have ‹□[λx φ{x}]↓›
      using 0 exist−nec[THEN →E] by blast
    moreover AOT-have ‹□[λx φ{x}]↓ → □∀x (¬[λx φ{x}]x ≡ ¬φ{x})›
      by(rule RM; safe intro!: GEN ≡I →I β→C(2) β←C(2) cqt:2)
    ultimately AOT-show ‹□∀x (¬[λx φ{x}]x ≡ ¬φ{x})›
      using →E by blast
  qed
qed
```

Auxiliary lemma: conjunctions of denoting relations denote.

```
AOT-theorem conjunction-denotes: ‹[λx φ{x}]↓ & [λx ψ{x}]↓ → [λx φ{x} & ψ{x}]↓›
```

proof(*rule* →*I*)
  AOT-assume *0*: ‹[λx φ{x}]↓ & [λx ψ{x}]↓›
  AOT-show ‹[λx φ{x} & ψ{x}]↓›
  proof (*rule safe−ext*[*axiom-inst*, *THEN* →*E*, *OF* &*I*])
    AOT-show ‹[λx [λx φ{x}]x & [λx ψ{x}]x]↓› **by** *cqt*:*2*
  next
    AOT-have ‹□([λx φ{x}]↓ & [λx ψ{x}]↓)›
      using *0 exist−nec*[*THEN* →*E*] &*E*
            KBasic:*3 df−simplify*:*2 intro−elim*:*3*:*b* **by** *blast*
    moreover AOT-have
      ‹□([λx φ{x}]↓ & [λx ψ{x}]↓) → □∀ x ([λx φ{x}]x & [λx ψ{x}]x ≡ φ{x} & ψ{x})›
      by(*rule RM*; *auto intro*!: *GEN* ≡*I* →*I cqt*:*2* &*I*
                  intro: β←*C*
                  dest: &*E* β→*C*)
    ultimately AOT-show ‹□∀ x ([λx φ{x}]x & [λx ψ{x}]x ≡ φ{x} & ψ{x})›
      using →*E* **by** *blast*
  qed
qed

AOT-register-rigid-restricted-type
  Ordinary: ‹O!κ›
proof
  AOT-modally-strict {
    AOT-show ‹∃ x O!x›
      by (*meson B*◇ *T*◇ *o−objects−exist*:*1* →*E*)
  }
next
  AOT-modally-strict {
    AOT-show ‹O!κ → κ↓› **for** κ
      by (*simp add*: →*I cqt*:*5*:*a*[*1*][*axiom-inst*, *THEN* →*E*, *THEN* &*E*(*2*)])
  }
next
  AOT-modally-strict {
    AOT-show ‹∀ α(O!α → □O!α)›
      by (*simp add*: *GEN oa−facts*:*1*)
  }
qed

AOT-register-variable-names
  Ordinary: *u v r t s*

In PLM this is defined in the Natural Numbers chapter, but since it is helpful for stating the comprehension principles, we already define it here.

AOT-define *eqE* :: ‹τ ⇒ τ ⇒ φ› (**infixl** ‹$\equiv_E$› *50*)
  eqE: ‹F $\equiv_E$ G $\equiv_{df}$ F↓ & G↓ & ∀ u ([F]u ≡ [G]u)›

Derive existence claims about relations from the axioms.

AOT-theorem *denotes-all*: ‹[λx ∀ G (□G $\equiv_E$ F → x[G])]↓›
    and *denotes-all-neg*: ‹[λx ∀ G (□G $\equiv_E$ F → ¬x[G])]↓›
proof −
  AOT-have *Aux*: ‹∀ F (□F $\equiv_E$ G → (x[F] ≡ x[G])), ¬(x[G] ≡ y[G])
  $\vdash_\Box$ ∃ F([F]x & ¬[F]y)› **for** *x y G*
  proof −
    AOT-modally-strict {
    AOT-assume *0*: ‹∀ F (□F $\equiv_E$ G → (x[F] ≡ x[G]))›
    AOT-assume *G-prop*: ‹¬(x[G] ≡ y[G])›
    {
      AOT-assume ‹¬∃ F([F]x & ¬[F]y)›
      AOT-hence *0*: ‹∀ F ¬([F]x & ¬[F]y)›
        by (*metis cqt−further*:*4* →*E*)
      AOT-have ‹∀ F ([F]x ≡ [F]y)›
      proof (*rule GEN*; *rule* ≡*I*; *rule* →*I*)
        fix *F*

AOT-assume ‹[F]x›
moreover AOT-have ‹¬([F]x & ¬[F]y)›
using *0[THEN ∀E(2)]* **by** *blast*
ultimately AOT-show ‹[F]y›
by (*metis &I raa−cor:1*)
next
fix *F*
AOT-assume ‹[F]y›
AOT-hence ‹¬[λx ¬[F]x]y›
by (*metis ¬¬I β→C(2)*)
moreover AOT-have ‹¬([λx ¬[F]x]x & ¬[λx ¬[F]x]y)›
apply (*rule 0[THEN ∀E(1)]*) **by** *cqt:2[lambda]*
ultimately AOT-have *1*: ‹¬[λx ¬[F]x]x›
by (*metis &I raa−cor:3*)
{
AOT-assume ‹¬[F]x›
AOT-hence ‹[λx ¬[F]x]x›
by (*auto intro!: β←C(1) cqt:2*)
AOT-hence ‹p & ¬p› **for** *p*
using *1* **by** (*metis raa−cor:3*)
}
AOT-thus ‹[F]x› **by** (*metis raa−cor:1*)
qed
AOT-hence ‹□∀F ([F]x ≡ [F]y)›
using *ind−nec[THEN →E]* **by** *blast*
AOT-hence ‹∀F □([F]x ≡ [F]y)›
by (*metis CBF →E*)
} **note** *indistI = this*
{
AOT-assume *G-prop*: ‹x[G] & ¬y[G]›
AOT-hence *Ax*: ‹A!x›
using *&E(1) ∃I(2) →E encoders−are−abstract* **by** *blast*

{
AOT-assume *Ay*: ‹A!y›
{
fix *F*
{
AOT-assume ‹∀u□([F]u ≡ [G]u)›
AOT-hence ‹□∀u([F]u ≡ [G]u)›
using *Ordinary.res−var−bound−reas[BF][THEN →E]* **by** *simp*
AOT-hence ‹□F $\equiv_E$ G›
by (*AOT-subst* ‹F $\equiv_E$ G› ‹∀u ([F]u ≡ [G]u)›)
(*auto intro!: eqE[THEN ≡Df, THEN ≡S(1), OF &I] cqt:2*)
AOT-hence ‹x[F] ≡ x[G]›
using *0[THEN ∀E(2)] ≡E →E* **by** *meson*
AOT-hence ‹x[F]›
using *G-prop &E ≡E* **by** *blast*
}
AOT-hence ‹∀u□([F]u ≡ [G]u) → x[F]›
by (*rule →I*)
}
AOT-hence *xprop*: ‹∀F(∀u□([F]u ≡ [G]u) → x[F])›
by (*rule GEN*)
moreover AOT-have *yprop*: ‹¬∀F(∀u□([F]u ≡ [G]u) → y[F])›
proof (*rule raa−cor:2*)
AOT-assume ‹∀F(∀u□([F]u ≡ [G]u) → y[F])›
AOT-hence ‹∀F(□∀u([F]u ≡ [G]u) → y[F])›
apply (*AOT-subst* ‹□∀u([F]u ≡ [G]u)› ‹∀u□([F]u ≡ [G]u)› **for**: *F*)
using *Ordinary.res−var−bound−reas[BF]*
Ordinary.res−var−bound−reas[CBF]
intro−elim:2 **apply** *presburger*
by *simp*

```
      AOT-hence A: ‹∀ F(□F ≡E G → y[F])›
        by (AOT-subst ‹F ≡E G› ‹∀ u ([F]u ≡ [G]u)› for: F)
           (auto intro!: eqE[THEN ≡Df, THEN ≡S(1), OF &I] cqt:2)
      moreover AOT-have ‹□G ≡E G›
        by (auto intro!: eqE[THEN ≡df I] cqt:2 RN &I GEN →I ≡I)
      ultimately AOT-have ‹y[G]› using ∀ E(2) →E by blast
      AOT-thus ‹p & ¬p› for p using G-prop &E by (metis raa−cor:3)
    qed
    AOT-have ‹∃ F([F]x & ¬[F]y)›
    proof(rule raa−cor:1)
      AOT-assume ‹¬∃ F([F]x & ¬[F]y)›
      AOT-hence indist: ‹∀ F □([F]x ≡ [F]y)› using indistI by blast
      AOT-have ‹∀ F(∀ u□([F]u ≡ [G]u) → y[F])›
        using indistinguishable-ord-enc-all[axiom-inst, THEN →E, OF &I,
                  OF &I, OF &I, OF cqt:2[const-var][axiom-inst],
                  OF Ax, OF Ay, OF indist, THEN ≡E(1), OF xprop].
      AOT-thus ‹∀ F(∀ u□([F]u ≡ [G]u) → y[F]) & ¬∀ F(∀ u□([F]u ≡ [G]u) → y[F])›
        using yprop &I by blast
    qed
  }
  moreover {
    AOT-assume notAy: ‹¬A!y›
    AOT-have ‹∃ F([F]x & ¬[F]y)›
      apply (rule ∃ I(1)[where τ=‹«A!»›])
      using Ax notAy &I apply blast
      by (simp add: oa−exist:2)
  }
  ultimately AOT-have ‹∃ F([F]x & ¬[F]y)›
    by (metis raa−cor:1)
}
moreover {
  AOT-assume G-prop: ‹¬x[G] & y[G]›
  AOT-hence Ay: ‹A!y›
    by (meson &E(2) encoders−are−abstract existential:2[const-var] →E)
  AOT-hence notOy: ‹¬O!y›
    using ≡E(1) oa−contingent:3 by blast
  {
    AOT-assume Ax: ‹A!x›
    {
      fix F
      {
        AOT-assume ‹□∀ u([F]u ≡ [G]u)›
        AOT-hence ‹□F ≡E G›
          by (AOT-subst ‹F ≡E G› ‹∀ u([F]u ≡ [G]u)›)
             (auto intro!: eqE[THEN ≡Df, THEN ≡S(1), OF &I] cqt:2)
        AOT-hence ‹x[F] ≡ x[G]›
          using 0[THEN ∀ E(2)] ≡E →E by meson
        AOT-hence ‹¬x[F]›
          using G-prop &E ≡E by blast
      }
      AOT-hence ‹□∀ u([F]u ≡ [G]u) → ¬x[F]›
        by (rule →I)
    }
    AOT-hence x-prop: ‹∀ F(□∀ u([F]u ≡ [G]u) → ¬x[F])›
      by (rule GEN)
    AOT-have x-prop: ‹¬∃ F(∀ u□([F]u ≡ [G]u) & x[F])›
    proof (rule raa−cor:2)
      AOT-assume ‹∃ F(∀ u □([F]u ≡ [G]u) & x[F])›
      then AOT-obtain F where F-prop: ‹∀ u □([F]u ≡ [G]u) & x[F]›
        using ∃ E[rotated] by blast
      AOT-have ‹□([F]u ≡ [G]u)› for u
        using F-prop[THEN &E(1), THEN Ordinary.∀ E].
      AOT-hence ‹∀ u □([F]u ≡ [G]u)›
```

```
        by (rule Ordinary.GEN)
      AOT-hence ‹□∀ u([F]u ≡ [G]u)›
        by (metis Ordinary.res−var−bound−reas[BF] →E)
      AOT-hence ‹¬x[F]›
        using x-prop[THEN ∀ E(2), THEN →E] by blast
      AOT-thus ‹p & ¬p› for p
        using F-prop[THEN &E(2)] by (metis raa−cor:3)
    qed
    AOT-have y-prop: ‹∃ F(∀ u □([F]u ≡ [G]u) & y[F])›
    proof (rule raa−cor:1)
      AOT-assume ‹¬∃ F (∀ u □([F]u ≡ [G]u) & y[F])›
      AOT-hence 0: ‹∀ F ¬(∀ u □([F]u ≡ [G]u) & y[F])›
        using cqt−further:4[THEN →E] by blast
      {
        fix F
        {
          AOT-assume ‹∀ u □([F]u ≡ [G]u)›
          AOT-hence ‹¬y[F]›
            using 0[THEN ∀ E(2)] &I raa−cor:1 by meson
        }
        AOT-hence ‹(∀ u □([F]u ≡ [G]u) → ¬y[F])›
          by (rule →I)
      }
      AOT-hence A: ‹∀ F(∀ u □([F]u ≡ [G]u) → ¬y[F])›
        by (rule GEN)
      moreover AOT-have ‹∀ u □([G]u ≡ [G]u)›
        by (simp add: RN oth−class−taut:3:a universal−cor →I)
      ultimately AOT-have ‹¬y[G]›
        using ∀ E(2) →E by blast
      AOT-thus ‹p & ¬p› for p
        using G-prop &E by (metis raa−cor:3)
    qed
    AOT-have ‹∃ F([F]x & ¬[F]y)›
    proof(rule raa−cor:1)
      AOT-assume ‹¬∃ F([F]x & ¬[F]y)›
      AOT-hence indist: ‹∀ F □([F]x ≡ [F]y)›
        using indistI by blast
      AOT-thus ‹∃ F(∀ u □([F]u ≡ [G]u) & x[F]) & ¬∃ F(∀ u □([F]u ≡ [G]u) & x[F])›
        using indistinguishable-ord-enc-ex[axiom-inst, THEN →E, OF &I,
                OF &I, OF &I, OF cqt:2[const-var][axiom-inst],
                OF Ax, OF Ay, OF indist, THEN ≡E(2), OF y-prop]
            x-prop &I by blast
    qed
  }
  moreover {
    AOT-assume notAx: ‹¬A!x›
    AOT-hence Ox: ‹O!x›
      using ∨E(3) oa−exist:3 by blast
    AOT-have ‹∃ F([F]x & ¬[F]y)›
      apply (rule ∃ I(1)[where τ=‹«O!»›])
      using Ox notOy &I apply blast
      by (simp add: oa−exist:1)
  }
  ultimately AOT-have ‹∃ F([F]x & ¬[F]y)›
    by (metis raa−cor:1)
 }
 ultimately AOT-show ‹∃ F([F]x & ¬[F]y)›
   using G-prop by (metis &I →I ≡I raa−cor:1)
}
qed

AOT-modally-strict {
  fix x y
```

```
AOT-assume indist: ‹∀ F ([F]x ≡ [F]y)›
AOT-hence nec-indist: ‹□∀ F ([F]x ≡ [F]y)›
  using ind−nec vdash−properties:10 by blast
AOT-hence indist-nec: ‹∀ F □([F]x ≡ [F]y)›
  using CBF vdash−properties:10 by blast
AOT-assume 0: ‹∀ G (□G ≡E F → x[G])›
AOT-hence 1: ‹∀ G (□∀ u ([G]u ≡ [F]u) → x[G])›
  by (AOT-subst (reverse) ‹∀ u ([G]u ≡ [F]u)› ‹G ≡E F› for: G)
     (auto intro!: eqE[THEN ≡Df, THEN ≡S(1), OF &I] cqt:2)
AOT-have ‹x[F]›
  by (safe intro!: 1[THEN ∀ E(2), THEN →E] GEN →I RN ≡I)
AOT-have ‹∀ G (□G ≡E F → y[G])›
proof(rule raa−cor:1)
  AOT-assume ‹¬∀ G (□G ≡E F → y[G])›
  AOT-hence ‹∃ G ¬(□G ≡E F → y[G])›
    using cqt−further:2 →E by blast
  then AOT-obtain G where G-prop: ‹¬(□G ≡E F → y[G])›
    using ∃ E[rotated] by blast
  AOT-hence 1: ‹□G ≡E F & ¬y[G]›
    by (metis ≡E(1) oth−class−taut:1:b)
  AOT-have xG: ‹x[G]›
    using 0[THEN ∀ E(2), THEN →E] 1[THEN &E(1)] by blast
  AOT-hence ‹x[G] & ¬y[G]›
    using 1[THEN &E(2)] &I by blast
  AOT-hence B: ‹¬(x[G] ≡ y[G])›
    using &E(2) ≡E(1) reductio−aa:1 xG by blast
  {
    fix H
    {
      AOT-assume ‹□H ≡E G›
      AOT-hence ‹□(H ≡E G & G ≡E F)›
        using 1 by (metis KBasic:3 con−dis−i−e:1 con−dis−i−e:2:a
                        intro−elim:3:b)
      moreover AOT-have ‹□(H ≡E G & G ≡E F) → □(H ≡E F)›
      proof(rule RM)
        AOT-modally-strict {
          AOT-show ‹H ≡E G & G ≡E F → H ≡E F›
          proof (safe intro!: →I eqE[THEN ≡df I] &I cqt:2 Ordinary.GEN)
            fix u
            AOT-assume ‹H ≡E G & G ≡E F›
            AOT-hence ‹∀ u ([H]u ≡ [G]u)› and ‹∀ u ([G]u ≡ [F]u)›
              using eqE[THEN ≡df E] &E by blast+
            AOT-thus ‹[H]u ≡ [F]u›
              by (auto dest!: Ordinary.∀ E dest: ≡E)
          qed
        }
      qed
      ultimately AOT-have ‹□(H ≡E F)›
        using →E by blast
      AOT-hence ‹x[H]›
        using 0[THEN ∀ E(2)] →E by blast
      AOT-hence ‹x[H] ≡ x[G]›
        using xG ≡I →I by blast
    }
    AOT-hence ‹□H ≡E G → (x[H] ≡ x[G])› by (rule →I)
  }
  AOT-hence A: ‹∀ H(□H ≡E G → (x[H] ≡ x[G]))›
    by (rule GEN)
  then AOT-obtain F where F-prop: ‹[F]x & ¬[F]y›
    using Aux[OF A, OF B] ∃ E[rotated] by blast
  moreover AOT-have ‹[F]y›
    using indist[THEN ∀ E(2), THEN ≡E(1), OF F-prop[THEN &E(1)]].
  AOT-thus ‹p & ¬p› for p
```

```
      using F-prop[THEN &E(2)] by (metis raa−cor:3)
  qed
} note 0 = this
AOT-modally-strict {
  fix x y
  AOT-assume ‹∀F ([F]x ≡ [F]y)›
  moreover AOT-have ‹∀F ([F]y ≡ [F]x)›
    by (metis calculation cqt−basic:11 ≡E(2))
  ultimately AOT-have ‹∀G (□G ≡E F → x[G]) ≡ ∀G (□G ≡E F → y[G])›
    using 0 ≡I →I by auto
} note 1 = this
AOT-show ‹[λx ∀G (□G ≡E F → x[G])]↓›
  by (safe intro!: RN GEN →I 1 kirchner−thm:2[THEN ≡E(2)])

AOT-modally-strict {
  fix x y
  AOT-assume indist: ‹∀F ([F]x ≡ [F]y)›
  AOT-hence nec-indist: ‹□∀F ([F]x ≡ [F]y)›
    using ind−nec vdash−properties:10 by blast
  AOT-hence indist-nec: ‹∀F □([F]x ≡ [F]y)›
    using CBF vdash−properties:10 by blast
  AOT-assume 0: ‹∀G (□G ≡E F → ¬x[G])›
  AOT-hence 1: ‹∀G (□∀u ([G]u ≡ [F]u) → ¬x[G])›
    by (AOT-subst (reverse) ‹∀u ([G]u ≡ [F]u)› ‹G ≡E F› for: G)
       (auto intro!: eqE[THEN ≡Df, THEN ≡S(1), OF &I] cqt:2)
  AOT-have ‹¬x[F]›
    by (safe intro!: 1[THEN ∀E(2), THEN →E] GEN →I RN ≡I)
  AOT-have ‹∀G (□G ≡E F → ¬y[G])›
  proof(rule raa−cor:1)
    AOT-assume ‹¬∀G (□G ≡E F → ¬y[G])›
    AOT-hence ‹∃G ¬(□G ≡E F → ¬y[G])›
      using cqt−further:2 →E by blast
    then AOT-obtain G where G-prop: ‹¬(□G ≡E F → ¬y[G])›
      using ∃E[rotated] by blast
    AOT-hence 1: ‹□G ≡E F & ¬¬y[G]›
      by (metis ≡E(1) oth−class−taut:1:b)
    AOT-hence yG: ‹y[G]›
      using G-prop →I raa−cor:3 by blast
    moreover AOT-hence 12: ‹¬x[G]›
      using 0[THEN ∀E(2), THEN →E] 1[THEN &E(1)] by blast
    ultimately AOT-have ‹¬x[G] & y[G]›
      using &I by blast
    AOT-hence B: ‹¬(x[G] ≡ y[G])›
      by (metis 12 ≡E(3) raa−cor:3 yG)
    {
      fix H
      {
        AOT-assume 3: ‹□H ≡E G›
        AOT-hence ‹□(H ≡E G & G ≡E F)›
          using 1
          by (metis KBasic:3 con−dis−i−e:1 →I intro−elim:3:b
                 reductio−aa:1 G-prop)
        moreover AOT-have ‹□(H ≡E G & G ≡E F) → □(H ≡E F)›
        proof (rule RM)
          AOT-modally-strict {
            AOT-show ‹H ≡E G & G ≡E F → H ≡E F›
            proof (safe intro!: →I eqE[THEN ≡df I] &I cqt:2 Ordinary.GEN)
              fix u
              AOT-assume ‹H ≡E G & G ≡E F›
              AOT-hence ‹∀u ([H]u ≡ [G]u)› and ‹∀u ([G]u ≡ [F]u)›
                using eqE[THEN ≡df E] &E by blast+
              AOT-thus ‹[H]u ≡ [F]u›
                by (auto dest!: Ordinary.∀E dest: ≡E)
```

```
              qed
            }
          qed
          ultimately AOT-have ‹□(H ≡E F)›
            using →E by blast
          AOT-hence ‹¬x[H]›
            using 0[THEN ∀E(2)] →E by blast
          AOT-hence ‹x[H] ≡ x[G]›
            using 12 ≡I →I by (metis raa−cor:3)
        }
        AOT-hence ‹□H ≡E G → (x[H] ≡ x[G])›
          by (rule →I)
      }
      AOT-hence A: ‹∀H(□H ≡E G → (x[H] ≡ x[G]))›
        by (rule GEN)
      then AOT-obtain F where F-prop: ‹[F]x & ¬[F]y›
        using Aux[OF A, OF B] ∃E[rotated] by blast
      moreover AOT-have ‹[F]y›
        using indist[THEN ∀E(2), THEN ≡E(1), OF F-prop[THEN &E(1)]].
      AOT-thus ‹p & ¬p› for p
        using F-prop[THEN &E(2)] by (metis raa−cor:3)
    qed
  } note 0 = this
  AOT-modally-strict {
    fix x y
    AOT-assume ‹∀F ([F]x ≡ [F]y)›
    moreover AOT-have ‹∀F ([F]y ≡ [F]x)›
      by (metis calculation cqt−basic:11 ≡E(2))
    ultimately AOT-have ‹∀G (□G ≡E F → ¬x[G]) ≡ ∀G (□G ≡E F → ¬y[G])›
      using 0 ≡I →I by auto
  } note 1 = this
  AOT-show ‹[λx ∀G (□G ≡E F → ¬x[G])]↓›
    by (safe intro!: RN GEN →I 1 kirchner−thm:2[THEN ≡E(2)])
qed
```

Reformulate the existence claims in terms of their negations.

```
AOT-theorem denotes-ex: ‹[λx ∃G (□G ≡E F & x[G])]↓›
proof (rule safe−ext[axiom-inst, THEN →E, OF &I])
  AOT-show ‹[λx ¬∀G (□G ≡E F → ¬x[G])]↓›
    using denotes-all-neg[THEN negation-denotes[THEN →E]].
next
  AOT-show ‹□∀x (¬∀G (□G ≡E F → ¬x[G]) ≡ ∃G (□G ≡E F & x[G]))›
    by (AOT-subst ‹□G ≡E F & x[G]› ‹¬(□G ≡E F → ¬x[G])› for: G x)
      (auto simp: conventions:1 rule−eq−df:1
            intro: oth−class−taut:4:b[THEN ≡E(2)]
                   intro−elim:3:f[OF cqt−further:3, OF oth−class−taut:3:b]
            intro!: RN GEN)
qed

AOT-theorem denotes-ex-neg: ‹[λx ∃G (□G ≡E F & ¬x[G])]↓›
proof (rule safe−ext[axiom-inst, THEN →E, OF &I])
  AOT-show ‹[λx ¬∀G (□G ≡E F → x[G])]↓›
    using denotes-all[THEN negation-denotes[THEN →E]].
next
  AOT-show ‹□∀x (¬∀G (□G ≡E F → x[G]) ≡ ∃G (□G ≡E F & ¬x[G]))›
    by (AOT-subst (reverse) ‹□G ≡E F & ¬x[G]› ‹¬(□G ≡E F → x[G])› for: G x)
      (auto simp: oth−class−taut:1:b
            intro: oth−class−taut:4:b[THEN ≡E(2)]
                   intro−elim:3:f[OF cqt−further:3, OF oth−class−taut:3:b]
            intro!: RN GEN)
qed
```

Derive comprehension principles.

```
AOT-theorem Comprehension-1:
  shows ‹□∀F∀G(□G ≡E F → (φ{F} ≡ φ{G})) → [λx ∃F (φ{F} & x[F])]↓›
proof(rule →I)
  AOT-assume assm: ‹□∀F∀G(□G ≡E F → (φ{F} ≡ φ{G}))›
  AOT-modally-strict {
    fix x y
    AOT-assume 0: ‹∀F∀G (□G ≡E F → (φ{F} ≡ φ{G}))›
    AOT-assume indist: ‹∀F ([F]x ≡ [F]y)›
    AOT-assume x-prop: ‹∃F (φ{F} & x[F])›
    then AOT-obtain F where F-prop: ‹φ{F} & x[F]›
      using ∃E[rotated] by blast
    AOT-hence ‹□F ≡E F & x[F]›
      by (auto intro!: RN eqE[THEN ≡df I] &I cqt:2 GEN ≡I →I dest: &E)
    AOT-hence ‹∃G(□G ≡E F & x[G])›
      by (rule ∃I)
    AOT-hence ‹[λx ∃G(□G ≡E F & x[G])]x›
      by (safe intro!: β←C denotes-ex cqt:2)
    AOT-hence ‹[λx ∃G(□G ≡E F & x[G])]y›
      using indist[THEN ∀E(1), OF denotes-ex, THEN ≡E(1)] by blast
    AOT-hence ‹∃G(□G ≡E F & y[G])›
      using β→C by blast
    then AOT-obtain G where ‹□G ≡E F & y[G]›
      using ∃E[rotated] by blast
    AOT-hence ‹φ{G} & y[G]›
      using 0[THEN ∀E(2), THEN ∀E(2), THEN →E, THEN ≡E(1)]
            F-prop[THEN &E(1)] &E &I by blast
    AOT-hence ‹∃F (φ{F} & y[F])›
      by (rule ∃I)
  } note 1 = this
  AOT-modally-strict {
    AOT-assume 0: ‹∀F∀G (□G ≡E F → (φ{F} ≡ φ{G}))›
    {
      fix x y
      {
        AOT-assume ‹∀F ([F]x ≡ [F]y)›
        moreover AOT-have ‹∀F ([F]y ≡ [F]x)›
          by (metis calculation cqt−basic:11 ≡E(1))
        ultimately AOT-have ‹∃F (φ{F} & x[F]) ≡ ∃F (φ{F} & y[F])›
          using 0 1[OF 0] ≡I →I by simp
      }
      AOT-hence ‹∀F ([F]x ≡ [F]y) → (∃F (φ{F} & x[F]) ≡ ∃F (φ{F} & y[F]))›
        using →I by blast
    }
    AOT-hence ‹∀x∀y(∀F ([F]x ≡ [F]y) → (∃F (φ{F} & x[F]) ≡  ∃F (φ{F} & y[F])))›
      by (auto intro!: GEN)
  } note 1 = this
  AOT-hence ‹⊢□ ∀F∀G (□G ≡E F → (φ{F} ≡ φ{G})) →
              ∀x∀y(∀F ([F]x ≡ [F]y) → (∃F (φ{F} & x[F]) ≡ ∃F (φ{F} & y[F])))›
    by (rule →I)
  AOT-hence ‹□∀F∀G (□G ≡E F → (φ{F} ≡ φ{G})) →
             □∀x∀y(∀F ([F]x ≡ [F]y) → (∃F (φ{F} & x[F]) ≡ ∃F (φ{F} & y[F])))›
    by (rule RM)
  AOT-hence ‹□∀x∀y(∀F ([F]x ≡ [F]y) → (∃F (φ{F} & x[F]) ≡ ∃F (φ{F} & y[F])))›
    using →E assm by blast
  AOT-thus ‹[λx ∃F (φ{F} & x[F])]↓›
    by (safe intro!: kirchner−thm:2[THEN ≡E(2)])
qed

AOT-theorem Comprehension-2:
  shows ‹□∀F∀G(□G ≡E F → (φ{F} ≡ φ{G})) → [λx ∃F (φ{F} & ¬x[F])]↓›
proof(rule →I)
  AOT-assume assm: ‹□∀F∀G(□G ≡E F → (φ{F} ≡ φ{G}))›
  AOT-modally-strict {
```

```
    fix x y
    AOT-assume 0: ‹∀ F∀ G (□G ≡E F → (φ{F} ≡ φ{G}))›
    AOT-assume indist: ‹∀ F ([F]x ≡ [F]y)›
    AOT-assume x-prop: ‹∃ F (φ{F} & ¬x[F])›
    then AOT-obtain F where F-prop: ‹φ{F} & ¬x[F]›
      using ∃ E[rotated] by blast
    AOT-hence ‹□F ≡E F & ¬x[F]›
      by (auto intro!: RN eqE[THEN ≡df I] &I cqt:2 GEN ≡I →I dest: &E)
    AOT-hence ‹∃ G(□G ≡E F & ¬x[G])›
      by (rule ∃ I)
    AOT-hence ‹[λx ∃ G(□G ≡E F & ¬x[G])]x›
      by (safe intro!: β←C denotes-ex-neg cqt:2)
    AOT-hence ‹[λx ∃ G(□G ≡E F & ¬x[G])]y›
      using indist[THEN ∀ E(1), OF denotes-ex-neg, THEN ≡E(1)] by blast
    AOT-hence ‹∃ G(□G ≡E F & ¬y[G])›
      using β→C by blast
    then AOT-obtain G where ‹□G ≡E F & ¬y[G]›
      using ∃ E[rotated] by blast
    AOT-hence ‹φ{G} & ¬y[G]›
      using 0[THEN ∀ E(2), THEN ∀ E(2), THEN →E, THEN ≡E(1)]
            F-prop[THEN &E(1)] &E &I by blast
    AOT-hence ‹∃ F (φ{F} & ¬y[F])›
      by (rule ∃ I)
  } note 1 = this
  AOT-modally-strict {
    AOT-assume 0: ‹∀ F∀ G (□G ≡E F → (φ{F} ≡ φ{G}))›
    {
      fix x y
      {
        AOT-assume ‹∀ F ([F]x ≡ [F]y)›
        moreover AOT-have ‹∀ F ([F]y ≡ [F]x)›
          by (metis calculation cqt−basic:11 ≡E(1))
        ultimately AOT-have ‹∃ F (φ{F} & ¬x[F]) ≡ ∃ F (φ{F} & ¬y[F])›
          using 0 1[OF 0] ≡I →I by simp
      }
      AOT-hence ‹∀ F ([F]x ≡ [F]y) → (∃ F (φ{F} & ¬x[F]) ≡ ∃ F (φ{F} & ¬y[F]))›
        using →I by blast
    }
    AOT-hence ‹∀ x∀ y(∀ F ([F]x ≡ [F]y) → (∃ F (φ{F} & ¬x[F]) ≡  ∃ F (φ{F} & ¬y[F])))›
      by (auto intro!: GEN)
  } note 1 = this
  AOT-hence ‹⊢□ ∀ F∀ G (□G ≡E F → (φ{F} ≡ φ{G})) →
              ∀ x∀ y(∀ F ([F]x ≡ [F]y) → (∃ F (φ{F} & ¬x[F]) ≡ ∃ F (φ{F} & ¬y[F])))›
    by (rule →I)
  AOT-hence ‹□∀ F∀ G (□G ≡E F → (φ{F} ≡ φ{G})) →
            □∀ x∀ y(∀ F ([F]x ≡ [F]y) → (∃ F (φ{F} & ¬x[F]) ≡ ∃ F (φ{F} & ¬y[F])))›
    by (rule RM)
  AOT-hence ‹□∀ x∀ y(∀ F ([F]x ≡ [F]y) → (∃ F (φ{F} & ¬x[F]) ≡ ∃ F (φ{F} & ¬y[F])))›
    using →E assm by blast
  AOT-thus ‹[λx ∃ F (φ{F} & ¬x[F])]↓›
    by (safe intro!: kirchner−thm:2[THEN ≡E(2)])
qed
```

Derived variants of the comprehension principles above.

```
AOT-theorem Comprehension-1':
  shows ‹□∀ F∀ G(□G ≡E F → (φ{F} ≡ φ{G})) → [λx ∀ F (x[F] → φ{F})]↓›
proof(rule →I)
  AOT-assume ‹□∀ F∀ G(□G ≡E F → (φ{F} ≡ φ{G}))›
  AOT-hence 0: ‹□∀ F∀ G(□G ≡E F → (¬φ{F} ≡ ¬φ{G}))›
    by (AOT-subst (reverse) ‹¬φ{F} ≡ ¬φ{G}› ‹φ{F} ≡ φ{G}› for: F G)
       (auto simp: oth−class−taut:4:b)
  AOT-show ‹[λx ∀ F (x[F] → φ{F})]↓›
  proof(rule safe−ext[axiom-inst, THEN →E, OF &I])
```

AOT-show ‹$[\lambda x\ \neg\exists F(\neg\varphi\{F\}\ \&\ x[F])]\downarrow$›
using *Comprehension-1*[*THEN* →*E*, *OF 0*, *THEN negation-denotes*[*THEN* →*E*]]**.**
next
AOT-show ‹$\Box\forall x\ (\neg\exists F\ (\neg\varphi\{F\}\ \&\ x[F]) \equiv \forall F\ (x[F] \rightarrow \varphi\{F\}))$›
by (*AOT-subst* (*reverse*) ‹$\neg\varphi\{F\}\ \&\ x[F]$› ‹$\neg(x[F] \rightarrow \varphi\{F\})$› **for**: *F x*)
(*auto simp*: *oth−class−taut*:*1*:*b*[*THEN intro−elim*:*3*:*e*,
OF oth−class−taut:*2*:*a*]
intro: *intro−elim*:*3*:*f*[*OF cqt−further*:*3*, *OF oth−class−taut*:*3*:*a*,
symmetric]
intro!: *RN GEN*)
qed
qed

AOT-theorem *Comprehension-2′*:
shows ‹$\Box\forall F\forall G(\Box G \equiv_E F \rightarrow (\varphi\{F\} \equiv \varphi\{G\})) \rightarrow [\lambda x\ \forall F\ (\varphi\{F\} \rightarrow x[F])]\downarrow$›
proof(*rule* →*I*)
AOT-assume *0*: ‹$\Box\forall F\forall G(\Box G \equiv_E F \rightarrow (\varphi\{F\} \equiv \varphi\{G\}))$›
AOT-show ‹$[\lambda x\ \forall F\ (\varphi\{F\} \rightarrow x[F])]\downarrow$›
proof(*rule safe−ext*[*axiom-inst*, *THEN* →*E*, *OF* &*I*])
AOT-show ‹$[\lambda x\ \neg\exists F(\varphi\{F\}\ \&\ \neg x[F])]\downarrow$›
using *Comprehension-2*[*THEN* →*E*, *OF 0*, *THEN negation-denotes*[*THEN* →*E*]]**.**
next
AOT-show ‹$\Box\forall x\ (\neg\exists F\ (\varphi\{F\}\ \&\ \neg x[F]) \equiv \forall F\ (\varphi\{F\} \rightarrow x[F]))$›
by (*AOT-subst* (*reverse*) ‹$\varphi\{F\}\ \&\ \neg x[F]$› ‹$\neg(\varphi\{F\} \rightarrow x[F])$› **for**: *F x*)
(*auto simp*: *oth−class−taut*:*1*:*b*
intro: *intro−elim*:*3*:*f*[*OF cqt−further*:*3*, *OF oth−class−taut*:*3*:*a*,
symmetric]
intro!: *RN GEN*)
qed
qed

Derive a combined comprehension principles.

AOT-theorem *Comprehension-3*:
‹$\Box\forall F\forall G(\Box G \equiv_E F \rightarrow (\varphi\{F\} \equiv \varphi\{G\})) \rightarrow [\lambda x\ \forall F\ (x[F] \equiv \varphi\{F\})]\downarrow$›
proof(*rule* →*I*)
AOT-assume *0*: ‹$\Box\forall F\forall G(\Box G \equiv_E F \rightarrow (\varphi\{F\} \equiv \varphi\{G\}))$›
AOT-show ‹$[\lambda x\ \forall F\ (x[F] \equiv \varphi\{F\})]\downarrow$›
proof(*rule safe−ext*[*axiom-inst*, *THEN* →*E*, *OF* &*I*])
AOT-show ‹$[\lambda x\ \forall F\ (x[F] \rightarrow \varphi\{F\})\ \&\ \forall F\ (\varphi\{F\} \rightarrow x[F])]\downarrow$›
by (*safe intro*!: *conjunction-denotes*[*THEN* →*E*, *OF* &*I*]
Comprehension-1′[*THEN* →*E*]
Comprehension-2′[*THEN* →*E*] *0*)
next
AOT-show ‹$\Box\forall x\ (\forall F\ (x[F] \rightarrow \varphi\{F\})\ \&\ \forall F\ (\varphi\{F\} \rightarrow x[F]) \equiv \forall F\ (x[F] \equiv \varphi\{F\}))$›
by (*auto intro*!: *RN GEN* ≡*I* →*I* &*I dest*: &*E* ∀*E*(*2*) →*E* ≡*E*(*1*,*2*))
qed
qed

notepad
begin

Verify that the original axioms are equivalent to $\vdash_\Box [\lambda x\ \exists G\ (\Box G \equiv_E F\ \&\ x[G])]\downarrow$ and $\vdash_\Box [\lambda x\ \exists G\ (\Box G \equiv_E F\ \&\ \neg x[G])]\downarrow$.

AOT-modally-strict {
fix *x y H*
AOT-have ‹$A!x\ \&\ A!y\ \&\ \forall F\ \Box([F]x \equiv [F]y) \rightarrow$
$(\forall G\ (\forall z\ (O!z \rightarrow \Box([G]z \equiv [H]z)) \rightarrow x[G]) \equiv$
$\forall G\ (\forall z\ (O!z \rightarrow \Box([G]z \equiv [H]z)) \rightarrow y[G]))$›
proof(*rule* →*I*)
{
fix *x y*
AOT-assume ‹$A!x$›
AOT-assume ‹$A!y$›

```
    AOT-assume indist: ‹∀ F □([F]x ≡ [F]y)›
    AOT-assume ‹∀ G (∀ u □([G]u ≡ [H]u) → x[G])›
    AOT-hence ‹∀ G (□∀ u ([G]u ≡ [H]u) → x[G])›
      using Ordinary.res−var−bound−reas[BF] Ordinary.res−var−bound−reas[CBF]
            intro−elim:2
      by (AOT-subst ‹□∀ u ([G]u ≡ [H]u)› ‹∀ u □([G]u ≡ [H]u)› for: G) auto
    AOT-hence ‹∀ G (□G ≡_E H → x[G])›
      by (AOT-subst ‹G ≡_E H› ‹∀ u ([G]u ≡ [H]u)› for: G)
         (safe intro!: eqE[THEN ≡Df, THEN ≡S(1), OF &I] cqt:2)
    AOT-hence ‹¬∃ G (□G ≡_E H & ¬x[G])›
      by (AOT-subst (reverse) ‹(□G ≡_E H & ¬x[G])› ‹¬(□G ≡_E H → x[G])› for: G)
        (auto simp: oth−class−taut:1:b cqt−further:3[THEN ≡E(1)])
    AOT-hence ‹¬[λx ∃ G (□G ≡_E H & ¬x[G])]x›
      by (auto intro: β→C)
    AOT-hence ‹¬[λx ∃ G (□G ≡_E H & ¬x[G])]y›
      using indist[THEN ∀ E(1), OF denotes-ex-neg,
                  THEN qml:2[axiom-inst, THEN →E],
                  THEN ≡E(3)] by blast
    AOT-hence ‹¬∃ G (□G ≡_E H & ¬y[G])›
      by (safe intro!: β←C denotes-ex-neg cqt:2)
    AOT-hence ‹∀ G ¬(□G ≡_E H & ¬y[G])›
      using cqt−further:4[THEN →E] by blast
    AOT-hence ‹∀ G(□G ≡_E H → y[G])›
      by (AOT-subst ‹□G ≡_E H → y[G]› ‹¬(□G ≡_E H & ¬y[G])› for: G)
        (auto simp: oth−class−taut:1:a)
    AOT-hence ‹∀ G (□∀ u([G]u ≡ [H]u) → y[G])›
      by (AOT-subst (reverse) ‹∀ u ([G]u ≡ [H]u)› ‹G ≡_E H› for: G)
        (safe intro!: eqE[THEN ≡Df, THEN ≡S(1), OF &I] cqt:2)
    AOT-hence ‹∀ G (∀ u □([G]u ≡ [H]u) → y[G])›
      using Ordinary.res−var−bound−reas[BF] Ordinary.res−var−bound−reas[CBF]
            intro−elim:2
      by (AOT-subst ‹∀ u □([G]u ≡ [H]u)› ‹□∀ u ([G]u ≡ [H]u)› for: G) auto
  } note 0 = this
  AOT-assume ‹A!x & A!y & ∀ F □([F]x ≡ [F]y)›
  AOT-hence ‹A!x› and ‹A!y› and ‹∀ F □([F]x ≡ [F]y)›
    using &E by blast+
  moreover AOT-have ‹∀ F □([F]y ≡ [F]x)›
    using calculation(3)
    apply (safe intro!: CBF[THEN →E] dest!: BF[THEN →E])
    using RM:3 cqt−basic:11 intro−elim:3:b by fast
  ultimately AOT-show ‹∀ G (∀ u □([G]u ≡ [H]u) → x[G]) ≡
                      ∀ G (∀ u □([G]u ≡ [H]u) → y[G])›
    using 0 by (auto intro!: ≡I →I)
qed

AOT-have ‹A!x & A!y & ∀ F □([F]x ≡ [F]y) →
(∃ G (∀ z (O!z → □([G]z ≡ [H]z)) & x[G]) ≡ ∃ G (∀ z (O!z → □([G]z ≡ [H]z)) & y[G]))›
proof(rule →I)
  {
    fix x y
    AOT-assume ‹A!x›
    AOT-assume ‹A!y›
    AOT-assume indist: ‹∀ F □([F]x ≡ [F]y)›
    AOT-assume x-prop: ‹∃ G (∀ u □([G]u ≡ [H]u) & x[G])›
    AOT-hence ‹∃ G (□∀ u ([G]u ≡ [H]u) & x[G])›
      using Ordinary.res−var−bound−reas[BF] Ordinary.res−var−bound−reas[CBF]
            intro−elim:2
      by (AOT-subst ‹□∀ u ([G]u ≡ [H]u)› ‹∀ u □([G]u ≡ [H]u)› for: G) auto
    AOT-hence ‹∃ G (□G ≡_E H & x[G])›
      by (AOT-subst ‹G ≡_E H› ‹∀ u ([G]u ≡ [H]u)› for: G)
        (safe intro!: eqE[THEN ≡Df, THEN ≡S(1), OF &I] cqt:2)
    AOT-hence ‹[λx ∃ G (□G ≡_E H & x[G])]x›
      by (safe intro!: β←C denotes-ex cqt:2)
```

```
      AOT-hence ‹[λx ∃ G (□G ≡E H & x[G])]y›
        using indist[THEN ∀ E(1), OF denotes-ex,
                     THEN qml:2[axiom-inst, THEN →E],
                     THEN ≡E(1)] by blast
      AOT-hence ‹∃ G (□G ≡E H & y[G])›
        by (rule β→C)
      AOT-hence ‹∃ G (□∀ u ([G]u ≡ [H]u) & y[G])›
        by (AOT-subst (reverse) ‹∀ u ([G]u ≡ [H]u)› ‹G ≡E H› for: G)
           (safe intro!: eqE[THEN ≡Df, THEN ≡S(1), OF &I] cqt:2)
      AOT-hence ‹∃ G (∀ u □([G]u ≡ [H]u) & y[G])›
        using Ordinary.res−var−bound−reas[BF]
              Ordinary.res−var−bound−reas[CBF]
              intro−elim:2
        by (AOT-subst ‹∀ u □([G]u ≡ [H]u)› ‹□∀ u ([G]u ≡ [H]u)› for: G) auto
    } note 0 = this
    AOT-assume ‹A!x & A!y & ∀ F □([F]x ≡ [F]y)›
    AOT-hence ‹A!x› and ‹A!y› and ‹∀ F □([F]x ≡ [F]y)›
      using &E by blast+
    moreover AOT-have ‹∀ F □([F]y ≡ [F]x)›
      using calculation(3)
      apply (safe intro!: CBF[THEN →E] dest!: BF[THEN →E])
      using RM:3 cqt−basic:11 intro−elim:3:b by fast
    ultimately AOT-show ‹∃ G (∀ u □([G]u ≡ [H]u) & x[G]) ≡
                         ∃ G (∀ u □([G]u ≡ [H]u) & y[G])›
      using 0 by (auto intro!: ≡I →I)
  qed
}
end
end
```

12 Possible Worlds

```
AOT-define Situation :: ‹τ ⇒ φ› (‹Situation'(-')›)
  situations: ‹Situation(x) ≡df A!x & ∀ F (x[F] → Propositional([F]))›

AOT-theorem T−sit: ‹TruthValue(x) → Situation(x)›
proof(rule →I)
  AOT-assume ‹TruthValue(x)›
  AOT-hence ‹∃ p TruthValueOf(x,p)›
    using T−value[THEN ≡df E] by blast
  then AOT-obtain p where ‹TruthValueOf(x,p)› using ∃ E[rotated] by blast
  AOT-hence ϑ: ‹A!x & ∀ F (x[F] ≡ ∃ q((q ≡ p) & F = [λy q]))›
    using tv−p[THEN ≡df E] by blast
  AOT-show ‹Situation(x)›
  proof(rule situations[THEN ≡df I]; safe intro!: &I GEN →I ϑ[THEN &E(1)])
    fix F
    AOT-assume ‹x[F]›
    AOT-hence ‹∃ q((q ≡ p) & F = [λy q])›
      using ϑ[THEN &E(2), THEN ∀ E(2)[where β=F], THEN ≡E(1)] by argo
    then AOT-obtain q where ‹(q ≡ p) & F = [λy q]› using ∃ E[rotated] by blast
    AOT-hence ‹∃ p F = [λy p]› using &E(2) ∃ I(2) by metis
    AOT-thus ‹Propositional([F])›
      by (metis ≡df I prop−prop1)
  qed
qed

AOT-theorem possit−sit:1: ‹Situation(x) ≡ □Situation(x)›
proof(rule ≡I; rule →I)
  AOT-assume ‹Situation(x)›
  AOT-hence 0: ‹A!x & ∀ F (x[F] → Propositional([F]))›
    using situations[THEN ≡df E] by blast
  AOT-have 1: ‹□(A!x & ∀ F (x[F] → Propositional([F])))›
  proof(rule KBasic:3[THEN ≡E(2)]; rule &I)
```

AOT-show ‹□A!x› **using** *0[THEN &E(1)]* **by** (*metis oa−facts:2[THEN →E]*)
next
AOT-have ‹∀ F (x[F] → Propositional([F])) → □∀ F (x[F] → Propositional([F]))›
by (*AOT-subst* ‹*Propositional*([F])› ‹∃ p (F = [λy p])› **for**: F :: ‹<κ>›)
(*auto simp*: *prop−prop1* ≡*Df enc−prop−nec*:*2*)
AOT-thus ‹□∀ F (x[F] → Propositional([F]))›
using *0[THEN &E(2)]* →*E* **by** *blast*
qed
AOT-show ‹□*Situation*(x)›
by (*AOT-subst* ‹*Situation*(x)› ‹A!x & ∀ F (x[F] → Propositional([F]))›)
(*auto simp*: *1* ≡*Df situations*)
next
AOT-show ‹*Situation*(x)› **if** ‹□*Situation*(x)›
using *qml*:*2*[*axiom-inst*, *THEN* →*E*, *OF that*].
qed

AOT-theorem *possit−sit*:*2*: ‹◊*Situation*(x) ≡ *Situation*(x)›
using *possit−sit*:*1*
by (*metis RE◊ S5Basic*:*2* ≡*E*(*1*) ≡*E*(*5*) *Commutativity of* ≡)

AOT-theorem *possit−sit*:*3*: ‹◊*Situation*(x) ≡ □*Situation*(x)›
using *possit−sit*:*1 possit−sit*:*2* **by** (*meson* ≡*E*(*5*))

AOT-theorem *possit−sit*:*4*: ‹𝓐*Situation*(x) ≡ *Situation*(x)›
by (*meson Act−Basic*:*5 Act−Sub*:*2 RA*[*2*] ≡*E*(*1*) ≡*E*(*6*) *possit−sit*:*2*)

AOT-theorem *possit−sit*:*5*: ‹*Situation*(∘p)›
proof (*safe intro*!: *situations*[*THEN* $\equiv_{df}$*I*] &*I GEN* →*I prop−prop1*[*THEN* $\equiv_{df}$*I*])
AOT-have ‹∃ F ∘p[F]›
using *tv−id*:*2*[*THEN prop−enc*[*THEN* $\equiv_{df}$*E*], *THEN* &*E*(*2*)]
existential:*1 prop−prop2*:*2* **by** *blast*
AOT-thus ‹A!∘p›
by (*safe intro*!: *encoders−are−abstract*[*unvarify x*, *THEN* →*E*]
t=*t−proper*:*2*[*THEN* →*E*, *OF ext−p−tv*:*3*])
next
fix *F*
AOT-assume ‹∘p[F]›
AOT-hence ‹ιx(A!x & ∀ F (x[F] ≡ ∃ q ((q ≡ p) & F = [λy q])))[F]›
using *tv−id*:*1 rule*=*E* **by** *fast*
AOT-hence ‹𝓐∃ q ((q ≡ p) & F = [λy q])›
using ≡*E*(*1*) *desc−nec−encode*:*1* **by** *fast*
AOT-hence ‹∃ q 𝓐((q ≡ p) & F = [λy q])›
by (*metis Act−Basic*:*10* ≡*E*(*1*))
then AOT-obtain *q* **where** ‹𝓐((q ≡ p) & F = [λy q])› **using** ∃ *E*[*rotated*] **by** *blast*
AOT-hence ‹𝓐F = [λy q]› **by** (*metis Act−Basic*:*2 con−dis−i−e*:*2*:*b intro−elim*:*3*:*a*)
AOT-hence ‹F = [λy q]›
using *id−act*:*1*[*unvarify β*, *THEN* ≡*E*(*2*)] **by** (*metis prop−prop2*:*2*)
AOT-thus ‹∃ p F = [λy p]›
using ∃ *I* **by** *fast*
qed

AOT-theorem *possit−sit*:*6*: ‹*Situation*(⊤)›
proof −
AOT-have *true-def*: ‹$\vdash_\Box$ ⊤ = ιx (A!x & ∀ F (x[F] ≡ ∃ p(p & F = [λy p])))›
by (*simp add*: *A−descriptions rule−id−df*:*1*[*zero*] *the−true*:*1*)
AOT-hence *true-den*: ‹$\vdash_\Box$ ⊤↓›
using *t*=*t−proper*:*1 vdash−properties*:*6* **by** *blast*
AOT-have ‹𝓐*TruthValue*(⊤)›
using *actual−desc*:*2*[*unvarify x*, *OF true-den*, *THEN* →*E*, *OF true-def*]
using *TV−lem2*:*1*[*unvarify x*, *OF true-den*, *THEN RA*[*2*],
THEN act−cond[*THEN* →*E*], *THEN* →*E*]
by *blast*
AOT-hence ‹𝓐*Situation*(⊤)›

using *T−sit[unvarify x, OF true-den, THEN RA[2],*
THEN act−cond[THEN →E], THEN →E] **by** *blast*
AOT-thus ‹*Situation*(⊤)›
using *possit−sit:4[unvarify x, OF true-den, THEN ≡E(1)]* **by** *blast*
qed

AOT-theorem *possit−sit:7*: ‹*Situation*(⊥)›
proof −
AOT-have *true-def*: ‹$\vdash_\Box$ ⊥ = ιx (A!x & ∀ F (x[F] ≡ ∃ p(¬p & F = [λy p]))))›
by (*simp add: A−descriptions rule−id−df:1[zero] the−true:2*)
AOT-hence *true-den*: ‹$\vdash_\Box$ ⊥↓›
using *t=t−proper:1 vdash−properties:6* **by** *blast*
AOT-have ‹**𝓐***TruthValue*(⊥)›
using *actual−desc:2[unvarify x, OF true-den, THEN →E, OF true-def]*
using *TV−lem2:2[unvarify x, OF true-den, THEN RA[2],*
THEN act−cond[THEN →E], THEN →E]
by *blast*
AOT-hence ‹**𝓐***Situation*(⊥)›
using *T−sit[unvarify x, OF true-den, THEN RA[2],*
THEN act−cond[THEN →E], THEN →E] **by** *blast*
AOT-thus ‹*Situation*(⊥)›
using *possit−sit:4[unvarify x, OF true-den, THEN ≡E(1)]* **by** *blast*
qed

AOT-register-rigid-restricted-type
Situation: ‹*Situation*(κ)›
proof
AOT-modally-strict {
fix *p*
AOT-obtain *x* **where** ‹*TruthValueOf(x,p)*›
by (*metis instantiation p−has−!tv:1*)
AOT-hence ‹∃ *p TruthValueOf(x,p)*› **by** (*rule* ∃ *I*)
AOT-hence ‹*TruthValue(x)*› **by** (*metis* $\equiv_{df}$*I T−value*)
AOT-hence ‹*Situation(x)*› **using** *T−sit[THEN →E]* **by** *blast*
AOT-thus ‹∃ *x Situation(x)*› **by** (*rule* ∃ *I*)
}
next
AOT-modally-strict {
AOT-show ‹*Situation*(κ) → κ↓› **for** κ
proof (*rule →I*)
AOT-assume ‹*Situation*(κ)›
AOT-hence ‹*A*!κ› **by** (*metis* $\equiv_{df}$*E &E(1) situations*)
AOT-thus ‹κ↓› **by** (*metis russell−axiom[exe,1].ψ-denotes-asm*)
qed
}
next
AOT-modally-strict {
AOT-show ‹∀ α(*Situation*(α) → □*Situation*(α))›
using *possit−sit:1[THEN conventions:3[THEN* $\equiv_{df}$*E],*
THEN &E(1)] GEN **by** *fast*
}
qed

AOT-register-variable-names
Situation: *s*

AOT-define *TruthInSituation* :: ‹τ ⇒ φ ⇒ φ› (‹(- ⊨/ -)› *[100, 40] 100*)
true−in−s: ‹*s* ⊨ *p* $\equiv_{df}$ *s***Σ***p*›

notepad
begin

fix *x p q*

```
  have ‹«x ⊨ p → q» = «(x ⊨ p) → q»›
    by simp
  have ‹«x ⊨ p & q» = «(x ⊨ p) & q»›
    by simp
  have ‹«x ⊨ ¬p» = «x ⊨ (¬p)»›
    by simp
  have ‹«x ⊨ □p» = «x ⊨ (□p)»›
    by simp
  have ‹«x ⊨ 𝓐p» = «x ⊨ (𝓐p)»›
    by simp
  have ‹«□x ⊨ p» = «□(x ⊨ p)»›
    by simp
  have ‹«¬x ⊨ p» = «¬(x ⊨ p)»›
    by simp
end


AOT-theorem lem1: ‹Situation(x) → (x ⊨ p ≡ x[λy p])›
proof (rule →I; rule ≡I; rule →I)
  AOT-assume ‹Situation(x)›
  AOT-assume ‹x ⊨ p›
  AOT-hence ‹xΣp›
    using true−in−s[THEN ≡df E] &E by blast
  AOT-thus ‹x[λy p]› using prop−enc[THEN ≡df E] &E by blast
next
  AOT-assume 1: ‹Situation(x)›
  AOT-assume ‹x[λy p]›
  AOT-hence ‹xΣp›
    using prop−enc[THEN ≡df I, OF &I, OF cqt:2(1)] by blast
  AOT-thus ‹x ⊨ p›
    using true−in−s[THEN ≡df I] 1 &I by blast
qed

AOT-theorem lem2:1: ‹s ⊨ p ≡ □s ⊨ p›
proof −
  AOT-have sit: ‹Situation(s)›
    by (simp add: Situation.ψ)
  AOT-have ‹s ⊨ p ≡ s[λy p]›
    using lem1[THEN →E, OF sit] by blast
  also AOT-have ‹... ≡ □s[λy p]›
    by (rule en−eq:2[1][unvarify F]) cqt:2[lambda]
  also AOT-have ‹... ≡ □s ⊨ p›
    using lem1[THEN RM, THEN →E, OF possit−sit:1[THEN ≡E(1), OF sit]]
    by (metis KBasic:6 ≡E(2) Commutativity of ≡ →E)
  finally show ?thesis.
qed

AOT-theorem lem2:2: ‹◇s ⊨ p ≡ s ⊨ p›
proof −
  AOT-have ‹□(s ⊨ p → □s ⊨ p)›
    using possit−sit:1[THEN ≡E(1), OF Situation.ψ]
          lem2:1[THEN conventions:3[THEN ≡df E, THEN &E(1)]]
          RM[OF →I, THEN →E] by blast
  thus ?thesis by (metis B◇ S5Basic:13 T◇ ≡I ≡E(1) →E)
qed

AOT-theorem lem2:3: ‹◇s ⊨ p ≡ □s ⊨ p›
  using lem2:1 lem2:2 by (metis ≡E(5))

AOT-theorem lem2:4: ‹𝓐(s ⊨ p) ≡ s ⊨ p›
proof −
  AOT-have ‹□(s ⊨ p → □s ⊨ p)›
    using possit−sit:1[THEN ≡E(1), OF Situation.ψ]
```

```
      lem2:1[THEN conventions:3[THEN ≡df E, THEN &E(1)]]
      RM[OF →I, THEN →E] by blast
  thus ?thesis
    using sc−eq−fur:2[THEN →E] by blast
qed

AOT-theorem lem2:5: ‹¬s ⊨ p ≡ □¬s ⊨ p›
  by (metis KBasic2:1 contraposition:1[2] →I ≡I ≡E(3) ≡E(4) lem2:2)

AOT-theorem sit−identity: ‹s = s′ ≡ ∀ p(s ⊨ p ≡ s′ ⊨ p)›
proof(rule ≡I; rule →I)
  AOT-assume ‹s = s′›
  moreover AOT-have ‹∀ p(s ⊨ p ≡ s ⊨ p)›
    by (simp add: oth−class−taut:3:a universal−cor)
  ultimately AOT-show ‹∀ p(s ⊨ p ≡ s′ ⊨ p)›
    using rule=E by fast
next
  AOT-assume a: ‹∀ p (s ⊨ p ≡ s′ ⊨ p)›
  AOT-show ‹s = s′›
  proof(safe intro!: ab−obey:1[THEN →E, THEN →E] &I GEN ≡I →I)
    AOT-show ‹A!s› using Situation.ψ ≡df E &E(1) situations by blast
  next
    AOT-show ‹A!s′› using Situation.ψ ≡df E &E(1) situations by blast
  next
    fix F
    AOT-assume 0: ‹s[F]›
    AOT-hence ‹∃ p (F = [λy p])›
      using Situation.ψ[THEN situations[THEN ≡df E], THEN &E(2),
                        THEN ∀ E(2)[where β=F], THEN →E]
          prop−prop1[THEN ≡df E] by blast
    then AOT-obtain p where F-def: ‹F = [λy p]›
      using ∃ E by metis
    AOT-hence ‹s[λy p]›
      using 0 rule=E by blast
    AOT-hence ‹s ⊨ p›
      using lem1[THEN →E, OF Situation.ψ, THEN ≡E(2)] by blast
    AOT-hence ‹s′ ⊨ p›
      using a[THEN ∀ E(2)[where β=p], THEN ≡E(1)] by blast
    AOT-hence ‹s′[λy p]›
      using lem1[THEN →E, OF Situation.ψ, THEN ≡E(1)] by blast
    AOT-thus ‹s′[F]›
      using F-def[symmetric] rule=E by blast
  next
    fix F
    AOT-assume 0: ‹s′[F]›
    AOT-hence ‹∃ p (F = [λy p])›
      using Situation.ψ[THEN situations[THEN ≡df E], THEN &E(2),
                        THEN ∀ E(2)[where β=F], THEN →E]
          prop−prop1[THEN ≡df E] by blast
    then AOT-obtain p where F-def: ‹F = [λy p]›
      using ∃ E by metis
    AOT-hence ‹s′[λy p]›
      using 0 rule=E by blast
    AOT-hence ‹s′ ⊨ p›
      using lem1[THEN →E, OF Situation.ψ, THEN ≡E(2)] by blast
    AOT-hence ‹s ⊨ p›
      using a[THEN ∀ E(2)[where β=p], THEN ≡E(2)] by blast
    AOT-hence ‹s[λy p]›
      using lem1[THEN →E, OF Situation.ψ, THEN ≡E(1)] by blast
    AOT-thus ‹s[F]›
      using F-def[symmetric] rule=E by blast
  qed
qed
```

AOT-define *PartOfSituation* :: ‹$\tau \Rightarrow \tau \Rightarrow \varphi$› (**infixl** ‹⊴› *80*)
sit−part−whole: ‹$s \trianglelefteq s' \equiv_{df} \forall p\ (s \models p \rightarrow s' \models p)$›

AOT-theorem *part*:*1*: ‹$s \trianglelefteq s$›
by (*rule sit−part−whole*[*THEN* $\equiv_{df}I$])
(*safe intro*!: &*I Situation*.ψ *GEN* →*I*)

AOT-theorem *part*:*2*: ‹$s \trianglelefteq s'$ & $s \neq s' \rightarrow \neg(s' \trianglelefteq s)$›
proof(*rule* →*I*; *frule* &*E*(*1*); *drule* &*E*(*2*); *rule raa−cor*:*2*)
AOT-assume *0*: ‹$s \trianglelefteq s'$›
AOT-hence *a*: ‹$s \models p \rightarrow s' \models p$› **for** *p*
using ∀*E*(*2*) *sit−part−whole*[*THEN* $\equiv_{df}E$] &*E* **by** *blast*
AOT-assume ‹$s' \trianglelefteq s$›
AOT-hence *b*: ‹$s' \models p \rightarrow s \models p$› **for** *p*
using ∀*E*(*2*) *sit−part−whole*[*THEN* $\equiv_{df}E$] &*E* **by** *blast*
AOT-have ‹$\forall p\ (s \models p \equiv s' \models p)$›
using *a b* **by** (*simp add*: ≡*I universal−cor*)
AOT-hence *1*: ‹$s = s'$›
using *sit−identity*[*THEN* ≡*E*(*2*)] **by** *metis*
AOT-assume ‹$s \neq s'$›
AOT-hence ‹$\neg(s = s')$›
by (*metis* $\equiv_{df}E$ =−*infix*)
AOT-thus ‹$s = s'$ & $\neg(s = s')$›
using *1* &*I* **by** *blast*
qed

AOT-theorem *part*:*3*: ‹$s \trianglelefteq s'$ & $s' \trianglelefteq s'' \rightarrow s \trianglelefteq s''$›
proof(*rule* →*I*; *frule* &*E*(*1*); *drule* &*E*(*2*);
safe intro!: &*I GEN* →*I sit−part−whole*[*THEN* $\equiv_{df}I$] *Situation*.ψ)
fix *p*
AOT-assume ‹$s \models p$›
moreover AOT-assume ‹$s \trianglelefteq s'$›
ultimately AOT-have ‹$s' \models p$›
using *sit−part−whole*[*THEN* $\equiv_{df}E$, *THEN* &*E*(*2*),
THEN ∀*E*(*2*)[**where** β=*p*], *THEN* →*E*] **by** *blast*
moreover AOT-assume ‹$s' \trianglelefteq s''$›
ultimately AOT-show ‹$s'' \models p$›
using *sit−part−whole*[*THEN* $\equiv_{df}E$, *THEN* &*E*(*2*),
THEN ∀*E*(*2*)[**where** β=*p*], *THEN* →*E*] **by** *blast*
qed

AOT-theorem *sit−identity2*:*1*: ‹$s = s' \equiv s \trianglelefteq s'$ & $s' \trianglelefteq s$›
proof (*safe intro*!: ≡*I* &*I* →*I*)
AOT-show ‹$s \trianglelefteq s'$› **if** ‹$s = s'$›
using *rule*=*E part*:*1 that* **by** *blast*
next
AOT-show ‹$s' \trianglelefteq s$› **if** ‹$s = s'$›
using *rule*=*E part*:*1 that*[*symmetric*] **by** *blast*
next
AOT-assume ‹$s \trianglelefteq s'$ & $s' \trianglelefteq s$›
AOT-thus ‹$s = s'$› **using** *part*:*2*[*THEN* →*E*, *OF* &*I*]
by (*metis* $\equiv_{df}I$ &*E*(*1*) &*E*(*2*) =−*infix raa−cor*:*3*)
qed

AOT-theorem *sit−identity2*:*2*: ‹$s = s' \equiv \forall s''\ (s'' \trianglelefteq s \equiv s'' \trianglelefteq s')$›
proof(*safe intro*!: ≡*I* →*I Situation*.*GEN sit−identity*[*THEN* ≡*E*(*2*)]
GEN[**where** ′*a*=o])
AOT-show ‹$s'' \trianglelefteq s'$› **if** ‹$s'' \trianglelefteq s$› **and** ‹$s = s'$› **for** s''
using *rule*=*E that* **by** *blast*
next
AOT-show ‹$s'' \trianglelefteq s$› **if** ‹$s'' \trianglelefteq s'$› **and** ‹$s = s'$› **for** s''
using *rule*=*E id-sym that* **by** *blast*

```
next
  AOT-show ‹s' ⊨ p› if ‹s ⊨ p› and ‹∀ s'' (s'' ⊴ s ≡ s'' ⊴ s')› for p
    using sit-part-whole[THEN ≡df E, THEN &E(2),
              OF that(2)[THEN Situation.∀ E, THEN ≡E(1), OF part:1],
              THEN ∀ E(2), THEN →E, OF that(1)].
next
  AOT-show ‹s ⊨ p› if ‹s' ⊨ p› and ‹∀ s'' (s'' ⊴ s ≡ s'' ⊴ s')› for p
    using sit-part-whole[THEN ≡df E, THEN &E(2),
          OF that(2)[THEN Situation.∀ E, THEN ≡E(2), OF part:1],
          THEN ∀ E(2), THEN →E, OF that(1)].
qed

AOT-define Persistent :: ‹φ ⇒ φ› (‹Persistent'(-')›)
  persistent: ‹Persistent(p) ≡df ∀ s (s ⊨ p → ∀ s' (s ⊴ s' → s' ⊨ p))›

AOT-theorem pers-prop: ‹∀ p Persistent(p)›
  by (safe intro!: GEN[where 'a=o] Situation.GEN persistent[THEN ≡df I] →I)
     (simp add: sit-part-whole[THEN ≡df E, THEN &E(2), THEN ∀ E(2), THEN →E])

AOT-define NullSituation :: ‹τ ⇒ φ› (‹NullSituation'(-')›)
  df-null-trivial:1: ‹NullSituation(s) ≡df ¬∃ p s ⊨ p›

AOT-define TrivialSituation :: ‹τ ⇒ φ› (‹TrivialSituation'(-')›)
  df-null-trivial:2: ‹TrivialSituation(s) ≡df ∀ p s ⊨ p›

AOT-theorem thm-null-trivial:1: ‹∃ !x NullSituation(x)›
proof (AOT-subst ‹NullSituation(x)› ‹A!x & ∀ F (x[F] ≡ F ≠ F)› for: x)
  AOT-modally-strict {
    AOT-show ‹NullSituation(x) ≡ A!x & ∀ F (x[F] ≡ F ≠ F)› for x
    proof (safe intro!: ≡I →I df-null-trivial:1[THEN ≡df I]
                dest!: df-null-trivial:1[THEN ≡df E])
      AOT-assume 0: ‹Situation(x) & ¬∃ p x ⊨ p›
      AOT-have 1: ‹A!x›
        using 0[THEN &E(1), THEN situations[THEN ≡df E], THEN &E(1)].
      AOT-have 2: ‹x[F] → ∃ p F = [λy p]› for F
        using 0[THEN &E(1), THEN situations[THEN ≡df E],
            THEN &E(2), THEN ∀ E(2)]
        by (metis ≡df E →I prop-prop1 →E)
      AOT-show ‹A!x & ∀ F (x[F] ≡ F ≠ F)›
      proof (safe intro!: &I 1 GEN ≡I →I)
        fix F
        AOT-assume ‹x[F]›
        moreover AOT-obtain p where ‹F = [λy p]›
          using calculation 2[THEN →E] ∃ E[rotated] by blast
        ultimately AOT-have ‹x[λy p]›
          by (metis rule=E)
        AOT-hence ‹x ⊨ p›
          using lem1[THEN →E, OF 0[THEN &E(1)], THEN ≡E(2)] by blast
        AOT-hence ‹∃ p (x ⊨ p)›
          by (rule ∃ I)
        AOT-thus ‹F ≠ F›
          using 0[THEN &E(2)] raa-cor:1 &I by blast
      next
        fix F :: ‹<κ> AOT-var›
        AOT-assume ‹F ≠ F›
        AOT-hence ‹¬(F = F)› by (metis ≡df E =-infix)
        moreover AOT-have ‹F = F›
          by (simp add: id-eq:1)
        ultimately AOT-show ‹x[F]› using &I raa-cor:1 by blast
      qed
    next
      AOT-assume 0: ‹A!x & ∀ F (x[F] ≡ F ≠ F)›
      AOT-hence ‹x[F] ≡ F ≠ F› for F
```

```
        using ∀E &E by blast
      AOT-hence 1: ‹¬x[F]› for F
        using ≡df E id−eq:1 =−infix reductio−aa:1 ≡E(1) by blast
      AOT-show ‹Situation(x) & ¬∃p x ⊨ p›
      proof (safe intro!: &I situations[THEN ≡df I] 0[THEN &E(1)] GEN →I)
        AOT-show ‹Propositional([F])› if ‹x[F]› for F
          using that 1 &I raa−cor:1 by fast
      next
        AOT-show ‹¬∃p x ⊨ p›
        proof(rule raa−cor:2)
          AOT-assume ‹∃p x ⊨ p›
          then AOT-obtain p where ‹x ⊨ p› using ∃E[rotated] by blast
          AOT-hence ‹x[λy p]›
            using ≡df E &E(1) ≡E(1) lem1 modus−tollens:1
                  raa−cor:3 true−in−s by fast
          moreover AOT-have ‹¬x[λy p]›
            by (rule 1[unvarify F]) cqt:2[lambda]
          ultimately AOT-show ‹p & ¬p› for p using &I raa−cor:1 by blast
        qed
      qed
    qed
  }
next
  AOT-show ‹∃!x ([A!]x & ∀F (x[F] ≡ F ≠ F))›
    by (simp add: A−objects!)
qed


AOT-theorem thm−null−trivial:2: ‹∃!x TrivialSituation(x)›
proof (AOT-subst ‹TrivialSituation(x)› ‹A!x & ∀F (x[F] ≡ ∃p F = [λy p])› for: x)
  AOT-modally-strict {
    AOT-show ‹TrivialSituation(x) ≡ A!x & ∀F (x[F] ≡ ∃p F = [λy p])› for x
    proof (safe intro!: ≡I →I df−null−trivial:2[THEN ≡df I]
                dest!: df−null−trivial:2[THEN ≡df E])
      AOT-assume 0: ‹Situation(x) & ∀p x ⊨ p›
      AOT-have 1: ‹A!x›
        using 0[THEN &E(1), THEN situations[THEN ≡df E], THEN &E(1)].
      AOT-have 2: ‹x[F] → ∃p F = [λy p]› for F
        using 0[THEN &E(1), THEN situations[THEN ≡df E],
              THEN &E(2), THEN ∀E(2)]
        by (metis ≡df E deduction−theorem prop−prop1 →E)
      AOT-show ‹A!x & ∀F (x[F] ≡ ∃p F = [λy p])›
      proof (safe intro!: &I 1 GEN ≡I →I 2)
        fix F
        AOT-assume ‹∃p F = [λy p]›
        then AOT-obtain p where ‹F = [λy p]›
          using ∃E[rotated] by blast
        moreover AOT-have ‹x ⊨ p›
          using 0[THEN &E(2)] ∀E by blast
        ultimately AOT-show ‹x[F]›
          by (metis 0 rule=E &E(1) id-sym ≡E(2) lem1
                  Commutativity of ≡ →E)
      qed
    next
      AOT-assume 0: ‹A!x & ∀F (x[F] ≡ ∃p F = [λy p])›
      AOT-hence 1: ‹x[F] ≡ ∃p F = [λy p]› for F
        using ∀E &E by blast
      AOT-have 2: ‹Situation(x)›
      proof (safe intro!: &I situations[THEN ≡df I] 0[THEN &E(1)] GEN →I)
        AOT-show ‹Propositional([F])› if ‹x[F]› for F
          using 1[THEN ≡E(1), OF that]
          by (metis ≡df I prop−prop1)
      qed
```

```
      AOT-show ‹Situation(x) & ∀p (x ⊨ p)›
      proof (safe intro!: &I 2 0[THEN &E(1)] GEN →I)
        AOT-have ‹x[λy p] ≡ ∃q [λy p] = [λy q]› for p
          by (rule 1[unvarify F, where τ=«[λy p]»]) cqt:2[lambda]
        moreover AOT-have ‹∃q [λy p] = [λy q]› for p
          by (rule ∃I(2)[where β=p])
             (simp add: rule=I:1 prop−prop2:2)
        ultimately AOT-have ‹x[λy p]› for p by (metis ≡E(2))
        AOT-thus ‹x ⊨ p› for p
          by (metis 2 ≡E(2) lem1 →E)
      qed
    qed
  }
next
  AOT-show ‹∃!x ([A!]x & ∀F (x[F] ≡ ∃p F = [λy p]))›
    by (simp add: A−objects!)
qed

AOT-theorem thm−null−trivial:3: ‹ιx NullSituation(x)↓›
  by (meson A−Exists:2 RA[2] ≡E(2) thm−null−trivial:1)

AOT-theorem thm−null−trivial:4: ‹ιx TrivialSituation(x)↓›
  using A−Exists:2 RA[2] ≡E(2) thm−null−trivial:2 by blast

AOT-define TheNullSituation :: ‹κ_s› (‹s_∅›)
  df−the−null−sit:1: ‹s_∅ =_df ιx NullSituation(x)›

AOT-define TheTrivialSituation :: ‹κ_s› (‹s_V›)
  df−the−null−sit:2: ‹s_V =_df ιx TrivialSituation(x)›

AOT-theorem null−triv−sc:1: ‹NullSituation(x) → □NullSituation(x)›
proof(safe intro!: →I dest!: df−null−trivial:1[THEN ≡_df E];
      frule &E(1); drule &E(2))
  AOT-assume 1: ‹¬∃p (x ⊨ p)›
  AOT-assume 0: ‹Situation(x)›
  AOT-hence ‹□Situation(x)› by (metis ≡E(1) possit−sit:1)
  moreover AOT-have ‹□¬∃p (x ⊨ p)›
  proof(rule raa−cor:1)
    AOT-assume ‹¬□¬∃p (x ⊨ p)›
    AOT-hence ‹◇∃p (x ⊨ p)›
      by (metis ≡_df I conventions:5)
    AOT-hence ‹∃p ◇(x ⊨ p)› by (metis BF◇ →E)
    then AOT-obtain p where ‹◇(x ⊨ p)› using ∃E[rotated] by blast
    AOT-hence ‹x ⊨ p›
      by (metis ≡E(1) lem2:2[unconstrain s, THEN →E, OF 0])
    AOT-hence ‹∃p x ⊨ p› using ∃I by fast
    AOT-thus ‹∃p x ⊨ p & ¬∃p x ⊨ p› using 1 &I by blast
  qed
  ultimately AOT-have 2: ‹□(Situation(x) & ¬∃p x ⊨ p)›
    by (metis KBasic:3 &I ≡E(2))
  AOT-show ‹□NullSituation(x)›
    by (AOT-subst ‹NullSituation(x)› ‹Situation(x) & ¬∃p x ⊨ p›)
       (auto simp: df−null−trivial:1 ≡Df 2)
qed


AOT-theorem null−triv−sc:2: ‹TrivialSituation(x) → □TrivialSituation(x)›
proof(safe intro!: →I dest!: df−null−trivial:2[THEN ≡_df E];
      frule &E(1); drule &E(2))
  AOT-assume 0: ‹Situation(x)›
  AOT-hence 1: ‹□Situation(x)› by (metis ≡E(1) possit−sit:1)
  AOT-assume ‹∀p x ⊨ p›
  AOT-hence ‹x ⊨ p› for p
```

using $\forall E$ **by** *blast*
AOT-hence ‹$\Box x \models p$› **for** p
using *0* $\equiv E(1)$ *lem2*:*1*[*unconstrain s*, *THEN* $\rightarrow E$] **by** *blast*
AOT-hence ‹$\forall p\ \Box x \models p$›
by (*rule GEN*)
AOT-hence ‹$\Box\forall p\ x \models p$›
by (*rule BF*[*THEN* $\rightarrow E$])
AOT-hence *2*: ‹$\Box$(*Situation*(x) & $\forall p\ x \models p$)›
using *1* **by** (*metis KBasic*:*3* &*I* $\equiv E(2)$)
AOT-show ‹$\Box$ *TrivialSituation*(x)›
by (*AOT-subst* ‹*TrivialSituation*(x)› ‹*Situation*(x) & $\forall p\ x \models p$›)
(*auto simp*: *df−null−trivial*:*2* $\equiv$*Df 2*)
qed

AOT-theorem *null−triv−sc*:*3*: ‹*NullSituation*($\mathbf{s}_\emptyset$)›
by (*safe intro*!: *df−the−null−sit*:*1*[*THEN* $=_{df}I(2)$] *thm−null−trivial*:*3*
rule=I:*1*[*OF thm−null−trivial*:*3*]
!*box−desc*:*2*[*THEN* $\rightarrow E$, *THEN* $\rightarrow E$, *rotated*, *OF thm−null−trivial*:*1*,
OF $\forall I$, *OF null−triv−sc*:*1*, *THEN* $\forall E(1)$, *THEN* $\rightarrow E$])

AOT-theorem *null−triv−sc*:*4*: ‹*TrivialSituation*($\mathbf{s}_V$)›
by (*safe intro*!: *df−the−null−sit*:*2*[*THEN* $=_{df}I(2)$] *thm−null−trivial*:*4*
rule=I:*1*[*OF thm−null−trivial*:*4*]
!*box−desc*:*2*[*THEN* $\rightarrow E$, *THEN* $\rightarrow E$, *rotated*, *OF thm−null−trivial*:*2*,
OF $\forall I$, *OF null−triv−sc*:*2*, *THEN* $\forall E(1)$, *THEN* $\rightarrow E$])

AOT-theorem *null−triv−facts*:*1*: ‹*NullSituation*(x) $\equiv$ *Null*(x)›
proof (*safe intro*!: $\equiv I$ $\rightarrow I$ *df−null−uni*:*1*[*THEN* $\equiv_{df}I$]
df−null−trivial:*1*[*THEN* $\equiv_{df}I$]
dest!: *df−null−uni*:*1*[*THEN* $\equiv_{df}E$] *df−null−trivial*:*1*[*THEN* $\equiv_{df}E$])
AOT-assume *0*: ‹*Situation*(x) & $\neg\exists p\ x \models p$›
AOT-have *1*: ‹$x[F] \rightarrow \exists p\ F = [\lambda y\ p]$› **for** F
using *0*[*THEN* &$E(1)$, *THEN situations*[*THEN* $\equiv_{df}E$], *THEN* &$E(2)$, *THEN* $\forall E(2)$]
by (*metis* $\equiv_{df}E$ *deduction−theorem prop−prop1* $\rightarrow E$)
AOT-show ‹$A!x$ & $\neg\exists F\ x[F]$›
proof (*safe intro*!: &*I* *0*[*THEN* &$E(1)$, *THEN situations*[*THEN* $\equiv_{df}E$],
THEN &$E(1)$];
rule raa−cor:*2*)
AOT-assume ‹$\exists F\ x[F]$›
then AOT-obtain F **where** *F-prop*: ‹$x[F]$›
using $\exists E$[*rotated*] **by** *blast*
AOT-hence ‹$\exists p\ F = [\lambda y\ p]$›
using *1*[*THEN* $\rightarrow E$] **by** *blast*
then AOT-obtain p **where** ‹$F = [\lambda y\ p]$›
using $\exists E$[*rotated*] **by** *blast*
AOT-hence ‹$x[\lambda y\ p]$›
by (*metis rule=E F-prop*)
AOT-hence ‹$x \models p$›
using *lem1*[*THEN* $\rightarrow E$, *OF* *0*[*THEN* &$E(1)$], *THEN* $\equiv E(2)$] **by** *blast*
AOT-hence ‹$\exists p\ x \models p$›
by (*rule* $\exists I$)
AOT-thus ‹$\exists p\ x \models p$ & $\neg\exists p\ x \models p$›
using *0*[*THEN* &$E(2)$] &*I* **by** *blast*
qed
next
AOT-assume *0*: ‹$A!x$ & $\neg\exists F\ x[F]$›
AOT-have ‹*Situation*(x)›
apply (*rule situations*[*THEN* $\equiv_{df}I$, *OF* &*I*, *OF* *0*[*THEN* &$E(1)$]]; *rule GEN*)
using *0*[*THEN* &$E(2)$] **by** (*metis* $\rightarrow I$ *existential*:*2*[*const-var*] *raa−cor*:*3*)
moreover AOT-have ‹$\neg\exists p\ x \models p$›
proof (*rule raa−cor*:*2*)
AOT-assume ‹$\exists p\ x \models p$›
then AOT-obtain p **where** ‹$x \models p$› **by** (*metis instantiation*)

```
    AOT-hence ‹x[λy p]› by (metis ≡df E &E(2) prop−enc true−in−s)
    AOT-hence ‹∃ F x[F]› by (rule ∃ I) cqt:2[lambda]
    AOT-thus ‹∃ F x[F] & ¬∃ F x[F]› using 0[THEN &E(2)] &I by blast
  qed
  ultimately AOT-show ‹Situation(x) & ¬∃ p x ⊨ p› using &I by blast
qed

AOT-theorem null−triv−facts:2: ‹s∅ = a∅›
  apply (rule =df I(2)[OF df−the−null−sit:1])
   apply (fact thm−null−trivial:3)
  apply (rule =df I(2)[OF df−null−uni−terms:1])
   apply (fact null−uni−uniq:3)
  apply (rule equiv−desc−eq:3[THEN →E])
  apply (rule &I)
   apply (fact thm−null−trivial:3)
  by (rule RN; rule GEN; rule null−triv−facts:1)

AOT-theorem null−triv−facts:3: ‹sV ≠ aV›
proof(rule =−infix[THEN ≡df I])
  AOT-have ‹Universal(aV)›
    by (simp add: null−uni−facts:4)
  AOT-hence 0: ‹aV[A!]›
    using df−null−uni:2[THEN ≡df E] &E ∀ E(1)
    by (metis cqt:5:a vdash−properties:10 vdash−properties:1[2])
  moreover AOT-have 1: ‹¬sV[A!]›
  proof(rule raa−cor:2)
    AOT-have ‹Situation(sV)›
      using ≡df E &E(1) df−null−trivial:2 null−triv−sc:4 by blast
    AOT-hence ‹∀ F (sV[F] → Propositional([F]))›
      by (metis ≡df E &E(2) situations)
    moreover AOT-assume ‹sV[A!]›
    ultimately AOT-have ‹Propositional(A!)›
      using ∀ E(1)[rotated, OF oa−exist:2] →E by blast
    AOT-thus ‹Propositional(A!) & ¬Propositional(A!)›
      using prop−in−f:4:d &I by blast
  qed
  AOT-show ‹¬(sV = aV)›
  proof (rule raa−cor:2)
    AOT-assume ‹sV = aV›
    AOT-hence ‹sV[A!]› using 0 rule=E id-sym by fast
    AOT-thus ‹sV[A!] & ¬sV[A!]› using 1 &I by blast
  qed
qed

definition ConditionOnPropositionalProperties :: ‹(<κ> ⇒ o) ⇒ bool› where
  cond−prop: ‹ConditionOnPropositionalProperties ≡ λ φ . ∀ v .
                    [v ⊨ ∀ F (φ{F} → Propositional([F]))]›

syntax ConditionOnPropositionalProperties :: ‹id-position ⇒ AOT-prop›
  (‹CONDITION'-ON'-PROPOSITIONAL'-PROPERTIES'(-')›)

AOT-theorem cond−prop[E]:
  assumes ‹CONDITION-ON-PROPOSITIONAL-PROPERTIES(φ)›
  shows ‹∀ F (φ{F} → Propositional([F]))›
  using assms[unfolded cond−prop] by auto

AOT-theorem cond−prop[I]:
  assumes ‹⊢□ ∀ F (φ{F} → Propositional([F]))›
  shows ‹CONDITION-ON-PROPOSITIONAL-PROPERTIES(φ)›
  using assms cond−prop by metis

AOT-theorem pre−comp−sit:
  assumes ‹CONDITION-ON-PROPOSITIONAL-PROPERTIES(φ)›
```

shows ‹(*Situation*(*x*) & ∀ *F* (*x*[*F*] ≡ φ{*F*})) ≡ (*A*!*x* & ∀ *F* (*x*[*F*] ≡ φ{*F*}))›
proof(*rule* ≡*I*; *rule* →*I*)
AOT-assume ‹*Situation*(*x*) & ∀ *F* (*x*[*F*] ≡ φ{*F*})›
AOT-thus ‹*A*!*x* & ∀ *F* (*x*[*F*] ≡ φ{*F*})›
using &*E* *situations*[*THEN* $\equiv_{df}$*E*] &*I* **by** *blast*
next
AOT-assume *0*: ‹*A*!*x* & ∀ *F* (*x*[*F*] ≡ φ{*F*})›
AOT-show ‹*Situation*(*x*) & ∀ *F* (*x*[*F*] ≡ φ{*F*})›
proof (*safe intro*!: *situations*[*THEN* $\equiv_{df}$*I*] &*I*)
AOT-show ‹*A*!*x*› **using** *0*[*THEN* &*E*(*1*)]**.**
next
AOT-show ‹∀ *F* (*x*[*F*] → *Propositional*([*F*]))›
proof(*rule GEN*; *rule* →*I*)
fix *F*
AOT-assume ‹*x*[*F*]›
AOT-hence ‹φ{*F*}›
using *0*[*THEN* &*E*(*2*)] ∀ *E* ≡*E* **by** *blast*
AOT-thus ‹*Propositional*([*F*])›
using *cond−prop*[*E*][*OF assms*] ∀ *E* →*E* **by** *blast*
qed
next
AOT-show ‹∀ *F* (*x*[*F*] ≡ φ{*F*})› **using** *0* &*E* **by** *blast*
qed
qed

AOT-theorem *comp−sit*:*1*:
assumes ‹*CONDITION-ON-PROPOSITIONAL-PROPERTIES*(φ)›
shows ‹∃ *s* ∀ *F*(*s*[*F*] ≡ φ{*F*})›
by (*AOT-subst* ‹*Situation*(*x*) & ∀ *F*(*x*[*F*] ≡ φ{*F*})› ‹*A*!*x* & ∀ *F* (*x*[*F*] ≡ φ{*F*})› **for**: *x*)
(*auto simp*: *pre−comp−sit*[*OF assms*] *A−objects*[**where** φ=φ, *axiom-inst*])

AOT-theorem *comp−sit*:*2*:
assumes ‹*CONDITION-ON-PROPOSITIONAL-PROPERTIES*(φ)›
shows ‹∃ !*s* ∀ *F*(*s*[*F*] ≡ φ{*F*})›
by (*AOT-subst* ‹*Situation*(*x*) & ∀ *F*(*x*[*F*] ≡ φ{*F*})› ‹*A*!*x* & ∀ *F* (*x*[*F*] ≡ φ{*F*})› **for**: *x*)
(*auto simp*: *assms pre−comp−sit pre−comp−sit*[*OF assms*] *A−objects*!)

AOT-theorem *can−sit−desc*:*1*:
assumes ‹*CONDITION-ON-PROPOSITIONAL-PROPERTIES*(φ)›
shows ‹ι*s*(∀ *F* (*s*[*F*] ≡ φ{*F*}))↓›
using *comp−sit*:*2*[*OF assms*] *A−Exists*:*2* *RA*[*2*] ≡*E*(*2*) **by** *blast*

AOT-theorem *can−sit−desc*:*2*:
assumes ‹*CONDITION-ON-PROPOSITIONAL-PROPERTIES*(φ)›
shows ‹ι*s*(∀ *F* (*s*[*F*] ≡ φ{*F*})) = ι*x*(*A*!*x* & ∀ *F* (*x*[*F*] ≡ φ{*F*}))›
by (*auto intro*!: *equiv−desc−eq*:*2*[*THEN* →*E*, *OF* &*I*,
OF can−sit−desc:*1*[*OF assms*]]
RA[*2*] *GEN pre−comp−sit*[*OF assms*])

AOT-theorem *strict−sit*:
assumes ‹*RIGID-CONDITION*(φ)›
and ‹*CONDITION-ON-PROPOSITIONAL-PROPERTIES*(φ)›
shows ‹*y* = ι*s*(∀ *F* (*s*[*F*] ≡ φ{*F*})) → ∀ *F* (*y*[*F*] ≡ φ{*F*})›
using *rule*=*E*[*rotated*, *OF can−sit−desc*:*2*[*OF assms*(*2*), *symmetric*]]
box−phi−a:*2*[*OF assms*(*1*)] →*E* →*I* &*E* **by** *fast*

AOT-define *actual* :: ‹τ ⇒ φ› (‹*Actual*′(-′)›)
‹*Actual*(*s*) $\equiv_{df}$ ∀ *p* (*s* ⊨ *p* → *p*)›

AOT-theorem *act−and−not−pos*: ‹∃ *s* (*Actual*(*s*) & ◇¬*Actual*(*s*))›
proof −

```
  AOT-obtain q1 where q1-prop: ‹q1 & ◇¬q1›
    by (metis ≡df E instantiation cont−tf:1 cont−tf−thm:1)
  AOT-have ‹∃ s (∀ F (s[F] ≡ F = [λy q1]))›
  proof (safe intro!: comp−sit:1 cond−prop[I] GEN →I)
    AOT-modally-strict {
      AOT-show ‹Propositional([F])› if ‹F = [λy q1]› for F
        using ≡df I existential:2[const-var] prop−prop1 that by fastforce
    }
  qed
  then AOT-obtain s1 where s-prop: ‹∀ F (s1[F] ≡ F = [λy q1])›
    using Situation.∃ E[rotated] by meson
  AOT-have ‹Actual(s1)›
  proof(safe intro!: actual[THEN ≡df I] &I GEN →I s-prop Situation.ψ)
    fix p
    AOT-assume ‹s1 ⊨ p›
    AOT-hence ‹s1[λy p]›
      by (metis ≡df E &E(2) prop−enc true−in−s)
    AOT-hence ‹[λy p] = [λy q1]›
      by (rule s-prop[THEN ∀ E(1), THEN ≡E(1), rotated]) cqt:2[lambda]
    AOT-hence ‹p = q1› by (metis ≡E(2) p−identity−thm2:3)
    AOT-thus ‹p› using q1-prop[THEN &E(1)] rule=E id-sym by fast
  qed
  moreover AOT-have ‹◇¬Actual(s1)›
  proof(rule raa−cor:1; drule KBasic:12[THEN ≡E(2)])
    AOT-assume ‹□Actual(s1)›
    AOT-hence ‹□(Situation(s1) & ∀ p (s1 ⊨ p → p))›
      using actual[THEN ≡Df, THEN conventions:3[THEN ≡df E],
                  THEN &E(1), THEN RM, THEN →E] by blast
    AOT-hence ‹□∀ p (s1 ⊨ p → p)›
      by (metis RM:1 Conjunction Simplification(2) →E)
    AOT-hence ‹∀ p □(s1 ⊨ p → p)›
      by (metis CBF vdash−properties:10)
    AOT-hence ‹□(s1 ⊨ q1 → q1)›
      using ∀ E by blast
    AOT-hence ‹□s1 ⊨ q1 → □q1›
      by (metis →E qml:1 vdash−properties:1[2])
    moreover AOT-have ‹s1 ⊨ q1›
      using s-prop[THEN ∀ E(1), THEN ≡E(2),
                  THEN lem1[THEN →E, OF Situation.ψ, THEN ≡E(2)]]
            rule=I:1 prop−prop2:2 by blast
    ultimately AOT-have ‹□q1›
      using ≡df E &E(1) ≡E(1) lem2:1 true−in−s →E by fast
    AOT-thus ‹◇¬q1 & ¬◇¬q1›
      using KBasic:12[THEN ≡E(1)] q1-prop[THEN &E(2)] &I by blast
  qed
  ultimately AOT-have ‹(Actual(s1) & ◇¬Actual(s1))›
    using s-prop &I by blast
  thus ?thesis
    by (rule Situation.∃ I)
qed

AOT-theorem actual−s:1: ‹∃ s Actual(s)›
proof −
  AOT-obtain s where ‹(Actual(s) & ◇¬Actual(s))›
    using act−and−not−pos Situation.∃ E[rotated] by meson
  AOT-hence ‹Actual(s)› using &E &I by metis
  thus ?thesis by (rule Situation.∃ I)
qed

AOT-theorem actual−s:2: ‹∃ s ¬Actual(s)›
proof(rule ∃ I(1)[where τ=‹«sV»›]; (rule &I)?)
  AOT-show ‹Situation(sV)›
    using ≡df E &E(1) df−null−trivial:2 null−triv−sc:4 by blast
```

next
  AOT-show ‹¬*Actual*($\mathbf{s}_V$)›
  proof(*rule raa−cor*:*2*)
    AOT-assume *0*: ‹*Actual*($\mathbf{s}_V$)›
    AOT-obtain p_1 **where** *notp*$_1$: ‹¬p_1›
      by (*metis* ∃*E* ∃*I*(*1*) *log−prop−prop*:*2 non−contradiction*)
    AOT-have ‹$\mathbf{s}_V$ ⊨ p_1›
      using *null−triv−sc*:*4*[*THEN* $\equiv_{df}$*E*[*OF df−null−trivial*:*2*], *THEN* &*E*(*2*)]
          ∀*E* **by** *blast*
    AOT-hence ‹p_1›
      using *0*[*THEN actual*[*THEN* $\equiv_{df}$*E*], *THEN* &*E*(*2*), *THEN* ∀*E*(*2*), *THEN* →*E*]
      by *blast*
    AOT-thus ‹*p* & ¬*p*› **for** *p* **using** *notp*$_1$ **by** (*metis raa−cor*:*3*)
  qed
next
  AOT-show ‹$\mathbf{s}_V$↓›
    using *df−the−null−sit*:*2 rule−id−df*:*2*:*b*[*zero*] *thm−null−trivial*:*4* **by** *blast*
qed

AOT-theorem *actual−s*:*3*: ‹∃*p*∀*s*(*Actual*(*s*) → ¬*s* ⊨ *p*)›
proof −
  AOT-obtain p_1 **where** *notp*$_1$: ‹¬p_1›
    by (*metis* ∃*E* ∃*I*(*1*) *log−prop−prop*:*2 non−contradiction*)
  AOT-have ‹∀*s* (*Actual*(*s*) → ¬(*s* ⊨ p_1))›
  proof (*rule Situation.GEN*; *rule* →*I*; *rule raa−cor*:*2*)
    fix *s*
    AOT-assume ‹*Actual*(*s*)›
    moreover AOT-assume ‹*s* ⊨ p_1›
    ultimately AOT-have ‹p_1›
      using *actual*[*THEN* $\equiv_{df}$*E*, *THEN* &*E*(*2*), *THEN* ∀*E*(*2*), *THEN* →*E*] **by** *blast*
    AOT-thus ‹p_1 & ¬p_1›
      using *notp*$_1$ &*I* **by** *simp*
  qed
  thus *?thesis* **by** (*rule* ∃*I*)
qed

AOT-theorem *comp*:
  ‹∃*s* (*s*′ ⊴ *s* & *s*′′ ⊴ *s* & ∀*s*′′′ (*s*′ ⊴ *s*′′′ & *s*′′ ⊴ *s*′′′ → *s* ⊴ *s*′′′))›
proof −
  have *cond-prop*: ‹*ConditionOnPropositionalProperties* (λ Π . «*s*′[Π] ∨ *s*′′[Π]»)›
  proof(*safe intro*!: *cond−prop*[*I*] *GEN oth−class−taut*:*8*:*c*[*THEN* →*E*, *THEN* →*E*];
        rule →*I*)
    AOT-modally-strict {
      fix *F*
      AOT-have ‹*Situation*(*s*′)›
        by (*simp add*: *Situation.restricted-var-condition*)
      AOT-hence ‹*s*′[*F*] → *Propositional*([*F*])›
        using *situations*[*THEN* $\equiv_{df}$*E*, *THEN* &*E*(*2*), *THEN* ∀*E*(*2*)] **by** *blast*
      moreover AOT-assume ‹*s*′[*F*]›
      ultimately AOT-show ‹*Propositional*([*F*])›
        using →*E* **by** *blast*
    }
  next
    AOT-modally-strict {
      fix *F*
      AOT-have ‹*Situation*(*s*′′)›
        by (*simp add*: *Situation.restricted-var-condition*)
      AOT-hence ‹*s*′′[*F*] → *Propositional*([*F*])›
        using *situations*[*THEN* $\equiv_{df}$*E*, *THEN* &*E*(*2*), *THEN* ∀*E*(*2*)] **by** *blast*
      moreover AOT-assume ‹*s*′′[*F*]›
      ultimately AOT-show ‹*Propositional*([*F*])›
        using →*E* **by** *blast*
    }

```
  qed
  AOT-obtain s₃ where ϑ: ‹∀F (s₃[F] ≡ s′[F] ∨ s′′[F])›
    using comp−sit:1[OF cond-prop] Situation.∃E[rotated] by meson
  AOT-have ‹s′ ⊴ s₃ & s′′ ⊴ s₃ & ∀s′′′ (s′ ⊴ s′′′ & s′′ ⊴ s′′′ → s₃ ⊴ s′′′)›
  proof(safe intro!: &I ≡df I[OF true−in−s] ≡df I[OF prop−enc]
                    Situation.GEN GEN[where ′a=o] →I
                    sit−part−whole[THEN ≡df I]
                    Situation.ψ cqt:2[const-var][axiom-inst])
    fix p
    AOT-assume ‹s′ |= p›
    AOT-hence ‹s′[λx p]›
      by (metis &E(2) prop−enc ≡df E true−in−s)
    AOT-thus ‹s₃[λx p]›
      using ϑ[THEN ∀E(1),OF prop−prop2:2, THEN ≡E(2), OF ∨I(1)] by blast
  next
    fix p
    AOT-assume ‹s′′ |= p›
    AOT-hence ‹s′′[λx p]›
      by (metis &E(2) prop−enc ≡df E true−in−s)
    AOT-thus ‹s₃[λx p]›
      using ϑ[THEN ∀E(1),OF prop−prop2:2, THEN ≡E(2), OF ∨I(2)] by blast
  next
    fix s p
    AOT-assume 0: ‹s′ ⊴ s & s′′ ⊴ s›
    AOT-assume ‹s₃ |= p›
    AOT-hence ‹s₃[λx p]›
      by (metis &E(2) prop−enc ≡df E true−in−s)
    AOT-hence ‹s′[λx p] ∨ s′′[λx p]›
      using ϑ[THEN ∀E(1),OF prop−prop2:2, THEN ≡E(1)] by blast
    moreover {
      AOT-assume ‹s′[λx p]›
      AOT-hence ‹s′ |= p›
        by (safe intro!: prop−enc[THEN ≡df I] true−in−s[THEN ≡df I] &I
                        Situation.ψ cqt:2[const-var][axiom-inst])
      moreover AOT-have ‹s′ |= p → s |= p›
        using sit−part−whole[THEN ≡df E, THEN &E(2)] 0[THEN &E(1)]
            ∀E(2) by blast
      ultimately AOT-have ‹s |= p›
        using →E by blast
      AOT-hence ‹s[λx p]›
        using true−in−s[THEN ≡df E] prop−enc[THEN ≡df E] &E by blast
    }
    moreover {
      AOT-assume ‹s′′[λx p]›
      AOT-hence ‹s′′ |= p›
        by (safe intro!: prop−enc[THEN ≡df I] true−in−s[THEN ≡df I] &I
                        Situation.ψ cqt:2[const-var][axiom-inst])
      moreover AOT-have ‹s′′ |= p → s |= p›
        using sit−part−whole[THEN ≡df E, THEN &E(2)] 0[THEN &E(2)]
            ∀E(2) by blast
      ultimately AOT-have ‹s |= p›
        using →E by blast
      AOT-hence ‹s[λx p]›
        using true−in−s[THEN ≡df E] prop−enc[THEN ≡df E] &E by blast
    }
    ultimately AOT-show ‹s[λx p]›
      by (metis ∨E(1) →I)
  qed
  thus ?thesis
    using Situation.∃I by fast
qed

AOT-theorem act−sit:1: ‹Actual(s) → (s |= p → [λy p]s)›
```

proof (*safe intro*!: $\rightarrow I$)
  AOT-assume ‹*Actual*(s)›
  AOT-hence p **if** ‹$s \models p$›
    using *actual*[*THEN* $\equiv_{df}E$, *THEN* &*E*(*2*), *THEN* $\forall E(2)$, *THEN* $\rightarrow E$] *that* **by** *blast*
  moreover AOT-assume ‹$s \models p$›
  ultimately AOT-have p **by** *blast*
  AOT-thus ‹$[\lambda y\ p]s$›
    by (*safe intro*!: $\beta{\leftarrow}C(1)$ *cqt*:*2*)
qed

AOT-theorem *act−sit*:*2*:
  ‹($Actual(s')$ & $Actual(s'')$) $\rightarrow \exists x$ ($Actual(x)$ & $s' \trianglelefteq x$ & $s'' \trianglelefteq x$)›
proof(*rule* $\rightarrow I$; *frule* &*E*(*1*); *drule* &*E*(*2*))
  AOT-assume *act-s′*: ‹$Actual(s')$›
  AOT-assume *act-s″*: ‹$Actual(s'')$›
  have *cond−prop*: ‹*ConditionOnPropositionalProperties*
                        ($\lambda\ \Pi$. «$\exists p$ ($\Pi = [\lambda y\ p]$ & ($s' \models p \vee s'' \models p$))»)›
  proof (*safe intro*!: *cond−prop*[*I*] $\forall I \rightarrow I$ *prop−prop1*[*THEN* $\equiv_{df}I$])
    AOT-modally-strict {
      fix β
      AOT-assume ‹$\exists p$ ($\beta = [\lambda y\ p]$ & ($s' \models p \vee s'' \models p$))›
      then AOT-obtain p **where** ‹$\beta = [\lambda y\ p]$› **using** $\exists E$[*rotated*] &*E* **by** *blast*
      AOT-thus ‹$\exists p\ \beta = [\lambda y\ p]$› **by** (*rule* $\exists I$)
    }
  qed
  have *rigid*: ‹*rigid-condition* ($\lambda\ \Pi$. «$\exists p$ ($\Pi = [\lambda y\ p]$ & ($s' \models p \vee s'' \models p$))»)›
  proof(*safe intro*!: *strict−can*:*1*[*I*] $\rightarrow I$ *GEN*)
    AOT-modally-strict {
      fix F
      AOT-assume ‹$\exists p$ ($F = [\lambda y\ p]$ & ($s' \models p \vee s'' \models p$))›
      then AOT-obtain p_1 **where** p_1*-prop*: ‹$F = [\lambda y\ p_1]$ & ($s' \models p_1 \vee s'' \models p_1$)›
        using $\exists E$[*rotated*] **by** *blast*
      AOT-hence ‹$\Box(F = [\lambda y\ p_1])$›
        using &*E*(*1*) *id−nec*:*2* *vdash−properties*:*10* **by** *blast*
      moreover AOT-have ‹$\Box(s' \models p_1 \vee s'' \models p_1)$›
      proof(*rule* $\vee E$; (*rule* $\rightarrow I$; *rule KBasic*:*15*[*THEN* $\rightarrow E$])*?*)
        AOT-show ‹$s' \models p_1 \vee s'' \models p_1$› **using** p_1*-prop* &*E* **by** *blast*
      next
        AOT-show ‹$\Box s' \models p_1 \vee \Box s'' \models p_1$› **if** ‹$s' \models p_1$›
          apply (*rule* $\vee I(1)$)
          using $\equiv_{df}E$ &*E*(*1*) $\equiv E(1)$ *lem2*:*1* *that true−in−s* **by** *blast*
      next
        AOT-show ‹$\Box s' \models p_1 \vee \Box s'' \models p_1$› **if** ‹$s'' \models p_1$›
          apply (*rule* $\vee I(2)$)
          using $\equiv_{df}E$ &*E*(*1*) $\equiv E(1)$ *lem2*:*1* *that true−in−s* **by** *blast*
      qed
      ultimately AOT-have ‹$\Box(F = [\lambda y\ p_1]$ & ($s' \models p_1 \vee s'' \models p_1$))›
        by (*metis KBasic*:*3* &*I* $\equiv E(2)$)
      AOT-hence ‹$\exists p\ \Box(F = [\lambda y\ p]$ & ($s' \models p \vee s'' \models p$))› **by** (*rule* $\exists I$)
      AOT-thus ‹$\Box\exists p$ ($F = [\lambda y\ p]$ & ($s' \models p \vee s'' \models p$))›
        using *Buridan*[*THEN* $\rightarrow E$] **by** *fast*
    }
  qed

  AOT-have *desc-den*: ‹$\iota s(\forall F$ ($s[F] \equiv \exists p$ ($F = [\lambda y\ p]$ & ($s' \models p \vee s'' \models p$))))$\downarrow$›
    by (*rule can−sit−desc*:*1*[*OF cond−prop*])
  AOT-obtain x_0
    where x_0*-prop1*: ‹$x_0 = \iota s(\forall F$ ($s[F] \equiv \exists p$ ($F = [\lambda y\ p]$ & ($s' \models p \vee s'' \models p$))))›
    by (*metis* (*no-types*, *lifting*) $\exists E$ *rule=I*:*1* *desc-den* $\exists I(1)$ *id-sym*)
  AOT-hence x_0*-sit*: ‹$Situation(x_0)$›
    using *actual−desc*:*3*[*THEN* $\rightarrow E$] *Act−Basic*:*2* &*E*(*1*) $\equiv E(1)$
          possit−sit:*4* **by** *blast*

```
  AOT-have 1: ‹∀ F (x0[F] ≡ ∃ p (F = [λy p] & (s' ⊨ p ∨ s'' ⊨ p)))›
    using strict−sit[OF rigid, OF cond−prop, THEN →E, OF x0-prop1].
  AOT-have 2: ‹(x0 ⊨ p) ≡ (s' ⊨ p ∨ s'' ⊨ p)› for p
  proof (rule ≡I; rule →I)
    AOT-assume ‹x0 ⊨ p›
    AOT-hence ‹x0[λy p]› using lem1[THEN →E, OF x0-sit, THEN ≡E(1)] by blast
    then AOT-obtain q where ‹[λy p] = [λy q] & (s' ⊨ q ∨ s'' ⊨ q)›
      using 1[THEN ∀ E(1)[where τ=«[λy p]»], OF prop−prop2:2, THEN ≡E(1)]
            ∃ E[rotated] by blast
    AOT-thus ‹s' ⊨ p ∨ s'' ⊨ p›
      by (metis rule=E &E(1) &E(2) ∨I(1) ∨I(2)
                ∨E(1) deduction−theorem id-sym ≡E(2) p−identity−thm2:3)
  next
    AOT-assume ‹s' ⊨ p ∨ s'' ⊨ p›
    AOT-hence ‹[λy p] = [λy p] & (s' ⊨ p ∨ s'' ⊨ p)›
      by (metis rule=I:1 &I prop−prop2:2)
    AOT-hence ‹∃ q ([λy p] = [λy q] & (s' ⊨ q ∨ s'' ⊨ q))›
      by (rule ∃ I)
    AOT-hence ‹x0[λy p]›
      using 1[THEN ∀ E(1), OF prop−prop2:2, THEN ≡E(2)] by blast
    AOT-thus ‹x0 ⊨ p›
      by (metis ≡df I &I ex:1:a prop−enc rule−ui:2[const-var]
                x0-sit true−in−s)
  qed

  AOT-have ‹Actual(x0) & s' ⊴ x0 & s'' ⊴ x0›
  proof(safe intro!: →I &I ∃ I(1) actual[THEN ≡df I] x0-sit GEN
                     sit−part−whole[THEN ≡df I])
    fix p
    AOT-assume ‹x0 ⊨ p›
    AOT-hence ‹s' ⊨ p ∨ s'' ⊨ p›
      using 2 ≡E(1) by metis
    AOT-thus ‹p›
      using act-s' act-s''
            actual[THEN ≡df E, THEN &E(2), THEN ∀ E(2), THEN →E]
      by (metis ∨E(3) reductio−aa:1)
  next
    AOT-show ‹x0 ⊨ p› if ‹s' ⊨ p› for p
      using 2[THEN ≡E(2), OF ∨I(1), OF that].
  next
    AOT-show ‹x0 ⊨ p› if ‹s'' ⊨ p› for p
      using 2[THEN ≡E(2), OF ∨I(2), OF that].
  next
    AOT-show ‹Situation(s')›
      using act-s'[THEN actual[THEN ≡df E]] &E by blast
  next
    AOT-show ‹Situation(s'')›
      using act-s''[THEN actual[THEN ≡df E]] &E by blast
  qed
  AOT-thus ‹∃ x (Actual(x) & s' ⊴ x & s'' ⊴ x)›
    by (rule ∃ I)
qed

AOT-define Consistent :: ‹τ ⇒ φ› (‹Consistent'(-')›)
  cons: ‹Consistent(s) ≡df ¬∃ p (s ⊨ p & s ⊨ ¬p)›

AOT-theorem sit−cons: ‹Actual(s) → Consistent(s)›
proof(safe intro!: →I cons[THEN ≡df I] &I Situation.ψ
           dest!: actual[THEN ≡df E]; frule &E(1); drule &E(2))
  AOT-assume 0: ‹∀ p (s ⊨ p → p)›
  AOT-show ‹¬∃ p (s ⊨ p & s ⊨ ¬p)›
  proof (rule raa−cor:2)
    AOT-assume ‹∃ p (s ⊨ p & s ⊨ ¬p)›
```

then AOT-obtain p **where** ‹$s \models p$ & $s \models \neg p$›
using $\exists$ *E*[*rotated*] **by** *blast*
AOT-hence ‹p & $\neg p$›
using *0*[*THEN* $\forall$ *E*(*1*)[**where** τ=‹«$\neg p$»›, *THEN* $\rightarrow$*E*], *OF log−prop−prop*:*2*]
0[*THEN* $\forall$ *E*(*2*)[**where** β=*p*], *THEN* $\rightarrow$*E*] &*E* &*I* **by** *blast*
AOT-thus ‹p & $\neg p$› **for** p **by** (*metis raa−cor*:*1*)
qed
qed

AOT-theorem *cons−rigid*:*1*: ‹$\neg$*Consistent*(s) $\equiv$ $\square\neg$*Consistent*(s)›
proof (*rule* $\equiv$*I*; *rule* $\rightarrow$*I*)
AOT-assume ‹$\neg$*Consistent*(s)›
AOT-hence ‹$\exists p$ ($s \models p$ & $s \models \neg p$)›
using *cons*[*THEN* $\equiv_{df}$*I*, *OF* &*I*, *OF Situation*.ψ]
by (*metis raa−cor*:*3*)
then AOT-obtain p **where** *p-prop*: ‹$s \models p$ & $s \models \neg p$›
using $\exists$ *E*[*rotated*] **by** *blast*
AOT-hence ‹$\square s \models p$›
using &*E*(*1*) $\equiv$*E*(*1*) *lem2*:*1* **by** *blast*
moreover AOT-have ‹$\square s \models \neg p$›
using *p-prop T*$\Diamond$ &*E* $\equiv$*E*(*1*)
modus−tollens:*1 raa−cor*:*3 lem2*:*3*[*unvarify p*]
log−prop−prop:*2* **by** *metis*
ultimately AOT-have ‹$\square$($s \models p$ & $s \models \neg p$)›
by (*metis KBasic*:*3* &*I* $\equiv$*E*(*2*))
AOT-hence ‹$\exists p$ $\square$($s \models p$ & $s \models \neg p$)›
by (*rule* $\exists$ *I*)
AOT-hence ‹$\square\exists p$($s \models p$ & $s \models \neg p$)›
by (*metis Buridan vdash−properties*:*10*)
AOT-thus ‹$\square\neg$*Consistent*(s)›
apply (*rule qml*:*1*[*axiom-inst*, *THEN* $\rightarrow$*E*, *THEN* $\rightarrow$*E*, *rotated*])
apply (*rule RN*)
using $\equiv_{df}$*E* &*E*(*2*) *cons deduction−theorem raa−cor*:*3* **by** *blast*
next
AOT-assume ‹$\square\neg$*Consistent*(s)›
AOT-thus ‹$\neg$*Consistent*(s)› **using** *qml*:*2*[*axiom-inst*, *THEN* $\rightarrow$*E*] **by** *auto*
qed

AOT-theorem *cons−rigid*:*2*: ‹$\Diamond$*Consistent*(x) $\equiv$ *Consistent*(x)›
proof(*rule* $\equiv$*I*; *rule* $\rightarrow$*I*)
AOT-assume *0*: ‹$\Diamond$*Consistent*(x)›
AOT-have ‹$\Diamond$(*Situation*(x) & $\neg\exists p$ ($x \models p$ & $x \models \neg p$))›
apply (*AOT-subst* ‹*Situation*(x) & $\neg\exists p$ ($x \models p$ & $x \models \neg p$)› ‹*Consistent*(x)›)
using *cons* $\equiv$*E*(*2*) *Commutativity of* $\equiv$ $\equiv$*Df* **apply** *blast*
by (*simp add*: *0*)
AOT-hence ‹$\Diamond$*Situation*(x)› **and** *1*: ‹$\Diamond\neg\exists p$ ($x \models p$ & $x \models \neg p$)›
using *RM*$\Diamond$ *Conjunction Simplification*(*1*) *Conjunction Simplification*(*2*)
modus−tollens:*1 raa−cor*:*3* **by** *blast*+
AOT-hence *2*: ‹*Situation*(x)› **by** (*metis* $\equiv$*E*(*1*) *possit−sit*:*2*)
AOT-have *3*: ‹$\neg\square\exists p$ ($x \models p$ & $x \models \neg p$)›
using *2* **using** *1 KBasic*:*11* $\equiv$*E*(*2*) **by** *blast*
AOT-show ‹*Consistent*(x)›
proof (*rule raa−cor*:*1*)
AOT-assume ‹$\neg$*Consistent*(x)›
AOT-hence ‹$\exists p$ ($x \models p$ & $x \models \neg p$)›
using *0* $\equiv_{df}$*E conventions*:*5 2 cons−rigid*:*1*[*unconstrain s*, *THEN* $\rightarrow$*E*]
modus−tollens:*1 raa−cor*:*3* $\equiv$*E*(*4*) **by** *meson*
then AOT-obtain p **where** ‹$x \models p$› **and** *4*: ‹$x \models \neg p$›
using $\exists$ *E*[*rotated*] &*E* **by** *blast*
AOT-hence ‹$\square x \models p$›
by (*metis 2* $\equiv$*E*(*1*) *lem2*:*1*[*unconstrain s*, *THEN* $\rightarrow$*E*])
moreover AOT-have ‹$\square x \models \neg p$›
using *4 lem2*:*1*[*unconstrain s*, *unvarify p*, *THEN* $\rightarrow$*E*]

```
      by (metis 2 ≡E(1) log-prop-prop:2)
    ultimately AOT-have ‹□(x ⊨ p & x ⊨ ¬p)›
      by (metis KBasic:3 &I ≡E(3) raa-cor:3)
    AOT-hence ‹∃ p □(x ⊨ p & x ⊨ ¬p)›
      by (metis existential:1 log-prop-prop:2)
    AOT-hence ‹□∃ p (x ⊨ p & x ⊨ ¬p)›
      by (metis Buridan vdash-properties:10)
    AOT-thus ‹p & ¬p› for p
      using 3 &I by (metis raa-cor:3)
  qed
next
  AOT-show ‹◇Consistent(x)› if ‹Consistent(x)›
    using T◇ that vdash-properties:10 by blast
qed

AOT-define possible :: ‹τ ⇒ φ› (‹Possible'(-')›)
  pos: ‹Possible(s) ≡df ◇Actual(s)›

AOT-theorem sit-pos:1: ‹Actual(s) → Possible(s)›
  apply(rule →I; rule pos[THEN ≡dfI]; rule &I)
  apply (meson ≡dfE actual &E(1))
  using T◇ vdash-properties:10 by blast

AOT-theorem sit-pos:2: ‹∃ p ((s ⊨ p) & ¬◇p) → ¬Possible(s)›
proof(rule →I)
  AOT-assume ‹∃ p ((s ⊨ p) & ¬◇p)›
  then AOT-obtain p where a: ‹(s ⊨ p) & ¬◇p›
    using ∃ E[rotated] by blast
  AOT-hence ‹□(s ⊨ p)›
    using &E by (metis T◇ ≡E(1) lem2:3 vdash-properties:10)
  moreover AOT-have ‹□¬p›
    using a[THEN &E(2)] by (metis KBasic2:1 ≡E(2))
  ultimately AOT-have ‹□(s ⊨ p & ¬p)›
    by (metis KBasic:3 &I ≡E(3) raa-cor:3)
  AOT-hence ‹∃ p □(s ⊨ p & ¬p)›
    by (rule ∃ I)
  AOT-hence 1: ‹□∃ q (s ⊨ q & ¬q)›
    by (metis Buridan vdash-properties:10)
  AOT-have ‹□¬∀ q (s ⊨ q → q)›
    apply (AOT-subst ‹s ⊨ q → q› ‹¬(s ⊨ q & ¬q)› for: q)
     apply (simp add: oth-class-taut:1:a)
    apply (AOT-subst ‹¬∀ q ¬(s ⊨ q & ¬q)› ‹∃ q (s ⊨ q & ¬q)›)
    by (auto simp: conventions:4 df-rules-formulas[3] df-rules-formulas[4] ≡I 1)
  AOT-hence 0: ‹¬◇∀ q (s ⊨ q → q)›
    by (metis ≡dfE conventions:5 raa-cor:3)
  AOT-show ‹¬Possible(s)›
    apply (AOT-subst ‹Possible(s)› ‹Situation(s) & ◇Actual(s)›)
     apply (simp add: pos ≡Df)
    apply (AOT-subst ‹Actual(s)› ‹Situation(s) & ∀ q (s ⊨ q → q)›)
     using actual ≡Df apply presburger
    by (metis 0 KBasic2:3 &E(2) raa-cor:3 vdash-properties:10)
qed

AOT-theorem pos-cons-sit:1: ‹Possible(s) → Consistent(s)›
  by (auto simp: sit-cons[THEN RM◇, THEN →E,
                          THEN cons-rigid:2[THEN ≡E(1)]]
          intro!: →I dest!: pos[THEN ≡dfE] &E(2))

AOT-theorem pos-cons-sit:2: ‹∃ s (Consistent(s) & ¬Possible(s))›
proof -
  AOT-obtain q₁ where ‹q₁ & ◇¬q₁›
    using ≡dfE instantiation cont-tf:1 cont-tf-thm:1 by blast
  have cond-prop: ‹ConditionOnPropositionalProperties
```

(λ Π . «Π = [λy q_1 & ¬q_1]»)›
  by (*auto intro*!: *cond−prop*[*I*] *GEN* →*I* *prop−prop1*[*THEN* ≡$_{df}$*I*]
      ∃ *I*(*1*)[**where** τ=‹«q_1 & ¬q_1»›, *rotated*, *OF log−prop−prop*:*2*])
have *rigid*: ‹*rigid-condition* (λ Π . «Π = [λy q_1 & ¬q_1]»)›
  by (*auto intro*!: *strict−can*:*1*[*I*] *GEN* →*I* *simp*: *id−nec*:*2*[*THEN* →*E*])

AOT-obtain *x* **where** *x-prop*: ‹*x* = ιs (∀ *F* (*s*[*F*] ≡ *F* = [λy q_1 & ¬q_1]))›
  using *ex*:*1*:*b*[*THEN* ∀ *E*(*1*), *OF can−sit−desc*:*1*, *OF cond−prop*]
    ∃ *E*[*rotated*] **by** *blast*
AOT-hence *0*: ‹𝓐(*Situation*(*x*) & ∀ *F* (*x*[*F*] ≡ *F* = [λy q_1 & ¬q_1]))›
  using →*E actual−desc*:*2* **by** *blast*
AOT-hence ‹𝓐(*Situation*(*x*))› **by** (*metis Act−Basic*:*2* &*E*(*1*) ≡*E*(*1*))
AOT-hence *s-sit*: ‹*Situation*(*x*)› **by** (*metis* ≡*E*(*1*) *possit−sit*:*4*)
AOT-have *s-enc-prop*: ‹∀ *F* (*x*[*F*] ≡ *F* = [λy q_1 & ¬q_1])›
  using *strict−sit*[*OF rigid*, *OF cond−prop*, *THEN* →*E*, *OF x-prop*]**.**
AOT-hence ‹*x*[λy q_1 & ¬q_1]›
  using ∀ *E*(*1*)[*rotated*, *OF prop−prop2*:*2*]
    rule=*I*:*1*[*OF prop−prop2*:*2*] ≡*E* **by** *blast*
AOT-hence ‹*x* ⊨ (q_1 & ¬q_1)›
  using *lem1*[*THEN* →*E*, *OF s-sit*, *unvarify p*, *THEN* ≡*E*(*2*), *OF log−prop−prop*:*2*]
  by *blast*
AOT-hence ‹□(*x* ⊨ (q_1 & ¬q_1))›
  using *lem2*:*1*[*unconstrain s*, *THEN* →*E*, *OF s-sit*, *unvarify p*,
      OF log−prop−prop:*2*, *THEN* ≡*E*(*1*)] **by** *blast*
moreover AOT-have ‹□(*x* ⊨ (q_1 & ¬q_1) → ¬*Actual*(*x*))›
proof(*rule RN*; *rule* →*I*; *rule raa−cor*:*2*)
  AOT-modally-strict {
    AOT-assume ‹*Actual*(*x*)›
    AOT-hence ‹∀ *p* (*x* ⊨ *p* → *p*)›
      using *actual*[*THEN* ≡$_{df}$*E*, *THEN* &*E*(*2*)] **by** *blast*
    moreover AOT-assume ‹*x* ⊨ (q_1 & ¬q_1)›
    ultimately AOT-show ‹q_1 & ¬q_1›
      using ∀ *E*(*1*)[*rotated*, *OF log−prop−prop*:*2*] →*E* **by** *metis*
  }
qed
ultimately AOT-have *nec-not-actual-s*: ‹□¬*Actual*(*x*)›
  using *qml*:*1*[*axiom-inst*, *THEN* →*E*, *THEN* →*E*] **by** *blast*
AOT-have *1*: ‹¬∃ *p* (*x* ⊨ *p* & *x* ⊨ ¬*p*)›
proof (*rule raa−cor*:*2*)
  AOT-assume ‹∃ *p* (*x* ⊨ *p* & *x* ⊨ ¬*p*)›
  then AOT-obtain *p* **where** ‹*x* ⊨ *p* & *x* ⊨ ¬*p*›
    using ∃ *E*[*rotated*] **by** *blast*
  AOT-hence ‹*x*[λy *p*] & *x*[λy ¬*p*]›
    using *lem1*[*unvarify p*, *THEN* →*E*, *OF log−prop−prop*:*2*,
      OF s-sit, *THEN* ≡*E*(*1*)] &*I* &*E* **by** *metis*
  AOT-hence ‹[λy *p*] = [λy q_1 & ¬q_1]› **and** ‹[λy ¬*p*] = [λy q_1 & ¬q_1]›
    by (*auto intro*!: *prop−prop2*:*2 s-enc-prop*[*THEN* ∀ *E*(*1*), *THEN* ≡*E*(*1*)]
      elim: &*E*)
  AOT-hence *i*: ‹[λy *p*] = [λy ¬*p*]› **by** (*metis rule*=*E id-sym*)
  {
    AOT-assume *0*: ‹*p*›
    AOT-have ‹[λy *p*]*x*› **for** *x*
      by (*auto intro*!: β←*C*(*1*) *cqt*:*2 0*)
    AOT-hence ‹[λy ¬*p*]*x*› **for** *x* **using** *i rule*=*E* **by** *fast*
    AOT-hence ‹¬*p*›
      using β→*C*(*1*) **by** *auto*
  }
  moreover {
    AOT-assume *0*: ‹¬*p*›
    AOT-have ‹[λy ¬*p*]*x*› **for** *x*
      by (*auto intro*!: β←*C*(*1*) *cqt*:*2 0*)
    AOT-hence ‹[λy *p*]*x*› **for** *x* **using** *i*[*symmetric*] *rule*=*E* **by** *fast*
    AOT-hence ‹*p*›

```
      using β→C(1) by auto
    }
    ultimately AOT-show ‹p & ¬p› for p by (metis raa−cor:1 raa−cor:3)
  qed
  AOT-have 2: ‹¬Possible(x)›
  proof(rule raa−cor:2)
    AOT-assume ‹Possible(x)›
    AOT-hence ‹◇Actual(x)›
      by (metis ≡df E &E(2) pos)
    moreover AOT-have ‹¬◇Actual(x)› using nec-not-actual-s
      using ≡df E conventions:5 reductio−aa:2 by blast
    ultimately AOT-show ‹◇Actual(x) & ¬◇Actual(x)› by (rule &I)
  qed
  show ?thesis
    by(rule ∃I(2)[where β=x]; safe intro!: &I 2 s-sit cons[THEN ≡df I] 1)
qed

AOT-theorem sit−classical:1: ‹∀p (s ⊨ p ≡ p) → ∀q(s ⊨ ¬q ≡ ¬s ⊨ q)›
proof(rule →I; rule GEN)
  fix q
  AOT-assume ‹∀p (s ⊨ p ≡ p)›
  AOT-hence ‹s ⊨ q ≡ q› and ‹s ⊨ ¬q ≡ ¬q›
    using ∀E(1)[rotated, OF log−prop−prop:2] by blast+
  AOT-thus ‹s ⊨ ¬q ≡ ¬s ⊨ q›
    by (metis deduction−theorem ≡I ≡E(1) ≡E(2) ≡E(4))
qed

AOT-theorem sit−classical:2:
  ‹∀p (s ⊨ p ≡ p) → ∀q∀r((s ⊨ (q → r)) ≡ (s ⊨ q → s ⊨ r))›
proof(rule →I; rule GEN; rule GEN)
  fix q r
  AOT-assume ‹∀p (s ⊨ p ≡ p)›
  AOT-hence ϑ: ‹s ⊨ q ≡ q› and ξ: ‹s ⊨ r ≡ r› and ζ: ‹(s ⊨ (q → r)) ≡ (q → r)›
    using ∀E(1)[rotated, OF log−prop−prop:2] by blast+
  AOT-show ‹(s ⊨ (q → r)) ≡ (s ⊨ q → s ⊨ r)›
  proof (safe intro!: ≡I →I)
    AOT-assume ‹s ⊨ (q → r)›
    moreover AOT-assume ‹s ⊨ q›
    ultimately AOT-show ‹s ⊨ r›
      using ϑ ξ ζ by (metis ≡E(1) ≡E(2) vdash−properties:10)
  next
    AOT-assume ‹s ⊨ q → s ⊨ r›
    AOT-thus ‹s ⊨ (q → r)›
      using ϑ ξ ζ by (metis deduction−theorem ≡E(1) ≡E(2) →E)
  qed
qed

AOT-theorem sit−classical:3:
  ‹∀p (s ⊨ p ≡ p) → ((s ⊨ ∀α φ{α}) ≡ ∀α s ⊨ φ{α})›
proof (rule →I)
  AOT-assume ‹∀p (s ⊨ p ≡ p)›
  AOT-hence ϑ: ‹s ⊨ φ{α} ≡ φ{α}› and ξ: ‹s ⊨ ∀α φ{α} ≡ ∀α φ{α}› for α
    using ∀E(1)[rotated, OF log−prop−prop:2] by blast+
  AOT-show ‹s ⊨ ∀α φ{α} ≡ ∀α s ⊨ φ{α}›
  proof (safe intro!: ≡I →I GEN)
    fix α
    AOT-assume ‹s ⊨ ∀α φ{α}›
    AOT-hence ‹φ{α}› using ξ ∀E(2) ≡E(1) by blast
    AOT-thus ‹s ⊨ φ{α}› using ϑ ≡E(2) by blast
  next
    AOT-assume ‹∀α s ⊨ φ{α}›
    AOT-hence ‹s ⊨ φ{α}› for α using ∀E(2) by blast
    AOT-hence ‹φ{α}› for α using ϑ ≡E(1) by blast
```

AOT-hence ‹$\forall\,\alpha\ \varphi\{\alpha\}$› **by** (*rule GEN*)
AOT-thus ‹$s \models \forall\,\alpha\ \varphi\{\alpha\}$› **using** ξ $\equiv E(2)$ **by** *blast*
qed
qed

AOT-theorem *sit−classical:4*: ‹$\forall\,p\ (s \models p \equiv p) \rightarrow \forall\,q\ (s \models \Box q \rightarrow \Box s \models q)$›
proof(*rule* $\rightarrow I$; *rule GEN*; *rule* $\rightarrow I$)
fix *q*
AOT-assume ‹$\forall\,p\ (s \models p \equiv p)$›
AOT-hence ϑ: ‹$s \models q \equiv q$› **and** ξ: ‹$s \models \Box q \equiv \Box q$›
using $\forall\,E(1)$[*rotated*, *OF log−prop−prop:2*] **by** *blast+*
AOT-assume ‹$s \models \Box q$›
AOT-hence ‹$\Box q$› **using** ξ $\equiv E(1)$ **by** *blast*
AOT-hence ‹q› **using** *qml:2*[*axiom-inst*, *THEN* $\rightarrow E$] **by** *blast*
AOT-hence ‹$s \models q$› **using** ϑ $\equiv E(2)$ **by** *blast*
AOT-thus ‹$\Box s \models q$› **using** $\equiv_{df} E$ $\&E(1)$ $\equiv E(1)$ *lem2:1 true−in−s* **by** *blast*
qed

AOT-theorem *sit−classical:5*:
‹$\forall\,p\ (s \models p \equiv p) \rightarrow \exists\,q(\Box(s \models q)\ \&\ \neg(s \models \Box\ q))$›
proof (*rule* $\rightarrow I$)
AOT-obtain *r* **where** *A*: ‹r› **and** ‹$\Diamond\neg r$›
by (*metis* $\&E(1)$ $\&E(2)$ $\equiv_{df} E$ *instantiation cont−tf:1 cont−tf−thm:1*)
AOT-hence *B*: ‹$\neg\Box r$›
using *KBasic:11* $\equiv E(2)$ **by** *blast*
moreover AOT-assume *asm*: ‹$\forall\ p\ (s \models p \equiv p)$›
AOT-hence ‹$s \models r$›
using $\forall\,E(2)$ *A* $\equiv E(2)$ **by** *blast*
AOT-hence *1*: ‹$\Box s \models r$›
using $\equiv_{df} E$ $\&E(1)$ $\equiv E(1)$ *lem2:1 true−in−s* **by** *blast*
AOT-have ‹$s \models \neg\Box r$›
using *asm*[*THEN* $\forall\,E(1)$[*rotated*, *OF log−prop−prop:2*], *THEN* $\equiv E(2)$] *B* **by** *blast*
AOT-hence ‹$\neg s \models \Box r$›
using *sit−classical:1*[*THEN* $\rightarrow E$, *OF asm*,
THEN $\forall\,E(1)$[*rotated*, *OF log−prop−prop:2*], *THEN* $\equiv E(1)$] **by** *blast*
AOT-hence ‹$\Box s \models r\ \&\ \neg s \models \Box r$›
using *1* $\&I$ **by** *blast*
AOT-thus ‹$\exists\,r\ (\Box s \models r\ \&\ \neg s \models \Box r)$›
by (*rule* $\exists\,I$)
qed

AOT-theorem *sit−classical:6*:
‹$\exists\,s\ \forall\,p\ (s \models p \equiv p)$›
proof −
have *cond−prop*: ‹*ConditionOnPropositionalProperties*
($\lambda\ \Pi$. «$\exists\,q\ (q\ \&\ \Pi = [\lambda y\ q])$»)›
proof (*safe intro*!: *cond−prop*[*I*] *GEN* $\rightarrow I$)
fix *F*
AOT-modally-strict {
AOT-assume ‹$\exists\,q\ (q\ \&\ F = [\lambda y\ q])$›
then AOT-obtain *q* **where** ‹$q\ \&\ F = [\lambda y\ q]$›
using $\exists\,E$[*rotated*] **by** *blast*
AOT-hence ‹$F = [\lambda y\ q]$›
using $\&E$ **by** *blast*
AOT-hence ‹$\exists\,q\ F = [\lambda y\ q]$›
by (*rule* $\exists\,I$)
AOT-thus ‹*Propositional*($[F]$)›
by (*metis* $\equiv_{df} I$ *prop−prop1*)
}
qed
AOT-have ‹$\exists\,s\ \forall\,F\ (s[F] \equiv \exists\,q\ (q\ \&\ F = [\lambda y\ q]))$›
using *comp−sit:1*[*OF cond−prop*].
then AOT-obtain s_0 **where** s_0-*prop*: ‹$\forall\,F\ (s_0[F] \equiv \exists\,q\ (q\ \&\ F = [\lambda y\ q]))$›

using *Situation*.$\exists E$[*rotated*] **by** *meson*
AOT-have ‹$\forall p\ (s_0 \models p \equiv p)$›
proof(*safe intro*!: *GEN* $\equiv I$ $\rightarrow I$)
fix p
AOT-assume ‹$s_0 \models p$›
AOT-hence ‹$s_0[\lambda y\ p]$›
using *lem1*[*THEN* $\rightarrow E$, *OF Situation*.ψ, *THEN* $\equiv E(1)$] **by** *blast*
AOT-hence ‹$\exists q\ (q\ \&\ [\lambda y\ p] = [\lambda y\ q])$›
using s_0-*prop*[*THEN* $\forall E(1)$[*rotated*, *OF prop−prop2*:*2*], *THEN* $\equiv E(1)$] **by** *blast*
then AOT-obtain q_1 **where** q_1-*prop*: ‹$q_1\ \&\ [\lambda y\ p] = [\lambda y\ q_1]$›
using $\exists E$[*rotated*] **by** *blast*
AOT-hence ‹$p = q_1$›
by (*metis* $\&E(2)$ $\equiv E(2)$ *p−identity−thm2*:*3*)
AOT-thus ‹p›
using q_1-*prop*[*THEN* $\&E(1)$] *rule=E id-sym* **by** *fast*
next
fix p
AOT-assume ‹p›
moreover AOT-have ‹$[\lambda y\ p] = [\lambda y\ p]$›
by (*simp add*: *rule=I*:*1*[*OF prop−prop2*:*2*])
ultimately AOT-have ‹$p\ \&\ [\lambda y\ p] = [\lambda y\ p]$›
using $\&I$ **by** *blast*
AOT-hence ‹$\exists q\ (q\ \&\ [\lambda y\ p] = [\lambda y\ q])$›
by (*rule* $\exists I$)
AOT-hence ‹$s_0[\lambda y\ p]$›
using s_0-*prop*[*THEN* $\forall E(1)$[*rotated*, *OF prop−prop2*:*2*], *THEN* $\equiv E(2)$] **by** *blast*
AOT-thus ‹$s_0 \models p$›
using *lem1*[*THEN* $\rightarrow E$, *OF Situation*.ψ, *THEN* $\equiv E(2)$] **by** *blast*
qed
AOT-hence ‹$\forall p\ (s_0 \models p \equiv p)$›
using $\&I$ **by** *blast*
AOT-thus ‹$\exists s\ \forall p\ (s \models p \equiv p)$›
by (*rule Situation*.$\exists I$)
qed

AOT-define *PossibleWorld* :: ‹$\tau \Rightarrow \varphi$› (‹*PossibleWorld*′(-)′›)
world:*1*: ‹$PossibleWorld(x) \equiv_{df} Situation(x)\ \&\ \Diamond\forall p(x \models p \equiv p)$›

AOT-theorem *world*:*2*: ‹$\exists x\ PossibleWorld(x)$›
proof −
AOT-obtain s **where** s-*prop*: ‹$\forall p\ (s \models p \equiv p)$›
using *sit−classical*:*6 Situation*.$\exists E$[*rotated*] **by** *meson*
AOT-have ‹$\forall p\ (s \models p \equiv p)$›
proof(*safe intro*!: *GEN* $\equiv I$ $\rightarrow I$)
fix p
AOT-assume ‹$s \models p$›
AOT-thus ‹p›
using s-*prop*[*THEN* $\forall E(2)$, *THEN* $\equiv E(1)$] **by** *blast*
next
fix p
AOT-assume ‹p›
AOT-thus ‹$s \models p$›
using s-*prop*[*THEN* $\forall E(2)$, *THEN* $\equiv E(2)$] **by** *blast*
qed
AOT-hence ‹$\Diamond\forall p\ (s \models p \equiv p)$›
by (*metis* $T\Diamond$[*THEN* $\rightarrow E$])
AOT-hence ‹$\Diamond\forall p\ (s \models p \equiv p)$›
using s-*prop* $\&I$ **by** *blast*
AOT-hence ‹$PossibleWorld(s)$›
using *world*:*1*[*THEN* $\equiv_{df} I$] *Situation*.ψ $\&I$ **by** *blast*
AOT-thus ‹$\exists x\ PossibleWorld(x)$›
by (*rule* $\exists I$)
qed

AOT-theorem *world*:*3*: ‹*PossibleWorld*(κ) → κ↓›
proof (*rule* →*I*)
  AOT-assume ‹*PossibleWorld*(κ)›
  AOT-hence ‹*Situation*(κ)›
    using *world*:*1*[*THEN* $\equiv_{df}E$] &*E* **by** *blast*
  AOT-hence ‹*A*!κ›
    by (*metis* $\equiv_{df}E$ &*E*(*1*) *situations*)
  AOT-thus ‹κ↓›
    by (*metis russell−axiom*[*exe*,*1*].ψ*-denotes-asm*)
qed

AOT-theorem *rigid−pw*:*1*: ‹*PossibleWorld*(*x*) ≡ □*PossibleWorld*(*x*)›
proof(*safe intro*!: ≡*I* →*I*)
  AOT-assume ‹*PossibleWorld*(*x*)›
  AOT-hence ‹*Situation*(*x*) & ◇∀ *p*(*x* ⊨ *p* ≡ *p*)›
    using *world*:*1*[*THEN* $\equiv_{df}E$] **by** *blast*
  AOT-hence ‹□*Situation*(*x*) & □◇∀ *p*(*x* ⊨ *p* ≡ *p*)›
    by (*metis S5Basic*:*1* &*I* &*E*(*1*) &*E*(*2*) ≡*E*(*1*) *possit−sit*:*1*)
  AOT-hence *0*: ‹□(*Situation*(*x*) & ◇∀ *p*(*x* ⊨ *p* ≡ *p*))›
    by (*metis KBasic*:*3* ≡*E*(*2*))
  AOT-show ‹□*PossibleWorld*(*x*)›
    by (*AOT-subst* ‹*PossibleWorld*(*x*)› ‹*Situation*(*x*) & ◇∀ *p*(*x* ⊨ *p* ≡ *p*)›)
      (*auto simp*: ≡*Df world*:*1 0*)
next
  AOT-show ‹*PossibleWorld*(*x*)› **if** ‹□*PossibleWorld*(*x*)›
    using *that qml*:*2*[*axiom-inst*, *THEN* →*E*] **by** *blast*
qed

AOT-theorem *rigid−pw*:*2*: ‹◇*PossibleWorld*(*x*) ≡ *PossibleWorld*(*x*)›
  using *rigid−pw*:*1*
  by (*meson RE*◇ *S5Basic*:*2* ≡*E*(*2*) ≡*E*(*6*) *Commutativity of* ≡)

AOT-theorem *rigid−pw*:*3*: ‹◇*PossibleWorld*(*x*) ≡ □*PossibleWorld*(*x*)›
  using *rigid−pw*:*1 rigid−pw*:*2* **by** (*meson* ≡*E*(*5*))

AOT-theorem *rigid−pw*:*4*: ‹***𝒜****PossibleWorld*(*x*) ≡ *PossibleWorld*(*x*)›
  by (*metis Act−Sub*:*3* →*I* ≡*I* ≡*E*(*6*) *nec−imp−act rigid−pw*:*1 rigid−pw*:*2*)

AOT-register-rigid-restricted-type
  PossibleWorld: ‹*PossibleWorld*(κ)›
proof
  AOT-modally-strict {
    AOT-show ‹∃ *x PossibleWorld*(*x*)› **using** *world*:*2*.
  }
next
  AOT-modally-strict {
    AOT-show ‹*PossibleWorld*(κ) → κ↓› **for** κ **using** *world*:*3*.
  }
next
  AOT-modally-strict {
    AOT-show ‹∀ α(*PossibleWorld*(α) → □*PossibleWorld*(α))›
      by (*meson GEN* →*I* ≡*E*(*1*) *rigid−pw*:*1*)
  }
qed
AOT-register-variable-names
  PossibleWorld: *w*

AOT-theorem *world−pos*: ‹*Possible*(*w*)›
proof (*safe intro*!: $\equiv_{df}E$[*OF world*:*1*, *OF PossibleWorld*.ψ, *THEN* &*E*(*1*)]
                pos[*THEN* $\equiv_{df}I$] &*I*)
  AOT-have ‹◇∀ *p* (*w* ⊨ *p* ≡ *p*)›
    using *world*:*1*[*THEN* $\equiv_{df}E$, *OF PossibleWorld*.ψ, *THEN* &*E*(*2*)].

```
  AOT-hence ‹◇∀p (w ⊨ p → p)›
  proof (rule RM◇[THEN →E, rotated]; safe intro!: →I GEN)
    AOT-modally-strict {
      fix p
      AOT-assume ‹∀p (w ⊨ p ≡ p)›
      AOT-hence ‹w ⊨ p ≡ p› using ∀E(2) by blast
      moreover AOT-assume ‹w ⊨ p›
      ultimately AOT-show p using ≡E(1) by blast
    }
  qed
  AOT-hence 0: ‹◇(Situation(w) & ∀p (w ⊨ p → p))›
    using world:1[THEN ≡dfE, OF PossibleWorld.ψ, THEN &E(1),
                THEN possit−sit:1[THEN ≡E(1)]]
    by (metis KBasic:16 &I vdash−properties:10)
  AOT-show ‹◇Actual(w)›
    by (AOT-subst ‹Actual(w)› ‹Situation(w) & ∀p (w ⊨ p → p)›)
      (auto simp: actual ≡Df 0)
qed

AOT-theorem world−cons:1: ‹Consistent(w)›
  using world−pos
  using pos−cons−sit:1[unconstrain s, THEN →E, THEN →E]
  by (meson ≡dfE &E(1) pos)

AOT-theorem world−cons:2: ‹¬TrivialSituation(w)›
proof(rule raa−cor:2)
  AOT-assume ‹TrivialSituation(w)›
  AOT-hence ‹Situation(w) & ∀p w ⊨ p›
    using df−null−trivial:2[THEN ≡dfE] by blast
  AOT-hence 0: ‹□w ⊨ (∃p (p & ¬p))›
    using &E
    by (metis Buridan◇ T◇ &E(2) ≡E(1) lem2:3[unconstrain s, THEN →E]
          log−prop−prop:2 rule−ui:1 universal−cor →E)
  AOT-have ‹◇∀p (w ⊨ p ≡ p)›
    using PossibleWorld.ψ world:1[THEN ≡dfE, THEN &E(2)] by metis
  AOT-hence ‹∀p ◇(w ⊨ p ≡ p)›
    using Buridan◇[THEN →E] by blast
  AOT-hence ‹◇(w ⊨ (∃p (p & ¬p)) ≡ (∃p (p & ¬p)))›
    by (metis log−prop−prop:2 rule−ui:1)
  AOT-hence ‹◇(w ⊨ (∃p (p & ¬p)) → (∃p (p & ¬p)))›
    using RM◇[THEN →E] →I ≡E(1) by meson
  AOT-hence ‹◇(∃p (p & ¬p))› using 0
    by (metis KBasic2:4 ≡E(1) →E)
  moreover AOT-have ‹¬◇(∃p (p & ¬p))›
    by (metis instantiation KBasic2:1 RN ≡E(1) raa−cor:2)
  ultimately AOT-show ‹◇(∃p (p & ¬p)) & ¬◇(∃p (p & ¬p))›
    using &I by blast
qed

AOT-theorem rigid−truth−at:1: ‹w ⊨ p ≡ □w ⊨ p›
  using lem2:1[unconstrain s, THEN →E,
              OF PossibleWorld.ψ[THEN world:1[THEN ≡dfE], THEN &E(1)]].

AOT-theorem rigid−truth−at:2: ‹◇w ⊨ p ≡ w ⊨ p›
  using lem2:2[unconstrain s, THEN →E,
              OF PossibleWorld.ψ[THEN world:1[THEN ≡dfE], THEN &E(1)]].

AOT-theorem rigid−truth−at:3: ‹◇w ⊨ p ≡ □w ⊨ p›
  using lem2:3[unconstrain s, THEN →E,
              OF PossibleWorld.ψ[THEN world:1[THEN ≡dfE], THEN &E(1)]].

AOT-theorem rigid−truth−at:4: ‹𝓐w ⊨ p ≡ w ⊨ p›
  using lem2:4[unconstrain s, THEN →E,
```

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              OF PossibleWorld.ψ[THEN world:1[THEN ≡df E], THEN &E(1)]].

AOT-theorem rigid−truth−at:5: ‹¬w ⊨ p ≡ □¬w ⊨ p›
  using lem2:5[unconstrain s, THEN →E,
              OF PossibleWorld.ψ[THEN world:1[THEN ≡df E], THEN &E(1)]].

AOT-define Maximal :: ‹τ ⇒ φ› (‹Maximal′(-′)›)
  max: ‹Maximal(s) ≡df ∀ p (s ⊨ p ∨ s ⊨ ¬p)›

AOT-theorem world−max: ‹Maximal(w)›
proof(safe intro!: PossibleWorld.ψ[THEN ≡df E[OF world:1], THEN &E(1)]
                GEN ≡df I[OF max] &I )
  fix q
  AOT-have ‹◇(w ⊨ q ∨ w ⊨ ¬q)›
  proof(rule RM◇[THEN →E]; (rule →I)?)
    AOT-modally-strict {
      AOT-assume ‹∀ p (w ⊨ p ≡ p)›
      AOT-hence ‹w ⊨ q ≡ q› and ‹w ⊨ ¬q ≡ ¬q›
        using ∀ E(1)[rotated, OF log−prop−prop:2] by blast+
      AOT-thus ‹w ⊨ q ∨ w ⊨ ¬q›
        by (metis ∨I(1) ∨I(2) ≡E(3) reductio−aa:1)
    }
  next
    AOT-show ‹◇∀ p (w ⊨ p ≡ p)›
      using PossibleWorld.ψ[THEN ≡df E[OF world:1], THEN &E(2)].
  qed
  AOT-hence ‹◇w ⊨ q ∨ ◇w ⊨ ¬q›
    using KBasic2:2[THEN ≡E(1)] by blast
  AOT-thus ‹w ⊨ q ∨ w ⊨ ¬q›
    using lem2:2[unconstrain s, THEN →E, unvarify p,
                OF PossibleWorld.ψ[THEN ≡df E[OF world:1], THEN &E(1)],
                THEN ≡E(1), OF log−prop−prop:2]
    by (metis ∨I(1) ∨I(2) ∨E(3) raa−cor:2)
qed

AOT-theorem world=maxpos:1: ‹Maximal(x) → □Maximal(x)›
proof (AOT-subst ‹Maximal(x)› ‹Situation(x) & ∀ p (x ⊨ p ∨ x ⊨ ¬p)›;
      safe intro!: max ≡Df →I; frule &E(1); drule &E(2))
  AOT-assume sit-x: ‹Situation(x)›
  AOT-hence nec-sit-x: ‹□Situation(x)›
    by (metis ≡E(1) possit−sit:1)
  AOT-assume ‹∀ p (x ⊨ p ∨ x ⊨ ¬p)›
  AOT-hence ‹x ⊨ p ∨ x ⊨ ¬p› for p
    using ∀ E(1)[rotated, OF log−prop−prop:2] by blast
  AOT-hence ‹□x ⊨ p ∨ □x ⊨ ¬p› for p
    using lem2:1[unconstrain s, THEN →E, OF sit-x, unvarify p,
                OF log−prop−prop:2, THEN ≡E(1)]
    by (metis ∨I(1) ∨I(2) ∨E(2) raa−cor:1)
  AOT-hence ‹□(x ⊨ p ∨ x ⊨ ¬p)› for p
    by (metis KBasic:15 →E)
  AOT-hence ‹∀ p □(x ⊨ p ∨ x ⊨ ¬p)›
    by (rule GEN)
  AOT-hence ‹□∀ p (x ⊨ p ∨ x ⊨ ¬p)›
    by (rule BF[THEN →E])
  AOT-thus ‹□(Situation(x) & ∀ p (x ⊨ p ∨ x ⊨ ¬p))›
    using nec-sit-x by (metis KBasic:3 &I ≡E(2))
qed

AOT-theorem world=maxpos:2: ‹PossibleWorld(x) ≡ Maximal(x) & Possible(x)›
proof(safe intro!: ≡I →I &I world−pos[unconstrain w, THEN →E]
                world−max[unconstrain w, THEN →E];
     frule &E(2); drule &E(1))
  AOT-assume pos-x: ‹Possible(x)›
```

```
  AOT-have ‹◇(Situation(x) & ∀p(x ⊨ p → p))›
    apply (AOT-subst (reverse) ‹Situation(x) & ∀p(x ⊨ p → p)› ‹Actual(x)›)
     using actual ≡Df apply presburger
    using ≡df E &E(2) pos pos-x by blast
  AOT-hence 0: ‹◇∀p(x ⊨ p → p)›
    by (metis KBasic2:3 &E(2) vdash−properties:6)
  AOT-assume max-x: ‹Maximal(x)›
  AOT-hence sit-x: ‹Situation(x)› by (metis ≡df E max-x &E(1) max)
  AOT-have ‹□Maximal(x)› using world=maxpos:1[THEN →E, OF max-x] by simp
  moreover AOT-have ‹□Maximal(x) → □(∀p(x ⊨ p → p) → ∀p (x ⊨ p ≡ p))›
  proof(safe intro!: →I RM GEN)
    AOT-modally-strict {
      fix p
      AOT-assume 0: ‹Maximal(x)›
      AOT-assume 1: ‹∀p (x ⊨ p → p)›
      AOT-show ‹x ⊨ p ≡ p›
      proof(safe intro!: ≡I →I 1[THEN ∀E(2), THEN →E]; rule raa−cor:1)
        AOT-assume ‹¬x ⊨ p›
        AOT-hence ‹x ⊨ ¬p›
          using 0[THEN ≡df E[OF max], THEN &E(2), THEN ∀E(2)]
                1 by (metis ∨E(2))
        AOT-hence ‹¬p›
          using 1[THEN ∀E(1), OF log−prop−prop:2, THEN →E] by blast
        moreover AOT-assume p
        ultimately AOT-show ‹p & ¬p› using &I by blast
      qed
    }
  qed
  ultimately AOT-have ‹□(∀p(x ⊨ p → p) → ∀p (x ⊨ p ≡ p))›
    using →E by blast
  AOT-hence ‹◇∀p(x ⊨ p → p) → ◇∀p(x ⊨ p ≡ p)›
    by (metis KBasic:13[THEN →E])
  AOT-hence ‹◇∀p(x ⊨ p ≡ p)›
    using 0 →E by blast
  AOT-thus ‹PossibleWorld(x)›
    using ≡df I[OF world:1, OF &I, OF sit-x] by blast
qed

AOT-define NecImpl :: ‹φ ⇒ φ ⇒ φ› (infixl ‹⇒› 26)
  nec−impl−p:1: ‹p ⇒ q ≡df □(p → q)›
AOT-define NecEquiv :: ‹φ ⇒ φ ⇒ φ› (infixl ‹⇔› 21)
  nec−impl−p:2: ‹p ⇔ q ≡df (p ⇒ q) & (q ⇒ p)›

AOT-theorem nec−equiv−nec−im: ‹p ⇔ q ≡ □(p ≡ q)›
proof(safe intro!: ≡I →I)
  AOT-assume ‹p ⇔ q›
  AOT-hence ‹(p ⇒ q)› and ‹(q ⇒ p)›
    using nec−impl−p:2[THEN ≡df E] &E by blast+
  AOT-hence ‹□(p → q)› and ‹□(q → p)›
    using nec−impl−p:1[THEN ≡df E] by blast+
  AOT-thus ‹□(p ≡ q)› by (metis KBasic:4 &I ≡E(2))
next
  AOT-assume ‹□(p ≡ q)›
  AOT-hence ‹□(p → q)› and ‹□(q → p)›
    using KBasic:4 &E ≡E(1) by blast+
  AOT-hence ‹(p ⇒ q)› and ‹(q ⇒ p)›
    using nec−impl−p:1[THEN ≡df I] by blast+
  AOT-thus ‹p ⇔ q›
    using nec−impl−p:2[THEN ≡df I] &I by blast
qed


AOT-theorem world-closed-lem-1-a:
```

‹$(s \models (\varphi \,\&\, \psi)) \rightarrow (\forall p\ (s \models p \equiv p) \rightarrow (s \models \varphi \,\&\, s \models \psi))$›
proof(*safe intro*!: →*I*)
  AOT-assume ‹$\forall\ p\ (s \models p \equiv p)$›
  AOT-hence ‹$s \models (\varphi \,\&\, \psi) \equiv (\varphi \,\&\, \psi)$› **and** ‹$s \models \varphi \equiv \varphi$› **and** ‹$s \models \psi \equiv \psi$›
    using ∀ *E*(*1*)[*rotated*, *OF log−prop−prop*:*2*] **by** *blast*+
  moreover AOT-assume ‹$s \models (\varphi \,\&\, \psi)$›
  ultimately AOT-show ‹$s \models \varphi \,\&\, s \models \psi$›
    by (*metis* &*I* &*E*(*1*) &*E*(*2*) ≡*E*(*1*) ≡*E*(*2*))
qed

AOT-theorem *world-closed-lem-1-b*:
  ‹$(s \models \varphi \,\&\, (\varphi \rightarrow q)) \rightarrow (\forall p\ (s \models p \equiv p) \rightarrow s \models q)$›
proof(*safe intro*!: →*I*)
  AOT-assume ‹$\forall\ p\ (s \models p \equiv p)$›
  AOT-hence ‹$s \models \varphi \equiv \varphi$› **for** φ
    using ∀ *E*(*1*)[*rotated*, *OF log−prop−prop*:*2*] **by** *blast*
  moreover AOT-assume ‹$s \models \varphi \,\&\, (\varphi \rightarrow q)$›
  ultimately AOT-show ‹$s \models q$›
    by (*metis* &*E*(*1*) &*E*(*2*) ≡*E*(*1*) ≡*E*(*2*) →*E*)
qed

AOT-theorem *world-closed-lem-1-c*:
  ‹$(s \models \varphi \,\&\, s \models (\varphi \rightarrow \psi)) \rightarrow (\forall p\ (s \models p \equiv p) \rightarrow s \models \psi)$›
proof(*safe intro*!: →*I*)
  AOT-assume ‹$\forall\ p\ (s \models p \equiv p)$›
  AOT-hence ‹$s \models \varphi \equiv \varphi$› **for** φ
    using ∀ *E*(*1*)[*rotated*, *OF log−prop−prop*:*2*] **by** *blast*
  moreover AOT-assume ‹$s \models \varphi \,\&\, s \models (\varphi \rightarrow \psi)$›
  ultimately AOT-show ‹$s \models \psi$›
    by (*metis* &*E*(*1*) &*E*(*2*) ≡*E*(*1*) ≡*E*(*2*) →*E*)
qed

AOT-theorem *world−closed−lem*:*1*[*0*]:
  ‹$q \rightarrow (\forall p\ (s \models p \equiv p) \rightarrow s \models q)$›
  by (*meson* →*I* ≡*E*(*2*) *log−prop−prop*:*2 rule−ui*:*1*)

AOT-theorem *world−closed−lem*:*1*[*1*]:
  ‹$s \models p_1 \,\&\, (p_1 \rightarrow q) \rightarrow (\forall p\ (s \models p \equiv p) \rightarrow s \models q)$›
  using *world-closed-lem-1-b*.

AOT-theorem *world−closed−lem*:*1*[*2*]:
  ‹$s \models p_1 \,\&\, s \models p_2 \,\&\, ((p_1 \,\&\, p_2) \rightarrow q) \rightarrow (\forall p\ (s \models p \equiv p) \rightarrow s \models q)$›
  using *world-closed-lem-1-b world-closed-lem-1-a*
  by (*metis* (*full-types*) &*I* &*E* →*I* →*E*)

AOT-theorem *world−closed−lem*:*1*[*3*]:
  ‹$s \models p_1 \,\&\, s \models p_2 \,\&\, s \models p_3 \,\&\, ((p_1 \,\&\, p_2 \,\&\, p_3) \rightarrow q) \rightarrow (\forall p\ (s \models p \equiv p) \rightarrow s \models q)$›
  using *world-closed-lem-1-b world-closed-lem-1-a*
  by (*metis* (*full-types*) &*I* &*E* →*I* →*E*)

AOT-theorem *world−closed−lem*:*1*[*4*]:
  ‹$s \models p_1 \,\&\, s \models p_2 \,\&\, s \models p_3 \,\&\, s \models p_4 \,\&\, ((p_1 \,\&\, p_2 \,\&\, p_3 \,\&\, p_4) \rightarrow q) \rightarrow$
   $(\forall p\ (s \models p \equiv p) \rightarrow s \models q)$›
  using *world-closed-lem-1-b world-closed-lem-1-a*
  by (*metis* (*full-types*) &*I* &*E* →*I* →*E*)

AOT-theorem *coherent*:*1*: ‹$w \models \neg p \equiv \neg w \models p$›
proof(*safe intro*!: ≡*I* →*I*)
  AOT-assume *1*: ‹$w \models \neg p$›
  AOT-show ‹$\neg w \models p$›
  proof(*rule raa−cor*:*2*)
    AOT-assume ‹$w \models p$›
    AOT-hence ‹$w \models p \,\&\, w \models \neg p$› **using** *1* &*I* **by** *blast*

```
    AOT-hence ‹∃ q (w ⊨ q & w ⊨ ¬q)› by (rule ∃ I)
    moreover AOT-have ‹¬∃ q (w ⊨ q & w ⊨ ¬q)›
      using world−cons:1[THEN ≡df E[OF cons], THEN &E(2)].
    ultimately AOT-show ‹∃ q (w ⊨ q & w ⊨ ¬q) & ¬∃ q (w ⊨ q & w ⊨ ¬q)›
      using &I by blast
  qed
next
  AOT-assume ‹¬w ⊨ p›
  AOT-thus ‹w ⊨ ¬p›
    using world−max[THEN ≡df E[OF max], THEN &E(2)]
    by (metis ∨E(2) log−prop−prop:2 rule−ui:1)
qed

AOT-theorem coherent:2: ‹w ⊨ p ≡ ¬w ⊨ ¬p›
  by (metis coherent:1 deduction−theorem ≡I ≡E(1) ≡E(2) raa−cor:3)

AOT-theorem act−world:1: ‹∃ w ∀ p (w ⊨ p ≡ p)›
proof −
  AOT-obtain s where s-prop: ‹∀ p (s ⊨ p ≡ p)›
    using sit−classical:6 Situation.∃ E[rotated] by meson
  AOT-hence ‹◇∀ p (s ⊨ p ≡ p)›
    by (metis T◇ vdash−properties:10)
  AOT-hence ‹PossibleWorld(s)›
    using world:1[THEN ≡df I] Situation.ψ &I by blast
  AOT-hence ‹PossibleWorld(s) & ∀ p (s ⊨ p ≡ p)›
    using &I s-prop by blast
  thus ?thesis by (rule ∃ I)
qed

AOT-theorem act−world:2: ‹∃!w Actual(w)›
proof −
  AOT-obtain w where w-prop: ‹∀ p (w ⊨ p ≡ p)›
    using act−world:1 PossibleWorld.∃ E[rotated] by meson
  AOT-have sit-s: ‹Situation(w)›
    using PossibleWorld.ψ world:1[THEN ≡df E, THEN &E(1)] by blast
  show ?thesis
  proof (safe intro!: uniqueness:1[THEN ≡df I] ∃ I(2) &I GEN →I
                    PossibleWorld.ψ actual[THEN ≡df I] sit-s
                    sit−identity[unconstrain s, unconstrain s′, THEN →E,
                                 THEN →E, THEN ≡E(2)] ≡I
                    w-prop[THEN ∀ E(2), THEN ≡E(1)])
    AOT-show ‹PossibleWorld(w)› using PossibleWorld.ψ.
  next
    AOT-show ‹Situation(w)›
      by (simp add: sit-s)
  next
    fix y p
    AOT-assume w-asm: ‹PossibleWorld(y) & Actual(y)›
    AOT-assume ‹w ⊨ p›
    AOT-hence p: ‹p›
      using w-prop[THEN ∀ E(2), THEN ≡E(1)] by blast
    AOT-show ‹y ⊨ p›
    proof(rule raa−cor:1)
      AOT-assume ‹¬y ⊨ p›
      AOT-hence ‹y ⊨ ¬p›
        by (metis coherent:1[unconstrain w, THEN →E] &E(1) ≡E(2) w-asm)
      AOT-hence ‹¬p›
        using w-asm[THEN &E(2), THEN actual[THEN ≡df E], THEN &E(2),
                    THEN ∀ E(1), rotated, OF log−prop−prop:2]
              →E by blast
      AOT-thus ‹p & ¬p› using p &I by blast
    qed
  next
```

```
    AOT-show ‹w ⊨ p› if ‹y ⊨ p› and ‹PossibleWorld(y) & Actual(y)› for p y
      using that(2)[THEN &E(2), THEN actual[THEN ≡df E], THEN &E(2),
                THEN ∀E(2), THEN →E, OF that(1)]
            w-prop[THEN ∀E(2), THEN ≡E(2)] by blast
  next
    AOT-show ‹Situation(y)› if ‹PossibleWorld(y) & Actual(y)› for y
      using that[THEN &E(1)] world:1[THEN ≡df E, THEN &E(1)] by blast
  next
    AOT-show ‹Situation(w)›
      using sit-s by blast
  qed(simp)
qed

AOT-theorem pre−walpha: ‹ιw Actual(w)↓›
  using A−Exists:2 RA[2] act−world:2 ≡E(2) by blast

AOT-define TheActualWorld :: ‹κs› (‹wα›)
  w−alpha: ‹wα =df ιw Actual(w)›


AOT-theorem true-in-truth-act-true: ‹⊤ ⊨ p ≡ 𝓐p›
proof(safe intro!: ≡I →I)
  AOT-have true-def: ‹⊢□ ⊤ = ιx (A!x & ∀F (x[F] ≡ ∃p(p & F = [λy p])))›
    by (simp add: A−descriptions rule−id−df:1[zero] the−true:1)
  AOT-hence true-den: ‹⊢□ ⊤↓›
    using t=t−proper:1 vdash−properties:6 by blast
  {
    AOT-assume ‹⊤ ⊨ p›
    AOT-hence ‹⊤[λy p]›
      by (metis ≡df E con−dis−i−e:2:b prop−enc true−in−s)
    AOT-hence ‹ιx(A!x & ∀F (x[F] ≡ ∃q (q & F = [λy q])))[λy p]›
      using rule=E true-def true-den by fast
    AOT-hence ‹𝓐∃q (q & [λy p] = [λy q])›
      using ≡E(1) desc−nec−encode:1[unvarify F] prop−prop2:2 by fast
    AOT-hence ‹∃q 𝓐(q & [λy p] = [λy q])›
      by (metis Act−Basic:10 ≡E(1))
    then AOT-obtain q where ‹𝓐(q & [λy p] = [λy q])›
      using ∃E[rotated] by blast
    AOT-hence actq: ‹𝓐q› and ‹𝓐[λy p] = [λy q]›
      using Act−Basic:2 intro−elim:3:a &E by blast+
    AOT-hence ‹[λy p] = [λy q]›
      using id−act:1[unvarify α β, THEN ≡E(2)] prop−prop2:2 by blast
    AOT-hence ‹p = q›
      by (metis intro−elim:3:b p−identity−thm2:3)
    AOT-thus ‹𝓐p›
      using actq rule=E id-sym by blast
  }
  {
    AOT-assume ‹𝓐p›
    AOT-hence ‹𝓐(p & [λy p] = [λy p])›
      by (auto intro!: Act−Basic:2[THEN ≡E(2)] &I
               intro: RA[2] =I(1)[OF prop−prop2:2])
    AOT-hence ‹∃q 𝓐(q & [λy p] = [λy q])›
      using ∃I by fast
    AOT-hence ‹𝓐∃q (q & [λy p] = [λy q])›
      by (metis Act−Basic:10 ≡E(2))
    AOT-hence ‹ιx(A!x & ∀F (x[F] ≡ ∃q (q & F = [λy q])))[λy p]›
      using ≡E(2) desc−nec−encode:1[unvarify F] prop−prop2:2 by fast
    AOT-hence ‹⊤[λy p]›
      using rule=E true-def true-den id-sym by fast
    AOT-thus ‹⊤ ⊨ p›
      by (safe intro!: true−in−s[THEN ≡df I] &I possit−sit:6
                   prop−enc[THEN ≡df I] true-den)
```

```
  }
qed

AOT-theorem T−world: ‹⊤ = w_α›
proof −
  AOT-have true-den: ‹⊢□ ⊤↓›
    using Situation.res−var:3 possit−sit:6 →E by blast
  AOT-have ‹𝒜∀ p (⊤ ⊨ p → p)›
  proof (safe intro!: logic−actual−nec:3[axiom-inst, THEN ≡E(2)] GEN
                      logic−actual−nec:2[axiom-inst, THEN ≡E(2)] →I)
    fix p
    AOT-assume ‹𝒜⊤ ⊨ p›
    AOT-hence ‹⊤ ⊨ p›
      using lem2:4[unconstrain s, unvarify β, OF true-den,
                   THEN →E, OF possit−sit:6] ≡E(1) by blast
    AOT-thus ‹𝒜p› using true-in-truth-act-true ≡E(1) by blast
  qed
  moreover AOT-have ‹𝒜(Situation(κ) & ∀ p (κ ⊨ p → p)) → 𝒜Actual(κ)› for κ
    using actual[THEN ≡Df, THEN conventions:3[THEN ≡_df E, THEN &E(2)],
              THEN RA[2], THEN act−cond[THEN →E]].
  ultimately AOT-have act-act-true: ‹𝒜Actual(⊤)›
    using possit−sit:4[unvarify x, OF true-den, THEN ≡E(2), OF possit−sit:6]
          Act−Basic:2[THEN ≡E(2), OF &I] →E by blast
  AOT-hence ‹◊Actual(⊤)› by (metis Act−Sub:3 vdash−properties:10)
  AOT-hence ‹Possible(⊤)›
    by (safe intro!: pos[THEN ≡_df I] &I possit−sit:6)
  moreover AOT-have ‹Maximal(⊤)›
  proof (safe intro!: max[THEN ≡_df I] &I possit−sit:6 GEN)
    fix p
    AOT-have ‹𝒜p ∨ 𝒜¬p›
      by (simp add: Act−Basic:1)
    moreover AOT-have ‹⊤ ⊨ p› if ‹𝒜p›
        using that true-in-truth-act-true[THEN ≡E(2)] by blast
    moreover AOT-have ‹⊤ ⊨ ¬p› if ‹𝒜¬p›
      using that true-in-truth-act-true[unvarify p, THEN ≡E(2)]
            log−prop−prop:2 by blast
    ultimately AOT-show ‹⊤ ⊨ p ∨ ⊤ ⊨ ¬p›
      using ∨I(3) →I by blast
  qed
  ultimately AOT-have ‹PossibleWorld(⊤)›
    by (safe intro!: world=maxpos:2[unvarify x, OF true-den, THEN ≡E(2)] &I)
  AOT-hence ‹𝒜PossibleWorld(⊤)›
    using rigid−pw:4[unvarify x, OF true-den, THEN ≡E(2)] by blast
  AOT-hence 1: ‹𝒜(PossibleWorld(⊤) & Actual(⊤))›
    using act-act-true Act−Basic:2 df−simplify:2 intro−elim:3:b by blast
  AOT-have ‹w_α = ιw(Actual(w))›
    using rule−id−df:1[zero][OF w−alpha, OF pre−walpha] by simp
  moreover AOT-have w-act-den: ‹w_α↓›
    using calculation t=t−proper:1 →E by blast
  ultimately AOT-have ‹∀ z (𝒜(PossibleWorld(z) & Actual(z)) → z = w_α)›
    using nec−hintikka−scheme[unvarify x] ≡E(1) &E by blast
  AOT-thus ‹⊤ = w_α›
    using ∀ E(1)[rotated, OF true-den] 1 →E by blast
qed

AOT-act-theorem truth−at−alpha:1: ‹p ≡ w_α = ιx (ExtensionOf(x, p))›
  by (metis rule=E T−world deduction−theorem ext−p−tv:3 id-sym ≡I
          ≡E(1) ≡E(2) q−True:1)

AOT-act-theorem truth−at−alpha:2: ‹p ≡ w_α ⊨ p›
proof −
  AOT-have ‹PossibleWorld(w_α)›
    using &E(1) pre−walpha rule−id−df:2:b[zero] →E
```

```
      w−alpha y−in:3 by blast
  AOT-hence sit-w-alpha: ‹Situation(wα)›
   by (metis ≡df E &E(1) world:1)
  AOT-have w-alpha-den: ‹wα↓›
   using pre−walpha rule−id−df:2:b[zero] w−alpha by blast
  AOT-have ‹p ≡ ⊤Σp›
   using q−True:3 by force
  moreover AOT-have ‹⊤ = wα›
   using T−world by auto
  ultimately AOT-have ‹p ≡ wαΣp›
   using rule=E by fast
  moreover AOT-have ‹wα Σ p ≡ wα ⊨ p›
   using lem1[unvarify x, OF w-alpha-den, THEN →E, OF sit-w-alpha]
   using ≡S(1) ≡E(1) Commutativity of ≡ ≡Df sit-w-alpha true−in−s by blast
  ultimately AOT-show ‹p ≡ wα ⊨ p›
   by (metis ≡E(5))
qed

AOT-theorem alpha−world:1: ‹PossibleWorld(wα)›
proof −
  AOT-have 0: ‹wα = ιw Actual(w)›
   using pre−walpha rule−id−df:1[zero] w−alpha by blast
  AOT-hence walpha-den: ‹wα↓›
   by (metis t=t−proper:1 vdash−properties:6)
  AOT-have ‹𝒜(PossibleWorld(wα) & Actual(wα))›
   by (rule actual−desc:2[unvarify x, OF walpha-den, THEN →E]) (fact 0)
  AOT-hence ‹𝒜PossibleWorld(wα)›
   by (metis Act−Basic:2 &E(1) ≡E(1))
  AOT-thus ‹PossibleWorld(wα)›
   using rigid−pw:4[unvarify x, OF walpha-den, THEN ≡E(1)]
   by blast
qed

AOT-theorem alpha−world:2: ‹Maximal(wα)›
proof −
  AOT-have ‹wα↓›
   using pre−walpha rule−id−df:2:b[zero] w−alpha by blast
  then AOT-obtain x where x-def: ‹x = wα›
   by (metis instantiation rule=I:1 existential:1 id-sym)
  AOT-hence ‹PossibleWorld(x)› using alpha−world:1 rule=E id-sym by fast
  AOT-hence ‹Maximal(x)› by (metis world−max[unconstrain w, THEN →E])
  AOT-thus ‹Maximal(wα)› using x-def rule=E by blast
qed

AOT-theorem t−at−alpha−strict: ‹wα ⊨ p ≡ 𝒜p›
proof −
  AOT-have 0: ‹wα = ιw Actual(w)›
   using pre−walpha rule−id−df:1[zero] w−alpha by blast
  AOT-hence walpha-den: ‹wα↓›
   by (metis t=t−proper:1 vdash−properties:6)
  AOT-have 1: ‹𝒜(PossibleWorld(wα) & Actual(wα))›
   by (rule actual−desc:2[unvarify x, OF walpha-den, THEN →E]) (fact 0)
  AOT-have walpha-sit: ‹Situation(wα)›
   by (meson ≡df E alpha−world:2 &E(1) max)
  {
   fix p
   AOT-have 2: ‹Situation(x) → (𝒜x ⊨ p ≡ x ⊨ p)› for x
    using lem2:4[unconstrain s] by blast
   AOT-assume ‹wα ⊨ p›
   AOT-hence θ: ‹𝒜wα ⊨ p›
    using 2[unvarify x, OF walpha-den, THEN →E, OF walpha-sit, THEN ≡E(2)]
    by argo
   AOT-have 3: ‹𝒜Actual(wα)›
```

```
      using 1 Act−Basic:2 &E(2) ≡E(1) by blast
    AOT-have ‹𝓐(Situation(wα) & ∀ q(wα |= q → q))›
      apply (AOT-subst (reverse) ‹Situation(wα) & ∀ q(wα |= q → q)› ‹Actual(wα)›)
       using actual ≡Df apply blast
      by (fact 3)
    AOT-hence ‹𝓐∀ q(wα |= q → q)› by (metis Act−Basic:2 &E(2) ≡E(1))
    AOT-hence ‹∀ q 𝓐(wα |= q → q)›
      using logic−actual−nec:3[axiom-inst, THEN ≡E(1)] by blast
    AOT-hence ‹𝓐(wα |= p → p)› using ∀ E(2) by blast
    AOT-hence ‹𝓐(wα |= p) → 𝓐p› by (metis act−cond vdash−properties:10)
    AOT-hence ‹𝓐p› using ϑ →E by blast
  }
  AOT-hence 2: ‹wα |= p → 𝓐p› for p by (rule →I)
  AOT-have walpha-sit: ‹Situation(wα)›
    using ≡df E alpha−world:2 &E(1) max by blast
  show ?thesis
  proof(safe intro!: ≡I →I 2)
    AOT-assume actp: ‹𝓐p›
    AOT-show ‹wα |= p›
    proof(rule raa−cor:1)
      AOT-assume ‹¬wα |= p›
      AOT-hence ‹wα |= ¬p›
        using alpha−world:2[THEN max[THEN ≡df E], THEN &E(2),
                              THEN ∀ E(1), OF log−prop−prop:2]
        by (metis ∨E(2))
      AOT-hence ‹𝓐¬p›
        using 2[unvarify p, OF log−prop−prop:2, THEN →E] by blast
      AOT-hence ‹¬𝓐p› by (metis ¬¬I Act−Sub:1 ≡E(4))
      AOT-thus ‹𝓐p & ¬𝓐p› using actp &I by blast
    qed
  qed
qed

AOT-act-theorem not−act: ‹w ≠ wα → ¬Actual(w)›
proof (rule →I; rule raa−cor:2)
  AOT-assume ‹w ≠ wα›
  AOT-hence 0: ‹¬(w = wα)› by (metis ≡df E =−infix)
  AOT-have walpha-den: ‹wα↓›
    using pre−walpha rule−id−df:2:b[zero] w−alpha by blast
  AOT-have walpha-sit: ‹Situation(wα)›
    using ≡df E alpha−world:2 &E(1) max by blast
  AOT-assume act-w: ‹Actual(w)›
  AOT-hence w-sit: ‹Situation(w)› by (metis ≡df E actual &E(1))
  AOT-have sid: ‹Situation(x′) → (w = x′ ≡ ∀ p (w |= p ≡ x′ |= p))› for x′
    using sit−identity[unconstrain s′, unconstrain s, THEN →E, OF w-sit]
    by blast
  AOT-have ‹w = wα›
  proof(safe intro!: GEN sid[unvarify x′, OF walpha-den, THEN →E,
                           OF walpha-sit, THEN ≡E(2)] ≡I →I)
    fix p
    AOT-assume ‹w |= p›
    AOT-hence ‹p›
      using actual[THEN ≡df E, OF act-w, THEN &E(2), THEN ∀ E(2), THEN →E]
      by blast
    AOT-hence ‹𝓐p›
      by (metis RA[1])
    AOT-thus ‹wα |= p›
      using t−at−alpha−strict[THEN ≡E(2)] by blast
  next
    fix p
    AOT-assume ‹wα |= p›
    AOT-hence ‹𝓐p›
      using t−at−alpha−strict[THEN ≡E(1)] by blast
```

    AOT-hence *p*: ‹*p*›
      using *logic−actual*[*act-axiom-inst*, *THEN* →*E*] **by** *blast*
    AOT-show ‹*w* ⊨ *p*›
    proof(*rule raa−cor*:*1*)
      AOT-assume ‹¬*w* ⊨ *p*›
      AOT-hence ‹*w* ⊨ ¬*p*›
        by (*metis coherent*:*1* ≡*E*(*2*))
      AOT-hence ‹¬*p*›
        using *actual*[*THEN* $\equiv_{df}$*E*, *OF act-w*, *THEN* &*E*(*2*), *THEN* ∀ *E*(*1*),
                  OF log−prop−prop:*2*, *THEN* →*E*] **by** *blast*
      AOT-thus ‹*p* & ¬*p*› **using** *p* &*I* **by** *blast*
    qed
  qed
  AOT-thus ‹$w = \mathbf{w}_\alpha$ & ¬($w = \mathbf{w}_\alpha$)› **using** *0* &*I* **by** *blast*
qed

AOT-act-theorem *w−alpha−part*: ‹*Actual*(*s*) ≡ $s \trianglelefteq \mathbf{w}_\alpha$›
proof(*safe intro*!: ≡*I* →*I sit−part−whole*[*THEN* $\equiv_{df}$*I*] &*I GEN*
          dest!: *sit−part−whole*[*THEN* $\equiv_{df}$*E*])
  AOT-show ‹*Situation*(*s*)› **if** ‹*Actual*(*s*)›
    using $\equiv_{df}$*E actual* &*E*(*1*) *that* **by** *blast*
next
  AOT-show ‹*Situation*($\mathbf{w}_\alpha$)›
    using $\equiv_{df}$*E alpha−world*:*2* &*E*(*1*) *max* **by** *blast*
next
  fix *p*
  AOT-assume ‹*Actual*(*s*)›
  moreover AOT-assume ‹*s* ⊨ *p*›
  ultimately AOT-have ‹*p*›
    using *actual*[*THEN* $\equiv_{df}$*E*, *THEN* &*E*(*2*), *THEN* ∀ *E*(*2*), *THEN* →*E*] **by** *blast*
  AOT-thus ‹$\mathbf{w}_\alpha$ ⊨ *p*›
     by (*metis* ≡*E*(*1*) *truth−at−alpha*:*2*)
next
  AOT-assume *0*: ‹*Situation*(*s*) & *Situation*($\mathbf{w}_\alpha$) & ∀ *p* (*s* ⊨ *p* → $\mathbf{w}_\alpha$ ⊨ *p*)›
  AOT-hence ‹*s* ⊨ *p* → $\mathbf{w}_\alpha$ ⊨ *p*› **for** *p*
    using &*E* ∀ *E*(*2*) **by** *blast*
  AOT-hence ‹*s* ⊨ *p* → *p*› **for** *p*
    by (*metis* →*I* ≡*E*(*2*) *truth−at−alpha*:*2* →*E*)
  AOT-hence ‹∀ *p* (*s* ⊨ *p* → *p*)› **by** (*rule GEN*)
  AOT-thus ‹*Actual*(*s*)›
    using *actual*[*THEN* $\equiv_{df}$*I*, *OF* &*I*, *OF 0*[*THEN* &*E*(*1*), *THEN* &*E*(*1*)]] **by** *blast*
qed

AOT-act-theorem *act−world2*:*1*: ‹$\mathbf{w}_\alpha$ ⊨ *p* ≡ [λ*y p*]$\mathbf{w}_\alpha$›
  apply (*AOT-subst* ‹[λ*y p*]$\mathbf{w}_\alpha$› *p*)
   apply (*rule beta−C−meta*[*THEN* →*E*, *OF prop−prop2*:*2*, *unvarify* $\nu_1\nu_n$])
  using *pre−walpha rule−id−df*:*2*:*b*[*zero*] *w−alpha* **apply** *blast*
  using ≡*E*(*2*) *Commutativity of* ≡ *truth−at−alpha*:*2* **by** *blast*

AOT-act-theorem *act−world2*:*2*: ‹*p* ≡ $\mathbf{w}_\alpha$ ⊨ [λ*y p*]$\mathbf{w}_\alpha$›
proof −
  AOT-have ‹*p* ≡ [λ*y p*]$\mathbf{w}_\alpha$›
    apply (*rule beta−C−meta*[*THEN* →*E*, *OF prop−prop2*:*2*,
                    unvarify $\nu_1\nu_n$, *symmetric*])
    using *pre−walpha rule−id−df*:*2*:*b*[*zero*] *w−alpha* **by** *blast*
  also AOT-have ‹. . . ≡ $\mathbf{w}_\alpha$ ⊨ [λ*y p*]$\mathbf{w}_\alpha$›
    by (*meson log−prop−prop*:*2 rule−ui*:*1 truth−at−alpha*:*2 universal−cor*)
  finally show *?thesis***.**
qed

AOT-theorem *fund−lem*:*1*: ‹◇*p* → ◇∃ *w* (*w* ⊨ *p*)›
proof (*rule RM*◇; *rule* →*I*; *rule raa−cor*:*1*)
  AOT-modally-strict {

```
    AOT-obtain w where w-prop: ‹∀ q (w ⊨ q ≡ q)›
      using act−world:1 PossibleWorld.∃ E[rotated] by meson
    AOT-assume p: ‹p›
    AOT-assume 0: ‹¬∃ w (w ⊨ p)›
    AOT-have ‹∀ w ¬(w ⊨ p)›
      apply (AOT-subst ‹PossibleWorld(x) → ¬x ⊨ p›
                ‹¬(PossibleWorld(x) & x ⊨ p)› for: x)
      apply (metis &I &E(1) &E(2) →I ≡I modus−tollens:2)
      using 0 cqt−further:4 vdash−properties:10 by blast
    AOT-hence ‹¬(w ⊨ p)›
      using PossibleWorld.ψ rule−ui:3 →E by blast
    AOT-hence ‹¬p›
      using w-prop[THEN ∀ E(2), THEN ≡E(2)]
      by (metis raa−cor:3)
    AOT-thus ‹p & ¬p›
      using p &I by blast
  }
qed

AOT-theorem fund−lem:2: ‹◇∃ w (w ⊨ p) → ∃ w (w ⊨ p)›
proof (rule →I)
  AOT-assume ‹◇∃ w (w ⊨ p)›
  AOT-hence ‹∃ w ◇(w ⊨ p)›
    using PossibleWorld.res−var−bound−reas[BF◇][THEN →E] by auto
  then AOT-obtain w where ‹◇(w ⊨ p)›
    using PossibleWorld.∃ E[rotated] by meson
  moreover AOT-have ‹Situation(w)›
    by (metis ≡df E &E(1) pos world−pos)
  ultimately AOT-have ‹w ⊨ p›
    using lem2:2[unconstrain s, THEN →E] ≡E by blast
  AOT-thus ‹∃ w w ⊨ p›
    by (rule PossibleWorld.∃ I)
qed

AOT-theorem fund−lem:3: ‹p → ∀ s(∀ q (s ⊨ q ≡ q) → s ⊨ p)›
proof(safe intro!: →I Situation.GEN)
  fix s
  AOT-assume ‹p›
  moreover AOT-assume ‹∀ q (s ⊨ q ≡ q)›
  ultimately AOT-show ‹s ⊨ p›
    using ∀ E(2) ≡E(2) by blast
qed

AOT-theorem fund−lem:4: ‹□p → □∀ s(∀ q (s ⊨ q ≡ q) → s ⊨ p)›
  using fund−lem:3 by (rule RM)

AOT-theorem fund−lem:5: ‹□∀ s φ{s} → ∀ s □φ{s}›
proof(safe intro!: →I Situation.GEN)
  fix s
  AOT-assume ‹□∀ s φ{s}›
  AOT-hence ‹∀ s □φ{s}›
    using Situation.res−var−bound−reas[CBF][THEN →E] by blast
  AOT-thus ‹□φ{s}›
    using Situation.∀ E by fast
qed
```

Note: not explicit in PLM.

```
AOT-theorem fund−lem:5[world]: ‹□∀ w φ{w} → ∀ w □φ{w}›
proof(safe intro!: →I PossibleWorld.GEN)
  fix w
  AOT-assume ‹□∀ w φ{w}›
  AOT-hence ‹∀ w □φ{w}›
    using PossibleWorld.res−var−bound−reas[CBF][THEN →E] by blast
```

```
  AOT-thus ‹□φ{w}›
    using PossibleWorld.∀E by fast
qed

AOT-theorem fund-lem:6: ‹∀w w ⊨ p → □∀w w ⊨ p›
proof(rule →I)
  AOT-assume ‹∀w (w ⊨ p)›
  AOT-hence 1: ‹PossibleWorld(w) → (w ⊨ p)› for w
    using ∀E(2) by blast
  AOT-show ‹□∀w w ⊨ p›
  proof(rule raa-cor:1)
    AOT-assume ‹¬□∀w w ⊨ p›
    AOT-hence ‹◇¬∀w w ⊨ p›
      by (metis KBasic:11 ≡E(1))
    AOT-hence ‹◇∃x (¬(PossibleWorld(x) → x ⊨ p))›
      apply (rule RM◇[THEN →E, rotated])
      by (simp add: cqt-further:2)
    AOT-hence ‹∃x ◇(¬(PossibleWorld(x) → x ⊨ p))›
      by (metis BF◇ vdash-properties:10)
    then AOT-obtain x where x-prop: ‹◇¬(PossibleWorld(x) → x ⊨ p)›
      using ∃E[rotated] by blast
    AOT-have ‹◇(PossibleWorld(x) & ¬x ⊨ p)›
      apply (AOT-subst ‹PossibleWorld(x) & ¬x ⊨ p›
                  ‹¬(PossibleWorld(x) → x ⊨ p)›)
       apply (meson ≡E(6) oth-class-taut:1:b oth-class-taut:3:a)
      by(fact x-prop)
    AOT-hence 2: ‹◇PossibleWorld(x) & ◇¬x ⊨ p›
      by (metis KBasic2:3 vdash-properties:10)
    AOT-hence ‹PossibleWorld(x)›
      using &E(1) ≡E(1) rigid-pw:2 by blast
    AOT-hence ‹□(x ⊨ p)›
      using 2[THEN &E(2)]  1[unconstrain w, THEN →E] →E
            rigid-truth-at:1[unconstrain w, THEN →E]
      by (metis ≡E(1))
    moreover AOT-have ‹¬□(x ⊨ p)›
      using 2[THEN &E(2)] by (metis ¬¬I KBasic:12 ≡E(4))
    ultimately AOT-show ‹p & ¬p› for p
      by (metis raa-cor:3)
  qed
qed

AOT-theorem fund-lem:7: ‹□∀w(w ⊨ p) → □p›
proof(rule RM; rule →I)
  AOT-modally-strict {
    AOT-obtain w where w-prop: ‹∀p (w ⊨ p ≡ p)›
      using act-world:1 PossibleWorld.∃E[rotated] by meson
    AOT-assume ‹∀w (w ⊨ p)›
    AOT-hence ‹w ⊨ p›
      using PossibleWorld.∀E by fast
    AOT-thus ‹p›
      using w-prop[THEN ∀E(2), THEN ≡E(1)] by blast
  }
qed

AOT-theorem fund:1: ‹◇p ≡ ∃w w ⊨ p›
proof (rule ≡I; rule →I)
  AOT-assume ‹◇p›
  AOT-thus ‹∃w w ⊨ p›
    by (metis fund-lem:1 fund-lem:2 →E)
next
  AOT-assume ‹∃w w ⊨ p›
  then AOT-obtain w where w-prop: ‹w ⊨ p›
    using PossibleWorld.∃E[rotated] by meson
```

AOT-hence ‹$\Diamond\forall p\ (w \models p \equiv p)$›
  using *world:1[THEN* $\equiv_{df}E$, *THEN* $\&E(2)$*] PossibleWorld.ψ* $\&E$ **by** *blast*
AOT-hence ‹$\forall p\ \Diamond(w \models p \equiv p)$›
  by (*metis Buridan*$\Diamond$ $\rightarrow E$)
AOT-hence *1*: ‹$\Diamond(w \models p \equiv p)$›
  by (*metis log−prop−prop:2 rule−ui:1*)
AOT-have ‹$\Diamond((w \models p \rightarrow p)\ \&\ (p \rightarrow w \models p))$›
  apply (*AOT-subst* ‹$(w \models p \rightarrow p)\ \&\ (p \rightarrow w \models p)$› ‹$w \models p \equiv p$›)
   apply (*meson conventions:3* $\equiv E(6)$ *oth−class−taut:3:a* $\equiv Df$)
  by (*fact 1*)
AOT-hence ‹$\Diamond(w \models p \rightarrow p)$›
  by (*metis RM*$\Diamond$ *Conjunction Simplification(1)* $\rightarrow E$)
moreover AOT-have ‹$\Box(w \models p)$›
  using *w-prop* **by** (*metis* $\equiv E(1)$ *rigid−truth−at:1*)
ultimately AOT-show ‹$\Diamond p$›
  by (*metis KBasic2:4* $\equiv E(1)$ $\rightarrow E$)
qed

AOT-theorem *fund:2*: ‹$\Box p \equiv \forall w\ (w \models p)$›
proof −
  AOT-have *0*: ‹$\forall w\ (w \models \neg p \equiv \neg w \models p)$›
    apply (*rule PossibleWorld.GEN*)
    using *coherent:1* **by** *blast*
  AOT-have ‹$\Diamond\neg p \equiv \exists w\ (w \models \neg p)$›
    using *fund:1[unvarify p, OF log−prop−prop:2]* **by** *blast*
  also AOT-have ‹$\ldots \equiv \exists w\ \neg(w \models p)$›
  proof(*safe intro*!: $\equiv I$ $\rightarrow I$)
    AOT-assume ‹$\exists w\ w \models \neg p$›
    then AOT-obtain *w* **where** *w-prop*: ‹$w \models \neg p$›
      using *PossibleWorld.∃E[rotated]* **by** *meson*
    AOT-hence ‹$\neg w \models p$›
      using *0[THEN PossibleWorld.∀E, THEN* $\equiv E(1)$*]* $\&E$ **by** *blast*
    AOT-thus ‹$\exists w\ \neg w \models p$›
      by (*rule PossibleWorld.∃I*)
  next
    AOT-assume ‹$\exists w\ \neg w \models p$›
    then AOT-obtain *w* **where** *w-prop*: ‹$\neg w \models p$›
      using *PossibleWorld.∃E[rotated]* **by** *meson*
    AOT-hence ‹$w \models \neg p$›
      using *0[THEN* $\forall E(2)$*, THEN* $\rightarrow E$*, THEN* $\equiv E(1)$*]* $\&E$
      by (*metis coherent:1* $\equiv E(2)$)
    AOT-thus ‹$\exists w\ w \models \neg p$›
      by (*rule PossibleWorld.∃I*)
  qed
  finally AOT-have ‹$\neg\Diamond\neg p \equiv \neg\exists w\ \neg w \models p$›
    by (*meson* $\equiv E(1)$ *oth−class−taut:4:b*)
  AOT-hence ‹$\Box p \equiv \neg\exists w\ \neg w \models p$›
    by (*metis KBasic:12* $\equiv E(5)$)
  also AOT-have ‹$\ldots \equiv \forall w\ w \models p$›
  proof(*safe intro*!: $\equiv I$ $\rightarrow I$)
    AOT-assume ‹$\neg\exists w\ \neg w \models p$›
    AOT-hence *0*: ‹$\forall x\ (\neg(PossibleWorld(x)\ \&\ \neg x \models p))$›
      by (*metis cqt−further:4* $\rightarrow E$)
    AOT-show ‹$\forall w\ w \models p$›
      apply (*AOT-subst* ‹$PossibleWorld(x) \rightarrow x \models p$›
                ‹$\neg(PossibleWorld(x)\ \&\ \neg x \models p)$› **for**: *x*)
       using *oth−class−taut:1:a* **apply** *presburger*
      by (*fact 0*)
  next
    AOT-assume *0*: ‹$\forall w\ w \models p$›
    AOT-have ‹$\forall x\ (\neg(PossibleWorld(x)\ \&\ \neg x \models p))$›
      by (*AOT-subst (reverse)* ‹$\neg(PossibleWorld(x)\ \&\ \neg x \models p)$›
                ‹$PossibleWorld(x) \rightarrow x \models p$› **for**: *x*)

(*auto simp*: *oth−class−taut*:*1*:*a 0*)
AOT-thus ‹¬∃ *w* ¬*w* ⊨ *p*›
by (*metis* ∃ *E raa−cor*:*3 rule−ui*:*3*)
qed
finally AOT-show ‹□*p* ≡ ∀ *w w* ⊨ *p*›.
qed

AOT-theorem *fund*:*3*: ‹¬◊*p* ≡ ¬∃ *w w* ⊨ *p*›
by (*metis* (*full-types*) *contraposition*:*1*[*1*] →*I fund*:*1* ≡*I* ≡*E*(*1*,*2*))

AOT-theorem *fund*:*4*: ‹¬□*p* ≡ ∃ *w* ¬*w* ⊨*p*›
apply (*AOT-subst* ‹∃ *w* ¬*w* ⊨ *p*› ‹¬ ∀ *w w* ⊨ *p*›)
apply (*AOT-subst* ‹*PossibleWorld*(*x*) → *x* ⊨ *p*›
‹¬(*PossibleWorld*(*x*) & ¬*x* ⊨ *p*)› **for**: *x*)
by (*auto simp add*: *oth−class−taut*:*1*:*a conventions*:*4* ≡*Df RN*
fund:*2 rule−sub−lem*:*1*:*a*)

AOT-theorem *nec−dia−w*:*1*: ‹□*p* ≡ ∃ *w w* ⊨ □*p*›
proof −
AOT-have ‹□*p* ≡ ◊□*p*›
using *S5Basic*:*2* **by** *blast*
also AOT-have ‹... ≡ ∃ *w w* ⊨ □*p*›
using *fund*:*1*[*unvarify p*, *OF log−prop−prop*:*2*] **by** *blast*
finally show *?thesis*.
qed

AOT-theorem *nec−dia−w*:*2*: ‹□*p* ≡ ∀ *w w* ⊨ □*p*›
proof −
AOT-have ‹□*p* ≡ □□*p*›
using *4 qml*:*2*[*axiom-inst*] ≡*I* **by** *blast*
also AOT-have ‹... ≡ ∀ *w w* ⊨ □*p*›
using *fund*:*2*[*unvarify p*, *OF log−prop−prop*:*2*] **by** *blast*
finally show *?thesis*.
qed

AOT-theorem *nec−dia−w*:*3*: ‹◊*p* ≡ ∃ *w w* ⊨ ◊*p*›
proof −
AOT-have ‹◊*p* ≡ ◊◊*p*›
by (*simp add*: *4*◊ *T*◊ ≡*I*)
also AOT-have ‹... ≡ ∃ *w w* ⊨ ◊*p*›
using *fund*:*1*[*unvarify p*, *OF log−prop−prop*:*2*] **by** *blast*
finally show *?thesis*.
qed

AOT-theorem *nec−dia−w*:*4*: ‹◊*p* ≡ ∀ *w w* ⊨ ◊*p*›
proof −
AOT-have ‹◊*p* ≡ □◊*p*›
by (*simp add*: *S5Basic*:*1*)
also AOT-have ‹... ≡ ∀ *w w* ⊨ ◊*p*›
using *fund*:*2*[*unvarify p*, *OF log−prop−prop*:*2*] **by** *blast*
finally show *?thesis*.
qed

AOT-theorem *conj−dist−w*:*1*: ‹*w* ⊨ (*p* & *q*) ≡ ((*w* ⊨ *p*) & (*w* ⊨ *q*))›
proof(*safe intro*!: ≡*I* →*I*)
AOT-assume ‹*w* ⊨ (*p* & *q*)›
AOT-hence *0*: ‹□*w* ⊨ (*p* & *q*)›
using *rigid−truth−at*:*1*[*unvarify p*, *THEN* ≡*E*(*1*), *OF log−prop−prop*:*2*]
by *blast*
AOT-modally-strict {
AOT-have ‹∀ *p* (*w* ⊨ *p* ≡ *p*) → ((*w* ⊨ (φ & ψ)) → (*w* ⊨ φ & *w* ⊨ ψ))› **for** *w* φ ψ
proof(*safe intro*!: →*I*)
AOT-assume ‹∀ *p* (*w* ⊨ *p* ≡ *p*)›

AOT-hence ‹$w \models (\varphi \ \& \ \psi) \equiv (\varphi \ \& \ \psi)$› **and** ‹$w \models \varphi \equiv \varphi$› **and** ‹$w \models \psi \equiv \psi$›
using $\forall E(1)$[*rotated*, *OF log−prop−prop*:*2*] **by** *blast+*
moreover AOT-assume ‹$w \models (\varphi \ \& \ \psi)$›
ultimately AOT-show ‹$w \models \varphi \ \& \ w \models \psi$›
by (*metis* &*I* &*E*(*1*) &*E*(*2*) ≡*E*(*1*) ≡*E*(*2*))
qed
}
AOT-hence ‹$\Diamond\forall p \ (w \models p \equiv p) \rightarrow \Diamond(w \models (\varphi \ \& \ \psi) \rightarrow w \models \varphi \ \& \ w \models \psi)$› **for** $w \ \varphi \ \psi$
by (*rule RM*◇)
moreover AOT-have *pos*: ‹$\Diamond\forall p \ (w \models p \equiv p)$›
using *world*:*1*[*THEN* $\equiv_{df}E$, *OF PossibleWorld*.ψ] &*E* **by** *blast*
ultimately AOT-have ‹$\Diamond(w \models (p \ \& \ q) \rightarrow w \models p \ \& \ w \models q)$› **using** →*E* **by** *blast*
AOT-hence ‹$\Diamond(w \models p) \ \& \ \Diamond(w \models q)$›
by (*metis 0 KBasic2*:*3 KBasic2*:*4* ≡*E*(*1*) *vdash−properties*:*10*)
AOT-thus ‹$w \models p \ \& \ w \models q$›
using *rigid−truth−at*:*2*[*unvarify p*, *THEN* ≡*E*(*1*), *OF log−prop−prop*:*2*]
&*E* &*I* **by** *meson*
next
AOT-assume ‹$w \models p \ \& \ w \models q$›
AOT-hence ‹$\Box w \models p \ \& \ \Box w \models q$›
using *rigid−truth−at*:*1*[*unvarify p*, *THEN* ≡*E*(*1*), *OF log−prop−prop*:*2*]
&*E* &*I* **by** *blast*
AOT-hence *0*: ‹$\Box(w \models p \ \& \ w \models q)$›
by (*metis KBasic*:*3* ≡*E*(*2*))
AOT-modally-strict {
AOT-have ‹$\forall p \ (w \models p \equiv p) \rightarrow ((w \models \varphi \ \& \ w \models \psi) \rightarrow (w \models (\varphi \ \& \ \psi)))$› **for** $w \ \varphi \ \psi$
proof(*safe intro*!: →*I*)
AOT-assume ‹$\forall \ p \ (w \models p \equiv p)$›
AOT-hence ‹$w \models (\varphi \ \& \ \psi) \equiv (\varphi \ \& \ \psi)$› **and** ‹$w \models \varphi \equiv \varphi$› **and** ‹$w \models \psi \equiv \psi$›
using $\forall E(1)$[*rotated*, *OF log−prop−prop*:*2*] **by** *blast+*
moreover AOT-assume ‹$w \models \varphi \ \& \ w \models \psi$›
ultimately AOT-show ‹$w \models (\varphi \ \& \ \psi)$›
by (*metis* &*I* &*E*(*1*) &*E*(*2*) ≡*E*(*1*) ≡*E*(*2*))
qed
}
AOT-hence ‹$\Diamond\forall p \ (w \models p \equiv p) \rightarrow \Diamond((w \models \varphi \ \& \ w \models \psi) \rightarrow w \models (\varphi \ \& \ \psi))$› **for** $w \ \varphi \ \psi$
by (*rule RM*◇)
moreover AOT-have *pos*: ‹$\Diamond\forall p \ (w \models p \equiv p)$›
using *world*:*1*[*THEN* $\equiv_{df}E$, *OF PossibleWorld*.ψ] &*E* **by** *blast*
ultimately AOT-have ‹$\Diamond((w \models p \ \& \ w \models q) \rightarrow w \models (p \ \& \ q))$›
using →*E* **by** *blast*
AOT-hence ‹$\Diamond(w \models (p \ \& \ q))$›
by (*metis 0 KBasic2*:*4* ≡*E*(*1*) *vdash−properties*:*10*)
AOT-thus ‹$w \models (p \ \& \ q)$›
using *rigid−truth−at*:*2*[*unvarify p*, *THEN* ≡*E*(*1*), *OF log−prop−prop*:*2*]
by *blast*
qed

AOT-theorem *conj−dist−w*:*2*: ‹$w \models (p \rightarrow q) \equiv ((w \models p) \rightarrow (w \models q))$›
proof(*safe intro*!: ≡*I* →*I*)
AOT-assume ‹$w \models (p \rightarrow q)$›
AOT-hence *0*: ‹$\Box w \models (p \rightarrow q)$›
using *rigid−truth−at*:*1*[*unvarify p*, *THEN* ≡*E*(*1*), *OF log−prop−prop*:*2*]
by *blast*
AOT-assume ‹$w \models p$›
AOT-hence *1*: ‹$\Box w \models p$›
by (*metis T*◇ ≡*E*(*1*) *rigid−truth−at*:*3* →*E*)
AOT-modally-strict {
AOT-have ‹$\forall p \ (w \models p \equiv p) \rightarrow ((w \models (\varphi \rightarrow \psi)) \rightarrow (w \models \varphi \rightarrow w \models \psi))$› **for** $w \ \varphi \ \psi$
proof(*safe intro*!: →*I*)
AOT-assume ‹$\forall \ p \ (w \models p \equiv p)$›
AOT-hence ‹$w \models (\varphi \rightarrow \psi) \equiv (\varphi \rightarrow \psi)$› **and** ‹$w \models \varphi \equiv \varphi$› **and** ‹$w \models \psi \equiv \psi$›
using $\forall E(1)$[*rotated*, *OF log−prop−prop*:*2*] **by** *blast+*

moreover AOT-assume ‹$w \models (\varphi \rightarrow \psi)$›
moreover AOT-assume ‹$w \models \varphi$›
ultimately AOT-show ‹$w \models \psi$›
by (*metis* $\equiv E(1)$ $\equiv E(2)$ $\rightarrow E$)
qed
}
AOT-hence ‹$\Diamond\forall p\ (w \models p \equiv p) \rightarrow \Diamond(w \models (\varphi \rightarrow \psi) \rightarrow (w \models \varphi \rightarrow w \models \psi))$› **for** $w\ \varphi\ \psi$
by (*rule RM*$\Diamond$)
moreover AOT-have *pos*: ‹$\Diamond\forall p\ (w \models p \equiv p)$›
using *world*:*1*[*THEN* $\equiv_{df}E$, *OF PossibleWorld*.ψ] &*E* **by** *blast*
ultimately AOT-have ‹$\Diamond(w \models (p \rightarrow q) \rightarrow (w \models p \rightarrow w \models q))$›
using $\rightarrow E$ **by** *blast*
AOT-hence ‹$\Diamond(w \models p \rightarrow w \models q)$›
by (*metis 0 KBasic2*:*4* $\equiv E(1)$ $\rightarrow E$)
AOT-hence ‹$\Diamond w \models q$›
by (*metis 1 KBasic2*:*4* $\equiv E(1)$ $\rightarrow E$)
AOT-thus ‹$w \models q$›
using *rigid*−*truth*−*at*:*2*[*unvarify p*, *THEN* $\equiv E(1)$, *OF log*−*prop*−*prop*:*2*]
&*E* &*I* **by** *meson*
next
AOT-assume ‹$w \models p \rightarrow w \models q$›
AOT-hence ‹$\neg(w \models p) \vee w \models q$›
by (*metis* $\vee I(1)$ $\vee I(2)$ *reductio*−*aa*:*1* $\rightarrow E$)
AOT-hence ‹$w \models \neg p \vee w \models q$›
by (*metis coherent*:*1* $\vee I(1)$ $\vee I(2)$ $\vee E(2)$ $\equiv E(2)$ *reductio*−*aa*:*1*)
AOT-hence *0*: ‹$\Box(w \models \neg p \vee w \models q)$›
using *rigid*−*truth*−*at*:*1*[*unvarify p*, *THEN* $\equiv E(1)$, *OF log*−*prop*−*prop*:*2*]
by (*metis KBasic*:*15* $\vee I(1)$ $\vee I(2)$ $\vee E(2)$ *reductio*−*aa*:*1* $\rightarrow E$)
AOT-modally-strict {
AOT-have ‹$\forall p\ (w \models p \equiv p) \rightarrow ((w \models \neg\varphi \vee w \models \psi) \rightarrow (w \models (\varphi \rightarrow \psi)))$› **for** $w\ \varphi\ \psi$
proof(*safe intro*!: $\rightarrow I$)
AOT-assume ‹$\forall\ p\ (w \models p \equiv p)$›
moreover AOT-assume ‹$w \models \neg\varphi \vee w \models \psi$›
ultimately AOT-show ‹$w \models (\varphi \rightarrow \psi)$›
by (*metis* $\vee E(2)$ $\rightarrow I$ $\equiv E(1)$ $\equiv E(2)$ *log*−*prop*−*prop*:*2*
reductio−*aa*:*1 rule*−*ui*:*1*)
qed
}
AOT-hence ‹$\Diamond\forall p\ (w \models p \equiv p) \rightarrow \Diamond((w \models \neg\varphi \vee w \models \psi) \rightarrow w \models (\varphi \rightarrow \psi))$› **for** $w\ \varphi\ \psi$
by (*rule RM*$\Diamond$)
moreover AOT-have *pos*: ‹$\Diamond\forall p\ (w \models p \equiv p)$›
using *world*:*1*[*THEN* $\equiv_{df}E$, *OF PossibleWorld*.ψ] &*E* **by** *blast*
ultimately AOT-have ‹$\Diamond((w \models \neg p \vee w \models q) \rightarrow w \models (p \rightarrow q))$›
using $\rightarrow E$ **by** *blast*
AOT-hence ‹$\Diamond(w \models (p \rightarrow q))$›
by (*metis 0 KBasic2*:*4* $\equiv E(1)$ $\rightarrow E$)
AOT-thus ‹$w \models (p \rightarrow q)$›
using *rigid*−*truth*−*at*:*2*[*unvarify p*, *THEN* $\equiv E(1)$, *OF log*−*prop*−*prop*:*2*]
by *blast*
qed

AOT-theorem *conj*−*dist*−*w*:*3*: ‹$w \models (p \vee q) \equiv ((w \models p) \vee (w \models q))$›
proof(*safe intro*!: $\equiv I$ $\rightarrow I$)
AOT-assume ‹$w \models (p \vee q)$›
AOT-hence *0*: ‹$\Box w \models (p \vee q)$›
using *rigid*−*truth*−*at*:*1*[*unvarify p*, *THEN* $\equiv E(1)$, *OF log*−*prop*−*prop*:*2*]
by *blast*
AOT-modally-strict {
AOT-have ‹$\forall p\ (w \models p \equiv p) \rightarrow ((w \models (\varphi \vee \psi)) \rightarrow (w \models \varphi \vee w \models \psi))$› **for** $w\ \varphi\ \psi$
proof(*safe intro*!: $\rightarrow I$)
AOT-assume ‹$\forall\ p\ (w \models p \equiv p)$›
AOT-hence ‹$w \models (\varphi \vee \psi) \equiv (\varphi \vee \psi)$› **and** ‹$w \models \varphi \equiv \varphi$› **and** ‹$w \models \psi \equiv \psi$›
using $\forall E(1)$[*rotated*, *OF log*−*prop*−*prop*:*2*] **by** *blast*+

```
      moreover AOT-assume ‹w ⊨ (φ ∨ ψ)›
      ultimately AOT-show ‹w ⊨ φ ∨ w ⊨ ψ›
        by (metis ∨I(1) ∨I(2) ∨E(3) ≡E(1) ≡E(2) reductio-aa:1)
    qed
  }
  AOT-hence ‹◇∀p (w ⊨ p ≡ p) → ◇(w ⊨ (φ ∨ ψ) → (w ⊨ φ ∨ w ⊨ ψ))› for w φ ψ
    by (rule RM◇)
  moreover AOT-have pos: ‹◇∀p (w ⊨ p ≡ p)›
    using world:1[THEN ≡df E, OF PossibleWorld.ψ] &E by blast
  ultimately AOT-have ‹◇(w ⊨ (p ∨ q) → (w ⊨ p ∨ w ⊨ q))› using →E by blast
  AOT-hence ‹◇(w ⊨ p ∨ w ⊨ q)›
    by (metis 0 KBasic2:4 ≡E(1) vdash-properties:10)
  AOT-hence ‹◇w ⊨ p ∨ ◇w ⊨ q›
    using KBasic2:2[THEN ≡E(1)] by blast
  AOT-thus ‹w ⊨ p ∨ w ⊨ q›
    using rigid-truth-at:2[unvarify p, THEN ≡E(1), OF log-prop-prop:2]
    by (metis ∨I(1) ∨I(2) ∨E(2) reductio-aa:1)
next
  AOT-assume ‹w ⊨ p ∨ w ⊨ q›
  AOT-hence 0: ‹□(w ⊨ p ∨ w ⊨ q)›
    using rigid-truth-at:1[unvarify p, THEN ≡E(1), OF log-prop-prop:2]
    by (metis KBasic:15 ∨I(1) ∨I(2) ∨E(2) reductio-aa:1 →E)
  AOT-modally-strict {
    AOT-have ‹∀p (w ⊨ p ≡ p) → ((w ⊨ φ ∨ w ⊨ ψ) → (w ⊨ (φ ∨ ψ)))› for w φ ψ
    proof(safe intro!: →I)
      AOT-assume ‹∀ p (w ⊨ p ≡ p)›
      moreover AOT-assume ‹w ⊨ φ ∨ w ⊨ ψ›
      ultimately AOT-show ‹w ⊨ (φ ∨ ψ)›
        by (metis ∨I(1) ∨I(2) ∨E(2) ≡E(1) ≡E(2)
                  log-prop-prop:2 reductio-aa:1 rule-ui:1)
    qed
  }
  AOT-hence ‹◇∀p (w ⊨ p ≡ p) → ◇((w ⊨ φ ∨ w ⊨ ψ) → w ⊨ (φ ∨ ψ))› for w φ ψ
    by (rule RM◇)
  moreover AOT-have pos: ‹◇∀p (w ⊨ p ≡ p)›
    using world:1[THEN ≡df E, OF PossibleWorld.ψ] &E by blast
  ultimately AOT-have ‹◇((w ⊨ p ∨ w ⊨ q) → w ⊨ (p ∨ q))›
    using →E by blast
  AOT-hence ‹◇(w ⊨ (p ∨ q))›
    by (metis 0 KBasic2:4 ≡E(1) →E)
  AOT-thus ‹w ⊨ (p ∨ q)›
    using rigid-truth-at:2[unvarify p, THEN ≡E(1), OF log-prop-prop:2]
    by blast
qed

AOT-theorem conj-dist-w:4: ‹w ⊨ (p ≡ q) ≡ ((w ⊨ p) ≡ (w ⊨ q))›
proof(rule ≡I; rule →I)
  AOT-assume ‹w ⊨ (p ≡ q)›
  AOT-hence 0: ‹□w ⊨ (p ≡ q)›
    using rigid-truth-at:1[unvarify p, THEN ≡E(1), OF log-prop-prop:2]
    by blast
  AOT-modally-strict {
    AOT-have ‹∀p (w ⊨ p ≡ p) → ((w ⊨ (φ ≡ ψ)) → (w ⊨ φ ≡ w ⊨ ψ))› for w φ ψ
    proof(safe intro!: →I)
      AOT-assume ‹∀ p (w ⊨ p ≡ p)›
      AOT-hence ‹w ⊨ (φ ≡ ψ) ≡ (φ ≡ ψ)› and ‹w ⊨ φ ≡ φ› and ‹w ⊨ ψ ≡ ψ›
        using ∀E(1)[rotated, OF log-prop-prop:2] by blast+
      moreover AOT-assume ‹w ⊨ (φ ≡ ψ)›
      ultimately AOT-show ‹w ⊨ φ ≡ w ⊨ ψ›
        by (metis ≡E(2) ≡E(5) Commutativity of ≡)
    qed
  }
  AOT-hence ‹◇∀p (w ⊨ p ≡ p) → ◇(w ⊨ (φ ≡ ψ) → (w ⊨ φ ≡ w ⊨ ψ))› for w φ ψ
```

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    by (rule RM◇)
  moreover AOT-have pos: ‹◇∀p (w ⊨ p ≡ p)›
    using world:1[THEN ≡dfE, OF PossibleWorld.ψ] &E by blast
  ultimately AOT-have ‹◇(w ⊨ (p ≡ q) → (w ⊨ p ≡ w ⊨ q))›
    using →E by blast
  AOT-hence 1: ‹◇(w ⊨ p ≡ w ⊨ q)›
    by (metis 0 KBasic2:4 ≡E(1) vdash−properties:10)
  AOT-have ‹◇((w ⊨ p → w ⊨ q) & (w ⊨ q → w ⊨ p))›
    apply (AOT-subst ‹(w ⊨ p → w ⊨ q) & (w ⊨ q → w ⊨ p)› ‹w ⊨ p ≡ w ⊨ q›)
     apply (meson ≡dfE conventions:3 →I df−rules−formulas[4] ≡I)
    by (fact 1)
  AOT-hence 2: ‹◇(w ⊨ p → w ⊨ q) & ◇(w ⊨ q → w ⊨ p)›
    by (metis KBasic2:3 vdash−properties:10)
  AOT-have ‹◇(¬w ⊨ p ∨ w ⊨ q)› and ‹◇(¬w ⊨ q ∨ w ⊨ p)›
     apply (AOT-subst (reverse) ‹¬w ⊨ p ∨ w ⊨ q› ‹w ⊨ p → w ⊨ q›)
      apply (simp add: oth−class−taut:1:c)
     apply (fact 2[THEN &E(1)])
    apply (AOT-subst (reverse) ‹¬w ⊨ q ∨ w ⊨ p› ‹w ⊨ q → w ⊨ p›)
     apply (simp add: oth−class−taut:1:c)
    by (fact 2[THEN &E(2)])
  AOT-hence ‹◇(¬w ⊨ p) ∨ ◇w ⊨ q› and ‹◇¬w ⊨ q ∨ ◇w ⊨ p›
    using KBasic2:2 ≡E(1) by blast+
  AOT-hence ‹¬□w ⊨ p ∨ ◇w ⊨ q› and ‹¬□w ⊨ q ∨ ◇w ⊨ p›
    by (metis KBasic:11 ∨I(1) ∨I(2) ∨E(2) ≡E(2) raa−cor:1)+
  AOT-thus ‹w ⊨ p ≡ w ⊨ q›
    using rigid−truth−at:2[unvarify p, THEN ≡E(1), OF log−prop−prop:2]
    by (metis ¬¬I T◇ ∨E(2) →I ≡I ≡E(1) rigid−truth−at:3)
next
  AOT-have ‹□PossibleWorld(w)›
    using ≡E(1) rigid−pw:1 PossibleWorld.ψ by blast
  moreover {
    fix p
    AOT-modally-strict {
      AOT-have ‹PossibleWorld(w) → (w ⊨ p → □w ⊨ p)›
        using rigid−truth−at:1 →I
        by (metis ≡E(1))
    }
    AOT-hence ‹□PossibleWorld(w) → □(w ⊨ p → □w ⊨ p)›
      by (rule RM)
  }
  ultimately AOT-have 1: ‹□(w ⊨ p → □w ⊨ p)› for p
    by (metis →E)
  AOT-assume ‹w ⊨ p ≡ w ⊨ q›
  AOT-hence 0: ‹□(w ⊨ p ≡ w ⊨ q)›
    using sc−eq−box−box:5[THEN →E, THEN qml:2[axiom-inst, THEN →E],
                          THEN →E, OF &I]
        by (metis 1)
  AOT-modally-strict {
    AOT-have ‹∀p (w ⊨ p ≡ p) → ((w ⊨ φ ≡ w ⊨ ψ) → (w ⊨ (φ ≡ ψ)))› for w φ ψ
    proof(safe intro!: →I)
      AOT-assume ‹∀ p (w ⊨ p ≡ p)›
      moreover AOT-assume ‹w ⊨ φ ≡ w ⊨ ψ›
      ultimately AOT-show ‹w ⊨ (φ ≡ ψ)›
        by (metis ≡E(2) ≡E(6) log−prop−prop:2 rule−ui:1)
    qed
  }
  AOT-hence ‹◇∀p (w ⊨ p ≡ p) → ◇((w ⊨ φ ≡ w ⊨ ψ) → w ⊨ (φ ≡ ψ))› for w φ ψ
    by (rule RM◇)
  moreover AOT-have pos: ‹◇∀p (w ⊨ p ≡ p)›
    using world:1[THEN ≡dfE, OF PossibleWorld.ψ] &E by blast
  ultimately AOT-have ‹◇((w ⊨ p ≡  w ⊨ q) → w ⊨ (p ≡ q))›
    using →E by blast
  AOT-hence ‹◇(w ⊨ (p ≡ q))›
```

by (*metis 0 KBasic2:4* ≡*E*(*1*) →*E*)
AOT-thus ‹$w \models (p \equiv q)$›
using *rigid−truth−at:2*[*unvarify p*, *THEN* ≡*E*(*1*), *OF log−prop−prop:2*]
by *blast*
qed

AOT-theorem *conj−dist−w:5*: ‹$w \models (\forall\alpha\ \varphi\{\alpha\}) \equiv (\forall\ \alpha\ (w \models \varphi\{\alpha\}))$›
proof(*safe intro*!: ≡*I* →*I GEN*)
AOT-assume ‹$w \models (\forall\alpha\ \varphi\{\alpha\})$›
AOT-hence *0*: ‹$\Box w \models (\forall\alpha\ \varphi\{\alpha\})$›
using *rigid−truth−at:1*[*unvarify p*, *THEN* ≡*E*(*1*), *OF log−prop−prop:2*]
by *blast*
AOT-modally-strict {
AOT-have ‹$\forall p\ (w \models p \equiv p) \to ((w \models (\forall\alpha\ \varphi\{\alpha\})) \to (\forall\alpha\ w \models \varphi\{\alpha\}))$› **for** w
proof(*safe intro*!: →*I GEN*)
AOT-assume ‹$\forall p\ (w \models p \equiv p)$›
moreover AOT-assume ‹$w \models (\forall\alpha\ \varphi\{\alpha\})$›
ultimately AOT-show ‹$w \models \varphi\{\alpha\}$› **for** α
by (*metis* ≡*E*(*1*) ≡*E*(*2*) *log−prop−prop:2 rule−ui:1 rule−ui:3*)
qed
}
AOT-hence ‹$\Diamond\forall p\ (w \models p \equiv p) \to \Diamond(w \models (\forall\alpha\ \varphi\{\alpha\}) \to (\forall\alpha\ w \models \varphi\{\alpha\}))$› **for** w
by (*rule RM*◇)
moreover AOT-have *pos*: ‹$\Diamond\forall p\ (w \models p \equiv p)$›
using *world:1*[*THEN* $\equiv_{df}E$, *OF PossibleWorld.ψ*] &*E* **by** *blast*
ultimately AOT-have ‹$\Diamond(w \models (\forall\alpha\ \varphi\{\alpha\}) \to (\forall\alpha\ w \models \varphi\{\alpha\}))$› **using** →*E* **by** *blast*
AOT-hence ‹$\Diamond(\forall\alpha\ w \models \varphi\{\alpha\})$›
by (*metis 0 KBasic2:4* ≡*E*(*1*) →*E*)
AOT-hence ‹$\forall\alpha\ \Diamond w \models \varphi\{\alpha\}$›
by (*metis Buridan*◇ →*E*)
AOT-thus ‹$w \models \varphi\{\alpha\}$› **for** α
using *rigid−truth−at:2*[*unvarify p*, *THEN* ≡*E*(*1*), *OF log−prop−prop:2*]
∀*E*(*2*) **by** *blast*
next
AOT-assume ‹$\forall\alpha\ w \models \varphi\{\alpha\}$›
AOT-hence ‹$w \models \varphi\{\alpha\}$› **for** α **using** ∀*E*(*2*) **by** *blast*
AOT-hence ‹$\Box w \models \varphi\{\alpha\}$› **for** α
using *rigid−truth−at:1*[*unvarify p*, *THEN* ≡*E*(*1*), *OF log−prop−prop:2*]
&*E* &*I* **by** *blast*
AOT-hence ‹$\forall\alpha\ \Box w \models \varphi\{\alpha\}$› **by** (*rule GEN*)
AOT-hence *0*: ‹$\Box\forall\alpha\ w \models \varphi\{\alpha\}$› **by** (*rule BF*[*THEN* →*E*])
AOT-modally-strict {
AOT-have ‹$\forall p\ (w \models p \equiv p) \to ((\forall\alpha\ w \models \varphi\{\alpha\}) \to (w \models (\forall\alpha\ \varphi\{\alpha\})))$› **for** w
proof(*safe intro*!: →*I*)
AOT-assume ‹$\forall\ p\ (w \models p \equiv p)$›
moreover AOT-assume ‹$\forall\alpha\ w \models \varphi\{\alpha\}$›
ultimately AOT-show ‹$w \models (\forall\alpha\ \varphi\{\alpha\})$›
by (*metis* ≡*E*(*1*) ≡*E*(*2*) *log−prop−prop:2 rule−ui:1*
rule−ui:3 universal−cor)
qed
}
AOT-hence ‹$\Diamond\forall p\ (w \models p \equiv p) \to \Diamond((\forall\alpha\ w \models \varphi\{\alpha\}) \to w \models (\forall\alpha\ \varphi\{\alpha\}))$› **for** w
by (*rule RM*◇)
moreover AOT-have *pos*: ‹$\Diamond\forall p\ (w \models p \equiv p)$›
using *world:1*[*THEN* $\equiv_{df}E$, *OF PossibleWorld.ψ*] &*E* **by** *blast*
ultimately AOT-have ‹$\Diamond((\forall\alpha\ w \models \varphi\{\alpha\}) \to w \models (\forall\alpha\ \varphi\{\alpha\}))$›
using →*E* **by** *blast*
AOT-hence ‹$\Diamond(w \models (\forall\alpha\ \varphi\{\alpha\}))$›
by (*metis 0 KBasic2:4* ≡*E*(*1*) →*E*)
AOT-thus ‹$w \models (\forall\alpha\ \varphi\{\alpha\})$›
using *rigid−truth−at:2*[*unvarify p*, *THEN* ≡*E*(*1*), *OF log−prop−prop:2*]
by *blast*
qed

AOT-theorem *conj−dist−w:6*: ‹$w \models (\exists\alpha\ \varphi\{\alpha\}) \equiv (\exists\ \alpha\ (w \models \varphi\{\alpha\}))$›
proof(*safe intro*!: $\equiv I$ $\rightarrow I$ *GEN*)
  AOT-assume ‹$w \models (\exists\alpha\ \varphi\{\alpha\})$›
  AOT-hence *0*: ‹$\Box w \models (\exists\alpha\ \varphi\{\alpha\})$›
    using *rigid−truth−at:1*[*unvarify p*, *THEN* $\equiv E(1)$, *OF log−prop−prop:2*]
    by *blast*
  AOT-modally-strict {
    AOT-have ‹$\forall p\ (w \models p \equiv p) \rightarrow ((w \models (\exists\alpha\ \varphi\{\alpha\})) \rightarrow (\exists\alpha\ w \models \varphi\{\alpha\}))$› **for** w
    proof(*safe intro*!: $\rightarrow I$ *GEN*)
      AOT-assume ‹$\forall p\ (w \models p \equiv p)$›
      moreover AOT-assume ‹$w \models (\exists\alpha\ \varphi\{\alpha\})$›
      ultimately AOT-show ‹$\exists\ \alpha\ (w \models \varphi\{\alpha\})$›
        by (*metis* $\exists E$ $\exists I(2)$ $\equiv E(1,2)$ *log−prop−prop:2 rule−ui:1*)
    qed
  }
  AOT-hence ‹$\Diamond\forall p\ (w \models p \equiv p) \rightarrow \Diamond(w \models (\exists\alpha\ \varphi\{\alpha\}) \rightarrow (\exists\alpha\ w \models \varphi\{\alpha\}))$› **for** w
    by (*rule RM*$\Diamond$)
  moreover AOT-have *pos*: ‹$\Diamond\forall p\ (w \models p \equiv p)$›
    using *world:1*[*THEN* $\equiv_{df}E$, *OF PossibleWorld.ψ*] &*E* **by** *blast*
  ultimately AOT-have ‹$\Diamond(w \models (\exists\alpha\ \varphi\{\alpha\}) \rightarrow (\exists\alpha\ w \models \varphi\{\alpha\}))$› **using** $\rightarrow E$ **by** *blast*
  AOT-hence ‹$\Diamond(\exists\alpha\ w \models \varphi\{\alpha\})$›
    by (*metis 0 KBasic2:4* $\equiv E(1)$ $\rightarrow E$)
  AOT-hence ‹$\exists\alpha\ \Diamond w \models \varphi\{\alpha\}$›
    by (*metis BF*$\Diamond$ $\rightarrow E$)
  then AOT-obtain α **where** ‹$\Diamond w \models \varphi\{\alpha\}$›
    using $\exists E$[*rotated*] **by** *blast*
  AOT-hence ‹$w \models \varphi\{\alpha\}$›
    using *rigid−truth−at:2*[*unvarify p*, *THEN* $\equiv E(1)$, *OF log−prop−prop:2*] **by** *blast*
  AOT-thus ‹$\exists\ \alpha\ w \models \varphi\{\alpha\}$› **by** (*rule* $\exists I$)
next
  AOT-assume ‹$\exists\alpha\ w \models \varphi\{\alpha\}$›
  then AOT-obtain α **where** ‹$w \models \varphi\{\alpha\}$› **using** $\exists E$[*rotated*] **by** *blast*
  AOT-hence ‹$\Box w \models \varphi\{\alpha\}$›
    using *rigid−truth−at:1*[*unvarify p*, *THEN* $\equiv E(1)$, *OF log−prop−prop:2*]
        &*E* &*I* **by** *blast*
  AOT-hence ‹$\exists\alpha\ \Box w \models \varphi\{\alpha\}$›
    by (*rule* $\exists I$)
  AOT-hence *0*: ‹$\Box\exists\alpha\ w \models \varphi\{\alpha\}$›
    by (*metis Buridan* $\rightarrow E$)
  AOT-modally-strict {
    AOT-have ‹$\forall p\ (w \models p \equiv p) \rightarrow ((\exists\alpha\ w \models \varphi\{\alpha\}) \rightarrow (w \models (\exists\alpha\ \varphi\{\alpha\})))$› **for** w
    proof(*safe intro*!: $\rightarrow I$)
      AOT-assume ‹$\forall\ p\ (w \models p \equiv p)$›
      moreover AOT-assume ‹$\exists\alpha\ w \models \varphi\{\alpha\}$›
      then AOT-obtain α **where** ‹$w \models \varphi\{\alpha\}$›
        using $\exists E$[*rotated*] **by** *blast*
      ultimately AOT-show ‹$w \models (\exists\alpha\ \varphi\{\alpha\})$›
        by (*metis* $\exists I(2)$ $\equiv E(1,2)$ *log−prop−prop:2 rule−ui:1*)
    qed
  }
  AOT-hence ‹$\Diamond\forall p\ (w \models p \equiv p) \rightarrow \Diamond((\exists\alpha\ w \models \varphi\{\alpha\}) \rightarrow w \models (\exists\alpha\ \varphi\{\alpha\}))$› **for** w
    by (*rule RM*$\Diamond$)
  moreover AOT-have *pos*: ‹$\Diamond\forall p\ (w \models p \equiv p)$›
    using *world:1*[*THEN* $\equiv_{df}E$, *OF PossibleWorld.ψ*] &*E* **by** *blast*
  ultimately AOT-have ‹$\Diamond((\exists\alpha\ w \models \varphi\{\alpha\}) \rightarrow w \models (\exists\alpha\ \varphi\{\alpha\}))$›
    using $\rightarrow E$ **by** *blast*
  AOT-hence ‹$\Diamond(w \models (\exists\alpha\ \varphi\{\alpha\}))$›
    by (*metis 0 KBasic2:4* $\equiv E(1)$ $\rightarrow E$)
  AOT-thus ‹$w \models (\exists\alpha\ \varphi\{\alpha\})$›
    using *rigid−truth−at:2*[*unvarify p*, *THEN* $\equiv E(1)$, *OF log−prop−prop:2*]
    by *blast*
qed

AOT-theorem *conj−dist−w:7*: ‹$(w \models \Box p) \rightarrow \Box w \models p$›
proof(*rule* →*I*)
  AOT-assume ‹$w \models \Box p$›
  AOT-hence ‹$\exists\, w\ w \models \Box p$› **by** (*rule PossibleWorld.∃ I*)
  AOT-hence ‹$\Diamond\Box p$› **using** *fund:1*[*unvarify p*, *OF log−prop−prop:2*, *THEN* ≡*E*(*2*)]
    by *blast*
  AOT-hence ‹$\Box p$›
    by (*metis 5*◊ →*E*)
  AOT-hence *1*: ‹$\Box\Box p$›
    by (*metis S5Basic:6* ≡*E*(*1*))
  AOT-have ‹$\Box\forall\, w\ w \models p$›
    by (*AOT-subst* (*reverse*) ‹$\forall\, w\ w \models p$› ‹$\Box p$›)
      (*auto simp add*: *fund:2 1*)
  AOT-hence ‹$\forall\, w\ \Box w \models p$›
    using *fund−lem:5*[*world*][*THEN* →*E*] **by** *simp*
  AOT-thus ‹$\Box w \models p$›
    using →*E PossibleWorld.*∀ *E* **by** *fast*
qed

AOT-theorem *conj−dist−w:8*: ‹$\exists\, w \exists\, p((\Box w \models p)\ \&\ \neg w \models \Box p)$›
proof −
  AOT-obtain *r* **where** *A*: *r* **and** ‹$\Diamond\neg r$›
    by (*metis* &*E*(*1*) &*E*(*2*) $\equiv_{df}E$ ∃ *E cont−tf:1 cont−tf−thm:1*)
  AOT-hence *B*: ‹$\neg\Box r$›
    by (*metis KBasic:11* ≡*E*(*2*))
  AOT-have ‹$\Diamond r$›
    using *A T*◊[*THEN* →*E*] **by** *simp*
  AOT-hence ‹$\exists\, w\ w \models r$›
    using *fund:1*[*THEN* ≡*E*(*1*)] **by** *blast*
  then AOT-obtain *w* **where** *w*: ‹$w \models r$›
    using *PossibleWorld.∃ E*[*rotated*] **by** *meson*
  AOT-hence ‹$\Box w \models r$›
    by (*metis T*◊ ≡*E*(*1*) *rigid−truth−at:3 vdash−properties:10*)
  moreover AOT-have ‹$\neg w \models \Box r$›
  proof(*rule raa−cor:2*)
    AOT-assume ‹$w \models \Box r$›
    AOT-hence ‹$\exists\, w\ w \models \Box r$›
      by (*rule PossibleWorld.∃ I*)
    AOT-hence ‹$\Box r$›
      by (*metis* ≡*E*(*2*) *nec−dia−w:1*)
    AOT-thus ‹$\Box r\ \&\ \neg\Box r$›
      using *B* &*I* **by** *blast*
  qed
  ultimately AOT-have ‹$\Box w \models r\ \&\ \neg w \models \Box r$›
    by (*rule* &*I*)
  AOT-hence ‹$\exists\, p\ (\Box w \models p\ \&\ \neg w \models \Box p)$›
    by (*rule* ∃ *I*)
  thus *?thesis*
    by (*rule PossibleWorld.∃ I*)
qed

AOT-theorem *conj−dist−w:9*: ‹$(\Diamond w \models p) \rightarrow w \models \Diamond p$›
proof(*rule* →*I*; *rule raa−cor:1*)
  AOT-assume ‹$\Diamond w \models p$›
  AOT-hence *0*: ‹$w \models p$›
    by (*metis* ≡*E*(*1*) *rigid−truth−at:2*)
  AOT-assume ‹$\neg w \models \Diamond p$›
  AOT-hence *1*: ‹$w \models \neg\Diamond p$›
    using *coherent:1*[*unvarify p*, *THEN* ≡*E*(*2*), *OF log−prop−prop:2*] **by** *blast*
  moreover AOT-have ‹$w \models (\neg\Diamond p \rightarrow \neg p)$›
    using *T*◊[*THEN contraposition:1*[*1*], *THEN RN*]
        fund:2[*unvarify p*, *OF log−prop−prop:2*, *THEN* ≡*E*(*1*), *THEN* ∀ *E*(*2*),

```
                THEN →E, rotated, OF PossibleWorld.ψ]
        by blast
  ultimately AOT-have ‹w ⊨ ¬p›
    using conj−dist−w:2[unvarify p q, OF log−prop−prop:2, OF log−prop−prop:2,
                       THEN ≡E(1), THEN →E]
    by blast
  AOT-hence ‹w ⊨ p & w ⊨ ¬p› using 0 &I by blast
  AOT-thus ‹p & ¬p›
    by (metis coherent:1 Conjunction Simplification(1,2) ≡E(4)
            modus−tollens:1 raa−cor:3)
qed

AOT-theorem conj−dist−w:10: ‹∃ w∃ p((w ⊨ ◇p) & ¬◇w ⊨ p)›
proof −
  AOT-obtain w where w: ‹∀ p (w ⊨ p ≡ p)›
    using act−world:1 PossibleWorld.∃ E[rotated] by meson
  AOT-obtain r where ‹¬r› and ‹◇r›
    using cont−tf−thm:2 cont−tf:2[THEN ≡df E] &E ∃ E[rotated] by metis
  AOT-hence ‹w ⊨ ¬r› and 0: ‹w ⊨ ◇r›
    using w[THEN ∀ E(1), OF log−prop−prop:2, THEN ≡E(2)] by blast+
  AOT-hence ‹¬w ⊨ r› using coherent:1[THEN ≡E(1)] by blast
  AOT-hence ‹¬◇w ⊨ r› by (metis ≡E(4) rigid−truth−at:2)
  AOT-hence ‹w ⊨ ◇r & ¬◇w ⊨ r› using 0 &I by blast
  AOT-hence ‹∃ p (w ⊨ ◇p & ¬◇w ⊨ p)› by (rule ∃ I)
  thus ?thesis by (rule PossibleWorld.∃ I)
qed

AOT-theorem two−worlds−exist:1: ‹∃ p(ContingentlyTrue(p)) → ∃ w (¬Actual(w))›
proof(rule →I)
  AOT-assume ‹∃ p ContingentlyTrue(p)›
  then AOT-obtain p where ‹ContingentlyTrue(p)›
    using ∃ E[rotated] by blast
  AOT-hence p: ‹p & ◇¬p›
    by (metis ≡df E cont−tf:1)
  AOT-hence ‹∃ w w ⊨ ¬p›
    using fund:1[unvarify p, OF log−prop−prop:2, THEN ≡E(1)] &E by blast
  then AOT-obtain w where w: ‹w ⊨ ¬p›
    using PossibleWorld.∃ E[rotated] by meson
  AOT-have ‹¬Actual(w)›
  proof(rule raa−cor:2)
    AOT-assume ‹Actual(w)›
    AOT-hence ‹w ⊨ p›
      using p[THEN &E(1)] actual[THEN ≡df E, THEN &E(2)]
      by (metis log−prop−prop:2 raa−cor:3 rule−ui:1 →E w)
    moreover AOT-have ‹¬(w ⊨ p)›
      by (metis coherent:1 ≡E(4) reductio−aa:2 w)
    ultimately AOT-show ‹w ⊨ p & ¬(w ⊨ p)›
      using &I by blast
  qed
  AOT-thus ‹∃ w ¬Actual(w)›
    by (rule PossibleWorld.∃ I)
qed


AOT-theorem two−worlds−exist:2: ‹∃ p(ContingentlyFalse(p)) → ∃ w (¬Actual(w))›
proof(rule →I)
  AOT-assume ‹∃ p ContingentlyFalse(p)›
  then AOT-obtain p where ‹ContingentlyFalse(p)›
    using ∃ E[rotated] by blast
  AOT-hence p: ‹¬p & ◇p›
    by (metis ≡df E cont−tf:2)
  AOT-hence ‹∃ w w ⊨ p›
    using fund:1[unvarify p, OF log−prop−prop:2, THEN ≡E(1)] &E by blast
```

```
  then AOT-obtain w where w: ‹w ⊨ p›
    using PossibleWorld.∃E[rotated] by meson
  moreover AOT-have ‹¬Actual(w)›
  proof(rule raa-cor:2)
    AOT-assume ‹Actual(w)›
    AOT-hence ‹w ⊨ ¬p›
      using p[THEN &E(1)] actual[THEN ≡df E, THEN &E(2)]
      by (metis log-prop-prop:2 raa-cor:3 rule-ui:1 →E w)
    moreover AOT-have ‹¬(w ⊨ p)›
      using calculation by (metis coherent:1 ≡E(4) reductio-aa:2)
    AOT-thus ‹w ⊨ p & ¬(w ⊨ p)›
      using &I w by metis
  qed
  AOT-thus ‹∃w ¬Actual(w)›
    by (rule PossibleWorld.∃I)
qed

AOT-theorem two-worlds-exist:3: ‹∃w ¬Actual(w)›
  using cont-tf-thm:1 two-worlds-exist:1 →E by blast

AOT-theorem two-worlds-exist:4: ‹∃w∃w'(w ≠ w')›
proof -
  AOT-obtain w where w: ‹Actual(w)›
    using act-world:2[THEN uniqueness:1[THEN ≡df E],
                     THEN cqt-further:5[THEN →E]]
          PossibleWorld.∃E[rotated] &E
    by blast
  moreover AOT-obtain w' where w': ‹¬Actual(w')›
    using two-worlds-exist:3 PossibleWorld.∃E[rotated] by meson
  AOT-have ‹¬(w = w')›
  proof(rule raa-cor:2)
    AOT-assume ‹w = w'›
    AOT-thus ‹p & ¬p› for p
      using w w' &E by (metis rule=E raa-cor:3)
  qed
  AOT-hence ‹w ≠ w'›
    by (metis ≡df I =-infix)
  AOT-hence ‹∃w' w ≠ w'›
    by (rule PossibleWorld.∃I)
  thus ?thesis
    by (rule PossibleWorld.∃I)
qed


AOT-theorem w-rel:1: ‹[λx φ{x}]↓ → [λx w ⊨ φ{x}]↓›
proof(rule →I)
  AOT-assume ‹[λx φ{x}]↓›
  AOT-hence ‹□[λx φ{x}]↓›
    by (metis exist-nec →E)
  moreover AOT-have
    ‹□[λx φ{x}]↓ → □∀x∀y(∀F([F]x ≡ [F]y) → ((w ⊨ φ{x}) ≡ ( w ⊨ φ{y})))›
  proof (rule RM; rule →I; rule GEN; rule GEN; rule →I)
    AOT-modally-strict {
      fix x y
      AOT-assume ‹[λx φ{x}]↓›
      AOT-hence ‹∀x∀y (∀F ([F]x ≡ [F]y) → □(φ{x} ≡ φ{y}))›
        using &E kirchner-thm-cor:1[THEN →E] by blast
      AOT-hence ‹∀F ([F]x ≡ [F]y) → □(φ{x} ≡ φ{y})›
        using ∀E(2) by blast
      moreover AOT-assume ‹∀F ([F]x ≡ [F]y)›
      ultimately AOT-have ‹□(φ{x} ≡ φ{y})›
        using →E by blast
```

```
    AOT-hence ‹∀ w (w ⊨ (φ{x} ≡ φ{y}))›
      using fund:2[unvarify p, OF log−prop−prop:2, THEN ≡E(1)] by blast
    AOT-hence ‹w ⊨ (φ{x} ≡ φ{y})›
      using ∀ E(2) using PossibleWorld.ψ →E by blast
    AOT-thus ‹(w ⊨ φ{x}) ≡ (w ⊨ φ{y})›
      using conj−dist−w:4[unvarify p q, OF log−prop−prop:2,
                          OF log−prop−prop:2, THEN ≡E(1)] by blast
  }
 qed
 ultimately AOT-have ‹□∀ x∀ y(∀ F([F]x ≡ [F]y) → ((w ⊨ φ{x}) ≡ ( w ⊨ φ{y})))›
   using →E by blast
 AOT-thus ‹[λx w ⊨ φ{x}]↓›
   using kirchner−thm:1[THEN ≡E(2)] by fast
qed

AOT-theorem w−rel:2: ‹[λx1...xn φ{x1...xn}]↓ → [λx1...xn w ⊨ φ{x1...xn}]↓›
proof(rule →I)
 AOT-assume ‹[λx1...xn φ{x1...xn}]↓›
 AOT-hence ‹□[λx1...xn φ{x1...xn}]↓›
   by (metis exist−nec →E)
 moreover AOT-have ‹□[λx1...xn φ{x1...xn}]↓ → □∀ x1...∀ xn∀ y1...∀ yn(
   ∀ F([F]x1...xn ≡ [F]y1...yn) → ((w ⊨ φ{x1...xn}) ≡ ( w ⊨ φ{y1...yn})))›
 proof (rule RM; rule →I; rule GEN; rule GEN; rule →I)
   AOT-modally-strict {
    fix x1xn y1yn
    AOT-assume ‹[λx1...xn φ{x1...xn}]↓›
    AOT-hence ‹∀ x1...∀ xn∀ y1...∀ yn (
      ∀ F ([F]x1...xn ≡ [F]y1...yn) → □(φ{x1...xn} ≡ φ{y1...yn}))›
      using &E kirchner−thm−cor:2[THEN →E] by blast
    AOT-hence ‹∀ F ([F]x1...xn ≡ [F]y1...yn) → □(φ{x1...xn} ≡ φ{y1...yn})›
      using ∀ E(2) by blast
    moreover AOT-assume ‹∀ F ([F]x1...xn ≡ [F]y1...yn)›
    ultimately AOT-have ‹□(φ{x1...xn} ≡ φ{y1...yn})›
      using →E by blast
    AOT-hence ‹∀ w (w ⊨ (φ{x1...xn} ≡ φ{y1...yn}))›
      using fund:2[unvarify p, OF log−prop−prop:2, THEN ≡E(1)] by blast
    AOT-hence ‹w ⊨ (φ{x1...xn} ≡ φ{y1...yn})›
      using ∀ E(2) using PossibleWorld.ψ →E by blast
    AOT-thus ‹(w ⊨ φ{x1...xn}) ≡ (w ⊨ φ{y1...yn})›
      using conj−dist−w:4[unvarify p q, OF log−prop−prop:2,
                          OF log−prop−prop:2, THEN ≡E(1)] by blast
  }
 qed
 ultimately AOT-have ‹□∀ x1...∀ xn∀ y1...∀ yn(
     ∀ F([F]x1...xn ≡ [F]y1...yn) → ((w ⊨ φ{x1...xn}) ≡ ( w ⊨ φ{y1...yn})))›
   using →E by blast
 AOT-thus ‹[λx1...xn w ⊨ φ{x1...xn}]↓›
   using kirchner−thm:2[THEN ≡E(2)] by fast
qed

AOT-theorem w−rel:3: ‹[λx1...xn w ⊨ [F]x1...xn]↓›
 by (rule w−rel:2[THEN →E]) cqt:2[lambda]

AOT-define WorldIndexedRelation :: ‹Π ⇒ τ ⇒ Π› (‹-.-›)
 w−index: ‹[F]w =df [λx1...xn w ⊨ [F]x1...xn]›

AOT-define Rigid :: ‹τ ⇒ φ› (‹Rigid′(-′)›)
 df−rigid−rel:1:
‹Rigid(F) ≡df F↓ & □∀ x1...∀ xn([F]x1...xn → □[F]x1...xn)›

AOT-define Rigidifies :: ‹τ ⇒ τ ⇒ φ› (‹Rigidifies′(-,-′)›)
 df−rigid−rel:2:
‹Rigidifies(F, G) ≡df Rigid(F) & ∀ x1...∀ xn([F]x1...xn ≡ [G]x1...xn)›
```

AOT-theorem *rigid−der:1*: ‹$[[F]_w]x_1...x_n \equiv w \models [F]x_1...x_n$›
  apply (*rule rule−id−df:2:b[2]*[**where** τ=λ (Π, κ). «$[\Pi]_\kappa$» **and**
                σ=λ(Π, κ). «$[\lambda x_1...x_n\ \kappa \models [\Pi]x_1...x_n]$»,
              simplified, *OF w−index*])
   apply (*fact w−rel:3*)
  apply (*rule beta−C−meta*[*THEN* →*E*])
  by (*fact w−rel:3*)

AOT-theorem *rigid−der:2*: ‹*Rigid*($[G]_w$)›
proof(*safe intro*!: $\equiv_{df}$*I*[*OF df−rigid−rel:1*] &*I*)
  AOT-show ‹$[G]_w\downarrow$›
    by (*rule rule−id−df:2:b[2]*[**where** τ=λ (Π, κ). «$[\Pi]_\kappa$» **and**
                σ=λ(Π, κ). «$[\lambda x_1...x_n\ \kappa \models [\Pi]x_1...x_n]$»,
              simplified, *OF w−index*])
       (*fact w−rel:3*)+
next
  AOT-have ‹$\Box\forall x_1...\forall x_n$ ($[[G]_w]x_1...x_n \rightarrow \Box[[G]_w]x_1...x_n$)›
  proof(*rule RN*; *safe intro*!: →*I GEN*)
    AOT-modally-strict {
      AOT-have *assms*: ‹*PossibleWorld*(*w*)› **using** *PossibleWorld*.ψ**.**
      AOT-hence *nec-pw-w*: ‹$\Box$*PossibleWorld*(*w*)›
        using ≡*E*(*1*) *rigid−pw:1* **by** *blast*
      fix $x_1 x_n$
      AOT-assume ‹$[[G]_w]x_1...x_n$›
      AOT-hence ‹$[\lambda x_1...x_n\ w \models [G]x_1...x_n]x_1...x_n$›
        using *rule−id−df:2:a[2]*[**where** τ=λ (Π, κ). «$[\Pi]_\kappa$» **and**
                σ=λ(Π, κ). «$[\lambda x_1...x_n\ \kappa \models [\Pi]x_1...x_n]$»,
                simplified, *OF w−index*, *OF w−rel:3*]
        by *fast*
      AOT-hence ‹$w \models [G]x_1...x_n$›
        by (*metis* β→*C*(*1*))
      AOT-hence ‹$\Box w \models [G]x_1...x_n$›
        using *rigid−truth−at:1*[*unvarify p*, *OF log−prop−prop:2*, *THEN* ≡*E*(*1*)]
        by *blast*
      moreover AOT-have ‹$\Box w \models [G]x_1...x_n \rightarrow \Box[\lambda x_1...x_n\ w \models [G]x_1...x_n]x_1...x_n$›
      proof (*rule RM*; *rule* →*I*)
        AOT-modally-strict {
          AOT-assume ‹$w \models [G]x_1...x_n$›
          AOT-thus ‹$[\lambda x_1...x_n\ w \models [G]x_1...x_n]x_1...x_n$›
            by (*auto intro*!: β←*C*(*1*) *simp*: *w−rel:3 cqt:2*)
        }
      qed
      ultimately AOT-have *1*: ‹$\Box[\lambda x_1...x_n\ w \models [G]x_1...x_n]x_1...x_n$›
        using →*E* **by** *blast*
      AOT-show ‹$\Box[[G]_w]x_1...x_n$›
        by (*rule rule−id−df:2:b[2]*[**where** τ=λ (Π, κ). «$[\Pi]_\kappa$» **and**
                σ=λ(Π, κ). «$[\lambda x_1...x_n\ \kappa \models [\Pi]x_1...x_n]$»,
              simplified, *OF w−index*])
          (*auto simp*: *1 w−rel:3*)
    }
  qed
  AOT-thus ‹$\Box\forall x_1...\forall x_n$ ($[[G]_w]x_1...x_n \rightarrow \Box[[G]_w]x_1...x_n$)›
    using →*E* **by** *blast*
qed

AOT-theorem *rigid−der:3*: ‹$\exists F$ *Rigidifies*(*F*, *G*)›
proof −
  AOT-obtain *w* **where** *w*: ‹$\forall p$ ($w \models p \equiv p$)›
    using *act−world:1 PossibleWorld*.$\exists$*E*[*rotated*] **by** *meson*
  show *?thesis*
  proof (*rule* $\exists$*I*(*1*)[**where** τ=‹«$[G]_w$»›])
    AOT-show ‹*Rigidifies*($[G]_w$, $[G]$)›

```
    proof(safe intro!: ≡_df I[OF df-rigid-rel:2] &I GEN)
      AOT-show ‹Rigid([G]_w)›
        using rigid-der:2 by blast
    next
      fix x1xn
      AOT-have ‹[[G]_w]x1...xn ≡ [λx1...xn w ⊨ [G]x1...xn]x1...xn›
      proof(rule ≡I; rule →I)
        AOT-assume ‹[[G]_w]x1...xn›
        AOT-thus ‹[λx1...xn w ⊨ [G]x1...xn]x1...xn›
          by (rule rule-id-df:2:a[2]
                  [where τ=λ (Π, κ). «[Π]_κ» and
                         σ=λ(Π, κ). «[λx1...xn κ ⊨ [Π]x1...xn]»,
                   simplified, OF w-index, OF w-rel:3])
      next
        AOT-assume ‹[λx1...xn w ⊨ [G]x1...xn]x1...xn›
        AOT-thus ‹[[G]_w]x1...xn›
          by (rule rule-id-df:2:b[2]
                  [where τ=λ (Π, κ). «[Π]_κ» and
                         σ=λ(Π, κ). «[λx1...xn κ ⊨ [Π]x1...xn]»,
                   simplified, OF w-index, OF w-rel:3])
      qed
      also AOT-have ‹... ≡ w ⊨ [G]x1...xn›
        by (rule beta-C-meta[THEN →E])
           (fact w-rel:3)
      also AOT-have ‹... ≡ [G]x1...xn›
        using w[THEN ∀E(1), OF log-prop-prop:2] by blast
      finally AOT-show ‹[[G]_w]x1...xn ≡ [G]x1...xn›.
    qed
  next
    AOT-show ‹[G]_w↓›
      by (rule rule-id-df:2:b[2][where τ=λ (Π, κ). «[Π]_κ»
                                 and σ=λ(Π, κ). «[λx1...xn κ ⊨ [Π]x1...xn]»,
                                simplified, OF w-index])
         (auto simp: w-rel:3)
  qed
qed

AOT-theorem rigid-rel-thms:1:
  ‹□(∀x1...∀xn ([F]x1...xn → □[F]x1...xn)) ≡ ∀x1...∀xn(◇[F]x1...xn → □[F]x1...xn)›
proof(safe intro!: ≡I →I GEN)
  fix x1xn
  AOT-assume ‹□∀x1...∀xn ([F]x1...xn → □[F]x1...xn)›
  AOT-hence ‹∀x1...∀xn □([F]x1...xn → □[F]x1...xn)›
    by (metis →E GEN RM cqt-orig:3)
  AOT-hence ‹□([F]x1...xn → □[F]x1...xn)›
    using ∀E(2) by blast
  AOT-hence ‹◇[F]x1...xn → □[F]x1...xn›
    by (metis ≡E(1) sc-eq-box-box:1)
  moreover AOT-assume ‹◇[F]x1...xn›
  ultimately AOT-show ‹□[F]x1...xn›
    using →E by blast
next
  AOT-assume ‹∀x1...∀xn (◇[F]x1...xn → □[F]x1...xn)›
  AOT-hence ‹◇[F]x1...xn → □[F]x1...xn› for x1xn
    using ∀E(2) by blast
  AOT-hence ‹□([F]x1...xn → □[F]x1...xn)› for x1xn
    by (metis ≡E(2) sc-eq-box-box:1)
  AOT-hence 0: ‹∀x1...∀xn □([F]x1...xn → □[F]x1...xn)›
    by (rule GEN)
  AOT-thus ‹□(∀x1...∀xn ([F]x1...xn → □[F]x1...xn))›
    using BF vdash-properties:10 by blast
qed
```

```
AOT-theorem rigid−rel−thms:2:
  ‹□(∀x1...∀xn ([F]x1...xn → □[F]x1...xn)) ≡ ∀x1...∀xn(□[F]x1...xn ∨ □¬[F]x1...xn)›
proof(safe intro!: ≡I →I)
  AOT-assume ‹□(∀x1...∀xn ([F]x1...xn → □[F]x1...xn))›
  AOT-hence 0: ‹∀x1...∀xn □([F]x1...xn → □[F]x1...xn)›
    using CBF[THEN →E] by blast
  AOT-show ‹∀x1...∀xn(□[F]x1...xn ∨ □¬[F]x1...xn)›
  proof(rule GEN)
    fix x1xn
    AOT-have 1: ‹□([F]x1...xn → □[F]x1...xn)›
      using 0[THEN ∀E(2)].
    AOT-hence 2: ‹◇[F]x1...xn → [F]x1...xn›
      using B◇ Hypothetical Syllogism K◇ vdash−properties:10 by blast
    AOT-have ‹[F]x1...xn ∨ ¬[F]x1...xn›
      using exc−mid.
    moreover {
      AOT-assume ‹[F]x1...xn›
      AOT-hence ‹□[F]x1...xn›
        using 1[THEN qml:2[axiom-inst, THEN →E], THEN →E] by blast
    }
    moreover {
      AOT-assume 3: ‹¬[F]x1...xn›
      AOT-have ‹□¬[F]x1...xn›
      proof(rule raa−cor:1)
        AOT-assume ‹¬□¬[F]x1...xn›
        AOT-hence ‹◇[F]x1...xn›
          by (AOT-subst-def conventions:5)
        AOT-hence ‹[F]x1...xn› using 2[THEN →E] by blast
        AOT-thus ‹[F]x1...xn & ¬[F]x1...xn›
          using 3 &I by blast
      qed
    }
    ultimately AOT-show ‹□[F]x1...xn ∨ □¬[F]x1...xn›
      by (metis ∨I(1,2) raa−cor:1)
  qed
next
  AOT-assume 0: ‹∀x1...∀xn(□[F]x1...xn ∨ □¬[F]x1...xn)›
  {
    fix x1xn
    AOT-have ‹□[F]x1...xn ∨ □¬[F]x1...xn› using 0[THEN ∀E(2)] by blast
    moreover {
      AOT-assume ‹□[F]x1...xn›
      AOT-hence ‹□□[F]x1...xn›
        using S5Basic:6[THEN ≡E(1)] by blast
      AOT-hence ‹□([F]x1...xn → □[F]x1...xn)›
        using KBasic:1[THEN →E] by blast
    }
    moreover {
      AOT-assume ‹□¬[F]x1...xn›
      AOT-hence ‹□([F]x1...xn → □[F]x1...xn)›
        using KBasic:2[THEN →E] by blast
    }
    ultimately AOT-have ‹□([F]x1...xn → □[F]x1...xn)›
      using con−dis−i−e:4:b raa−cor:1 by blast
  }
  AOT-hence ‹∀x1...∀xn □([F]x1...xn → □[F]x1...xn)›
    by (rule GEN)
  AOT-thus ‹□(∀x1...∀xn ([F]x1...xn → □[F]x1...xn))›
    using BF[THEN →E] by fast
qed

AOT-theorem rigid−rel−thms:3: ‹Rigid(F) ≡ ∀x1...∀xn (□[F]x1...xn ∨ □¬[F]x1...xn)›
  by (AOT-subst-thm df−rigid−rel:1[THEN ≡Df, THEN ≡S(1), OF cqt:2(1)];
```

AOT-subst-thm rigid−rel−thms:2)
(*simp add*: *oth−class−taut*:*3*:*a*)

13 Natural Numbers

AOT-define *CorrelatesOneToOne* :: ‹τ ⇒ τ ⇒ τ ⇒ φ› (‹- |: - $_{1-1}$⟷ -›)
1−1−cor: ‹R |: F $_{1-1}$⟷ G $\equiv_{df}$ R↓ & F↓ & G↓ &
∀x ([F]x → ∃!y([G]y & [R]xy)) &
∀y ([G]y → ∃!x([F]x & [R]xy))›

AOT-define *MapsTo* :: ‹τ ⇒ τ ⇒ τ ⇒ φ› (‹- |: - ⟶ -›)
fFG:*1*: ‹R |: F ⟶ G $\equiv_{df}$ R↓ & F↓ & G↓ & ∀x ([F]x → ∃!y([G]y & [R]xy))›

AOT-define *MapsToOneToOne* :: ‹τ ⇒ τ ⇒ τ ⇒ φ› (‹- |: - $_{1-1}$⟶ -›)
fFG:*2*: ‹R |: F $_{1-1}$⟶ G $\equiv_{df}$
R |: F ⟶ G & ∀x∀y∀z (([F]x & [F]y & [G]z) → ([R]xz & [R]yz → x = y))›

AOT-define *MapsOnto* :: ‹τ ⇒ τ ⇒ τ ⇒ φ› (‹- |: - $⟶_{onto}$ -›)
fFG:*3*: ‹R |: F $⟶_{onto}$ G $\equiv_{df}$ R |: F ⟶ G & ∀y ([G]y → ∃x([F]x & [R]xy))›

AOT-define *MapsOneToOneOnto* :: ‹τ ⇒ τ ⇒ τ ⇒ φ› (‹- |: - $_{1-1}⟶_{onto}$ -›)
fFG:*4*: ‹R |: F $_{1-1}⟶_{onto}$ G $\equiv_{df}$ R |: F $_{1-1}$⟶ G & R |: F $⟶_{onto}$ G›

AOT-theorem *eq−1−1*: ‹R |: F $_{1-1}$⟷ G ≡ R |: F $_{1-1}⟶_{onto}$ G›
proof(*rule* ≡*I*; *rule* →*I*)
AOT-assume ‹R |: F $_{1-1}$⟷ G›
AOT-hence *A*: ‹∀x ([F]x → ∃!y([G]y & [R]xy))›
and *B*: ‹∀y ([G]y → ∃!x([F]x & [R]xy))›
using $\equiv_{df}$*E*[*OF 1−1−cor*] &*E* **by** *blast+*
AOT-have *C*: ‹R |: F ⟶ G›
proof (*rule* $\equiv_{df}$*I*[*OF fFG*:*1*]; *rule* &*I*)
AOT-show ‹R↓ & F↓ & G↓›
using *cqt*:*2*[*const-var*][*axiom-inst*] &*I* **by** *metis*
next
AOT-show ‹∀x ([F]x → ∃!y([G]y & [R]xy))› **by** (*rule A*)
qed
AOT-show ‹R |: F $_{1-1}⟶_{onto}$ G›
proof (*rule* $\equiv_{df}$*I*[*OF fFG*:*4*]; *rule* &*I*)
AOT-show ‹R |: F $_{1-1}$⟶ G›
proof (*rule* $\equiv_{df}$*I*[*OF fFG*:*2*]; *rule* &*I*)
AOT-show ‹R |: F ⟶ G› **using** *C*.
next
AOT-show ‹∀x∀y∀z ([F]x & [F]y & [G]z → ([R]xz & [R]yz → x = y))›
proof(*rule GEN*; *rule GEN*; *rule GEN*; *rule* →*I*; *rule* →*I*)
fix *x y z*
AOT-assume *1*: ‹[F]x & [F]y & [G]z›
moreover AOT-assume *2*: ‹[R]xz & [R]yz›
ultimately AOT-have *3*: ‹∃!x ([F]x & [R]xz)›
using *B* &*E* ∀*E* →*E* **by** *fast*
AOT-show ‹x = y›
by (*rule uni−most*[*THEN* →*E*, *OF 3*, *THEN* ∀*E*(*2*)[**where** β=x],
THEN ∀*E*(*2*)[**where** β=y], *THEN* →*E*])
(*metis* &*I* &*E 1 2*)
qed
qed
next
AOT-show ‹R |: F $⟶_{onto}$ G›
proof (*rule* $\equiv_{df}$*I*[*OF fFG*:*3*]; *rule* &*I*)
AOT-show ‹R |: F ⟶ G› **using** *C*.
next
AOT-show ‹∀y ([G]y → ∃x ([F]x & [R]xy))›
proof(*rule GEN*; *rule* →*I*)

```
      fix y
      AOT-assume ‹[G]y›
      AOT-hence ‹∃!x ([F]x & [R]xy)›
        using B[THEN ∀E(2), THEN →E] by blast
      AOT-hence ‹∃x ([F]x & [R]xy & ∀β (([F]β & [R]βy) → β = x))›
        using uniqueness:1[THEN ≡df E] by blast
      then AOT-obtain x where ‹[F]x & [R]xy›
        using ∃E[rotated] &E by blast
      AOT-thus ‹∃x ([F]x & [R]xy)› by (rule ∃I)
    qed
  qed
 qed
next
  AOT-assume ‹R |: F 1−1⟶onto G›
  AOT-hence ‹R |: F 1−1⟶ G› and ‹R |: F ⟶onto G›
    using ≡df E[OF fFG:4] &E by blast+
  AOT-hence C: ‹R |: F ⟶ G›
    and D: ‹∀x∀y∀z ([F]x & [F]y & [G]z → ([R]xz & [R]yz → x = y))›
    and E: ‹∀y ([G]y → ∃x ([F]x & [R]xy))›
    using ≡df E[OF fFG:2] ≡df E[OF fFG:3] &E by blast+
  AOT-show ‹R |: F 1−1⟷ G›
  proof(rule 1−1−cor[THEN ≡df I]; safe intro!: &I cqt:2[const-var][axiom-inst])
    AOT-show ‹∀x ([F]x → ∃!y ([G]y & [R]xy))›
      using ≡df E[OF fFG:1, OF C] &E by blast
  next
    AOT-show ‹∀y ([G]y → ∃!x ([F]x & [R]xy))›
    proof (rule GEN; rule →I)
      fix y
      AOT-assume 0: ‹[G]y›
      AOT-hence ‹∃x ([F]x & [R]xy)›
        using E ∀E →E by fast
      then AOT-obtain a where a-prop: ‹[F]a & [R]ay›
        using ∃E[rotated] by blast
      moreover AOT-have ‹∀z ([F]z & [R]zy → z = a)›
      proof (rule GEN; rule →I)
        fix z
        AOT-assume ‹[F]z & [R]zy›
        AOT-thus ‹z = a›
          using D[THEN ∀E(2)[where β=z], THEN ∀E(2)[where β=a],
                THEN ∀E(2)[where β=y], THEN →E, THEN →E]
              a-prop 0 &E &I by metis
      qed
      ultimately AOT-have ‹∃x ([F]x & [R]xy & ∀z ([F]z & [R]zy → z = x))›
        using &I ∃I(2) by fast
      AOT-thus ‹∃!x ([F]x & [R]xy)›
        using uniqueness:1[THEN ≡df I] by fast
    qed
  qed
qed
```

We have already introduced the restricted type of Ordinary objects in the Extended Relation Comprehension theory. However, make sure all variable names are defined as expected (avoiding conflicts with situations of possible world theory).

```
AOT-register-variable-names
  Ordinary: u v r t s

AOT-theorem equi:1: ‹∃!u φ{u} ≡ ∃u (φ{u} & ∀v (φ{v} → v =E u))›
proof(rule ≡I; rule →I)
  AOT-assume ‹∃!u φ{u}›
  AOT-hence ‹∃!x (O!x & φ{x})›.
  AOT-hence ‹∃x (O!x & φ{x} & ∀β (O!β & φ{β} → β = x))›
    using uniqueness:1[THEN ≡df E] by blast
  then AOT-obtain x where x-prop: ‹O!x & φ{x} & ∀β (O!β & φ{β} → β = x)›
```

```
    using ∃E[rotated] by blast
  {
    fix β
    AOT-assume beta-ord: ‹O!β›
    moreover AOT-assume ‹φ{β}›
    ultimately AOT-have ‹β = x›
      using x-prop[THEN &E(2), THEN ∀E(2)[where β=β]] &I →E by blast
    AOT-hence ‹β =E x›
      using ord-=E=:1[THEN →E, OF ∨I(1)[OF beta-ord],
                       THEN qml:2[axiom-inst, THEN →E],
                       THEN ≡E(1)]
      by blast
  }
  AOT-hence ‹(O!β → (φ{β} → β =E x))› for β
    using →I by blast
  AOT-hence ‹∀β(O!β → (φ{β} → β =E x))›
    by (rule GEN)
  AOT-hence ‹O!x & φ{x} & ∀y (O!y → (φ{y} → y =E x))›
    using x-prop[THEN &E(1)] &I by blast
  AOT-hence ‹O!x & (φ{x} & ∀y (O!y → (φ{y} → y =E x)))›
    using &E &I by meson
  AOT-thus ‹∃u (φ{u} & ∀v (φ{v} → v =E u))›
    using ∃I by fast
next
  AOT-assume ‹∃u (φ{u} & ∀v (φ{v} → v =E u))›
  AOT-hence ‹∃x (O!x & (φ{x} & ∀y (O!y → (φ{y} → y =E x))))›
    by blast
  then AOT-obtain x where x-prop: ‹O!x & (φ{x} & ∀y (O!y → (φ{y} → y =E x)))›
    using ∃E[rotated] by blast
  AOT-have ‹∀y ([O!]y & φ{y} → y = x)›
  proof(rule GEN; rule →I)
    fix y
    AOT-assume ‹O!y & φ{y}›
    AOT-hence ‹y =E x›
      using x-prop[THEN &E(2), THEN &E(2), THEN ∀E(2)[where β=y]]
          →E &E by blast
    AOT-thus ‹y = x›
      using ord-=E=:1[THEN →E, OF ∨I(2)[OF x-prop[THEN &E(1)]],
                       THEN qml:2[axiom-inst, THEN →E], THEN ≡E(2)] by blast
  qed
  AOT-hence ‹[O!]x & φ{x} & ∀y ([O!]y & φ{y} → y = x)›
    using x-prop &E &I by meson
  AOT-hence ‹∃x ([O!]x & φ{x} & ∀y ([O!]y & φ{y} → y = x))›
    by (rule ∃I)
  AOT-hence ‹∃!x (O!x & φ{x})›
    by (rule uniqueness:1[THEN ≡dfI])
  AOT-thus ‹∃!u φ{u}›.
qed

AOT-define CorrelatesEOneToOne :: ‹τ ⇒ τ ⇒ τ ⇒ φ› (‹- |: - 1-1⟷E -›)
  equi:2: ‹R |: F 1-1⟷E G ≡df R↓ & F↓ & G↓ &
                              ∀u ([F]u → ∃!v([G]v & [R]uv)) &
                              ∀v ([G]v → ∃!u([F]u & [R]uv))›

AOT-define EquinumerousE :: ‹τ ⇒ τ ⇒ φ› (infixl ‹≈E› 50)
  equi:3: ‹F ≈E G ≡df ∃R (R |: F 1-1⟷E G)›
```

Note: not explicitly in PLM.

```
AOT-theorem eq-den-1: ‹Π↓› if ‹Π ≈E Π'›
proof -
  AOT-have ‹∃R (R |: Π 1-1⟷E Π')›
    using equi:3[THEN ≡dfE] that by blast
  then AOT-obtain R where ‹R |: Π 1-1⟷E Π'›
```

```
    using ∃E[rotated] by blast
  AOT-thus ‹Π↓›
    using equi:2[THEN ≡dfE] &E by blast
qed
```

Note: not explicitly in PLM.

```
AOT-theorem eq-den-2: ‹Π'↓› if ‹Π ≈E Π'›
proof -
  AOT-have ‹∃R (R |: Π 1-1⟷E Π')›
    using equi:3[THEN ≡dfE] that by blast
  then AOT-obtain R where ‹R |: Π 1-1⟷E Π'›
    using ∃E[rotated] by blast
  AOT-thus ‹Π'↓›
    using equi:2[THEN ≡dfE] &E by blast+
qed

AOT-theorem eq-part:1: ‹F ≈E F›
proof (safe intro!: &I GEN →I cqt:2[const-var][axiom-inst]
                    ≡dfI[OF equi:3] ≡dfI[OF equi:2] ∃I(1))
  fix x
  AOT-assume 1: ‹O!x›
  AOT-assume 2: ‹[F]x›
  AOT-show ‹∃!v ([F]v & x =E v)›
  proof(rule equi:1[THEN ≡E(2)];
        rule ∃I(2)[where β=x];
        safe dest!: &E(2)
             intro!:  &I →I 1 2 Ordinary.GEN ord=Eequiv:1[THEN →E, OF 1])
    AOT-show ‹v =E x› if ‹x =E v› for v
      by (metis that ord=Eequiv:2[THEN →E])
  qed
next
  fix y
  AOT-assume 1: ‹O!y›
  AOT-assume 2: ‹[F]y›
  AOT-show ‹∃!u ([F]u & u =E y)›
    by(safe dest!: &E(2)
           intro!: equi:1[THEN ≡E(2)] ∃I(2)[where β=y]
                  &I →I 1 2 GEN ord=Eequiv:1[THEN →E, OF 1])
qed(auto simp: =E[denotes])


AOT-theorem eq-part:2: ‹F ≈E G → G ≈E F›
proof (rule →I)
  AOT-assume ‹F ≈E G›
  AOT-hence ‹∃R R |: F 1-1⟷E G›
    using equi:3[THEN ≡dfE] by blast
  then AOT-obtain R where ‹R |: F 1-1⟷E G›
    using ∃E[rotated] by blast
  AOT-hence 0: ‹R↓ & F↓ & G↓ & ∀u ([F]u → ∃!v([G]v & [R]uv)) &
                          ∀v ([G]v → ∃!u([F]u & [R]uv))›
    using equi:2[THEN ≡dfE] by blast

  AOT-have ‹[λxy [R]yx]↓ & G↓ & F↓ & ∀u ([G]u → ∃!v([F]v & [λxy [R]yx]uv)) &
                          ∀v ([F]v → ∃!u([G]u & [λxy [R]yx]uv))›
  proof (AOT-subst ‹[λxy [R]yx]yx› ‹[R]xy› for: x y;
        (safe intro!: &I cqt:2[const-var][axiom-inst] 0[THEN &E(2)]
                    0[THEN &E(1), THEN &E(2)]; cqt:2[lambda])?)
    AOT-modally-strict {
      AOT-have ‹[λxy [R]yx]xy› if ‹[R]yx› for y x
        by (auto intro!: β←C(1) cqt:2
                simp: &I ex:1:a prod-denotesI rule-ui:3 that)
      moreover AOT-have ‹[R]yx› if ‹[λxy [R]yx]xy› for y x
        using β→C(1)[where φ=λ(x,y). - (x,y) and κ1κn=(-,-),
```

simplified, *OF that*, *simplified*].
ultimately AOT-show ‹$[\lambda xy\ [R]yx]\alpha\beta \equiv [R]\beta\alpha$› **for** α β
by (*metis deduction−theorem* ≡*I*)
}
qed
AOT-hence ‹$[\lambda xy\ [R]yx]$ |: $G\ {}_{1-1}{\longleftrightarrow}_E\ F$›
using *equi*:*2*[*THEN* $\equiv_{df}I$] **by** *blast*
AOT-hence ‹$\exists\, R\ R$ |: $G\ {}_{1-1}{\longleftrightarrow}_E\ F$›
by (*rule* ∃ *I*(*1*)) *cqt*:*2*[*lambda*]
AOT-thus ‹$G \approx_E F$›
using *equi*:*3*[*THEN* $\equiv_{df}I$] **by** *blast*
qed

Note: not explicitly in PLM.

AOT-theorem *eq−part*:*2*[*terms*]: ‹$\Pi \approx_E \Pi' \rightarrow \Pi' \approx_E \Pi$›
using *eq−part*:*2*[*unvarify F G*] *eq-den-1 eq-den-2* →*I* **by** *meson*
declare *eq−part*:*2*[*terms*][*THEN* →*E*, *sym*]

AOT-theorem *eq−part*:*3*: ‹$(F \approx_E G\ \&\ G \approx_E H) \rightarrow F \approx_E H$›
proof (*rule* →*I*)
AOT-assume ‹$F \approx_E G\ \&\ G \approx_E H$›
then AOT-obtain R_1 **and** R_2 **where**
‹R_1 |: $F\ {}_{1-1}{\longleftrightarrow}_E\ G$›
and ‹R_2 |: $G\ {}_{1-1}{\longleftrightarrow}_E\ H$›
using *equi*:*3*[*THEN* $\equiv_{df}E$] &*E* ∃ *E*[*rotated*] **by** *metis*
AOT-hence ϑ: ‹$\forall\, u\ ([F]u \rightarrow \exists!v([G]v\ \&\ [R_1]uv))\ \&\ \forall\, v\ ([G]v \rightarrow \exists!u([F]u\ \&\ [R_1]uv))$›
and ξ: ‹$\forall\, u\ ([G]u \rightarrow \exists!v([H]v\ \&\ [R_2]uv))\ \&\ \forall\, v\ ([H]v \rightarrow \exists!u([G]u\ \&\ [R_2]uv))$›
using *equi*:*2*[*THEN* $\equiv_{df}E$, *THEN* &*E*(*2*)]
equi:*2*[*THEN* $\equiv_{df}E$, *THEN* &*E*(*1*), *THEN* &*E*(*2*)]
&*I* **by** *blast*+
AOT-have ‹$\exists\, R\ R = [\lambda xy\ O!x\ \&\ O!y\ \&\ \exists\, v\ ([G]v\ \&\ [R_1]xv\ \&\ [R_2]vy)]$›
by (*rule free−thms*:*3*[*lambda*]) *cqt-2-lambda-inst-prover*
then AOT-obtain *R* **where** *R-def*: ‹$R = [\lambda xy\ O!x\ \&\ O!y\ \&\ \exists\, v\ ([G]v\ \&\ [R_1]xv\ \&\ [R_2]vy)]$›
using ∃ *E*[*rotated*] **by** *blast*
AOT-have *1*: ‹$\exists!v\ (([H]v\ \&\ [R]uv))$› **if** *a*: ‹$[O!]u$› **and** *b*: ‹$[F]u$› **for** *u*
proof (*rule* ≡*E*(*2*)[*OF equi*:*1*])
AOT-obtain *b* **where**
b-prop: ‹$[O!]b\ \&\ ([G]b\ \&\ [R_1]ub\ \&\ \forall\, v\ ([G]v\ \&\ [R_1]uv \rightarrow v =_E b))$›
using ϑ[*THEN* &*E*(*1*), *THEN* ∀ *E*(*2*), *THEN* →*E*, *THEN* →*E*,
OF a b, *THEN* ≡*E*(*1*)[*OF equi*:*1*]]
∃ *E*[*rotated*] **by** *blast*
AOT-obtain *c* **where**
c-prop: $[O!]c\ \&\ ([H]c\ \&\ [R_2]bc\ \&\ \forall\, v\ ([H]v\ \&\ [R_2]bv \rightarrow v =_E c))$
using ξ[*THEN* &*E*(*1*), *THEN* ∀ *E*(*2*)[**where** β=*b*], *THEN* →*E*,
OF b-prop[*THEN* &*E*(*1*)], *THEN* →*E*,
OF b-prop[*THEN* &*E*(*2*), *THEN* &*E*(*1*), *THEN* &*E*(*1*)],
THEN ≡*E*(*1*)[*OF equi*:*1*]]
∃ *E*[*rotated*] **by** *blast*
AOT-show ‹$\exists\, v\ ([H]v\ \&\ [R]uv\ \&\ \forall\, v'\ ([H]v'\ \&\ [R]uv' \rightarrow v' =_E v))$›
proof (*safe intro*!: &*I GEN* →*I* ∃ *I*(*2*)[**where** β=*c*])
AOT-show ‹$O!c$› **using** *c-prop* &*E* **by** *blast*
next
AOT-show ‹$[H]c$› **using** *c-prop* &*E* **by** *blast*
next
AOT-have *0*: ‹$[O!]u\ \&\ [O!]c\ \&\ \exists\, v\ ([G]v\ \&\ [R_1]uv\ \&\ [R_2]vc)$›
by (*safe intro*!: &*I a c-prop*[*THEN* &*E*(*1*)] ∃ *I*(*2*)[**where** β=*b*]
b-prop[*THEN* &*E*(*1*)] *b-prop*[*THEN* &*E*(*2*), *THEN* &*E*(*1*)]
c-prop[*THEN* &*E*(*2*), *THEN* &*E*(*1*), *THEN* &*E*(*2*)])
AOT-show ‹$[R]uc$›
by (*auto intro*: *rule*=*E*[*rotated*, *OF R-def*[*symmetric*]]
intro!: $\beta{\leftarrow}C(1)$ *cqt*:*2*
simp: &*I ex*:*1*:*a prod-denotesI rule−ui*:*3 0*)
next

```
      fix x
      AOT-assume ordx: ‹O!x›
      AOT-assume ‹[H]x & [R]ux›
      AOT-hence hx: ‹[H]x› and ‹[R]ux› using &E by blast+
      AOT-hence ‹[λxy O!x & O!y & ∃ v ([G]v & [R1]xv & [R2]vy)]ux›
        using rule=E[rotated, OF R-def] by fast
      AOT-hence ‹O!u & O!x & ∃ v ([G]v & [R1]uv & [R2]vx)›
        by (rule β→C(1)[where φ=λ(κ,κ'). - κ κ' and κ1κn=(-,-), simplified])
      then AOT-obtain z where z-prop: ‹O!z & ([G]z & [R1]uz & [R2]zx)›
        using &E ∃ E[rotated] by blast
      AOT-hence ‹z =E b›
        using b-prop[THEN &E(2), THEN &E(2), THEN ∀ E(2)[where β=z]]
        using &E →E by metis
      AOT-hence ‹z = b›
        by (metis =E−simple:2[THEN →E])
      AOT-hence ‹[R2]bx›
        using z-prop[THEN &E(2), THEN &E(2)] rule=E by fast
      AOT-thus ‹x =E c›
        using c-prop[THEN &E(2), THEN &E(2), THEN ∀ E(2)[where β=x],
                  THEN →E, THEN →E, OF ordx]
            hx &I by blast
    qed
  qed
  AOT-have 2: ‹∃!u (([F]u & [R]uv))› if a: ‹[O!]v› and b: ‹[H]v› for v
  proof (rule ≡E(2)[OF equi:1])
    AOT-obtain b where
      b-prop: ‹[O!]b & ([G]b & [R2]bv & ∀ u ([G]u & [R2]uv → u =E b))›
      using ξ[THEN &E(2), THEN ∀ E(2), THEN →E, THEN →E,
            OF a b, THEN ≡E(1)[OF equi:1]]
          ∃ E[rotated] by blast
    AOT-obtain c where
      c-prop: [O!]c & ([F]c & [R1]cb & ∀ v ([F]v & [R1]vb → v =E c))
      using ϑ[THEN &E(2), THEN ∀ E(2)[where β=b], THEN →E,
            OF b-prop[THEN &E(1)], THEN →E,
            OF b-prop[THEN &E(2), THEN &E(1), THEN &E(1)],
            THEN ≡E(1)[OF equi:1]]
    ∃ E[rotated] by blast
    AOT-show ‹∃ u ([F]u & [R]uv & ∀ v' ([F]v' & [R]v'v → v' =E u))›
    proof (safe intro!: &I GEN →I ∃ I(2)[where β=c])
      AOT-show ‹O!c› using c-prop &E by blast
    next
      AOT-show ‹[F]c› using c-prop &E by blast
    next
      AOT-have ‹[O!]c & [O!]v & ∃ u ([G]u & [R1]cu & [R2]uv)›
        by (safe intro!: &I a ∃ I(2)[where β=b]
                  c-prop[THEN &E(1)] b-prop[THEN &E(1)]
                  b-prop[THEN &E(2), THEN &E(1), THEN &E(1)]
                  b-prop[THEN &E(2), THEN &E(1), THEN &E(2)]
                  c-prop[THEN &E(2), THEN &E(1), THEN &E(2)])
      AOT-thus ‹[R]cv›
        by (auto intro: rule=E[rotated, OF R-def[symmetric]]
               intro!: β←C(1) cqt:2
               simp: &I ex:1:a prod-denotesI rule−ui:3)
    next
      fix x
      AOT-assume ordx: ‹O!x›
      AOT-assume ‹[F]x & [R]xv›
      AOT-hence hx: ‹[F]x› and ‹[R]xv› using &E by blast+
      AOT-hence ‹[λxy O!x & O!y & ∃ v ([G]v & [R1]xv & [R2]vy)]xv›
        using rule=E[rotated, OF R-def] by fast
      AOT-hence ‹O!x & O!v & ∃ u ([G]u & [R1]xu & [R2]uv)›
        by (rule β→C(1)[where φ=λ(κ,κ'). - κ κ' and κ1κn=(-,-), simplified])
      then AOT-obtain z where z-prop: ‹O!z & ([G]z & [R1]xz & [R2]zv)›
```

using &E ∃ E[*rotated*] **by** *blast*
AOT-hence ‹$z =_E b$›
using *b-prop*[*THEN* &E(*2*), *THEN* &E(*2*), *THEN* ∀ E(*2*)[**where** β=z]]
using &E →E &I **by** *metis*
AOT-hence ‹$z = b$›
by (*metis* =E−*simple*:*2*[*THEN* →E])
AOT-hence ‹$[R_1]xb$›
using *z-prop*[*THEN* &E(*2*), *THEN* &E(*1*), *THEN* &E(*2*)] *rule*=E **by** *fast*
AOT-thus ‹$x =_E c$›
using *c-prop*[*THEN* &E(*2*), *THEN* &E(*2*), *THEN* ∀ E(*2*)[**where** β=x],
THEN →E, *THEN* →E, *OF* *ordx*]
hx &I **by** *blast*
qed
qed
AOT-show ‹$F \approx_E H$›
apply (*rule* *equi*:*3*[*THEN* $\equiv_{df}I$])
apply (*rule* ∃ I(*2*)[**where** β=R])
by (*auto* *intro*!: *1* *2* *equi*:*2*[*THEN* $\equiv_{df}I$] &I *cqt*:*2*[*const-var*][*axiom-inst*]
Ordinary.GEN →I *Ordinary*.ψ)
qed

Note: not explicitly in PLM.

AOT-theorem *eq*−*part*:*3*[*terms*]: ‹$\Pi \approx_E \Pi''$› **if** ‹$\Pi \approx_E \Pi'$› **and** ‹$\Pi' \approx_E \Pi''$›
using *eq*−*part*:*3*[*unvarify* F G H, *THEN* →E] *eq-den-1* *eq-den-2* →I &I
by (*metis* *that*(*1*) *that*(*2*))
declare *eq*−*part*:*3*[*terms*][*trans*]

AOT-theorem *eq*−*part*:*4*: ‹$F \approx_E G \equiv \forall H\ (H \approx_E F \equiv H \approx_E G)$›
proof(*rule* ≡I; *rule* →I)
AOT-assume *0*: ‹$F \approx_E G$›
AOT-hence *1*: ‹$G \approx_E F$› **using** *eq*−*part*:*2*[*THEN* →E] **by** *blast*
AOT-show ‹$\forall H\ (H \approx_E F \equiv H \approx_E G)$›
proof (*rule* *GEN*; *rule* ≡I; *rule* →I)
AOT-show ‹$H \approx_E G$› **if** ‹$H \approx_E F$› **for** H **using** *0*
by (*meson* &I *eq*−*part*:*3* *that* *vdash*−*properties*:*6*)
next
AOT-show ‹$H \approx_E F$› **if** ‹$H \approx_E G$› **for** H **using** *1*
by (*metis* &I *eq*−*part*:*3* *that* *vdash*−*properties*:*6*)
qed
next
AOT-assume ‹$\forall H\ (H \approx_E F \equiv H \approx_E G)$›
AOT-hence ‹$F \approx_E F \equiv F \approx_E G$› **using** ∀ E **by** *blast*
AOT-thus ‹$F \approx_E G$› **using** *eq*−*part*:*1* ≡E **by** *blast*
qed

AOT-define *MapsE* :: ‹$\tau \Rightarrow \tau \Rightarrow \tau \Rightarrow \varphi$› (‹- |: - ⟶$E$ -›)
equi−*rem*:*1*:
‹R |: F ⟶E G $\equiv_{df}$ R↓ & F↓ & G↓ & ∀ u ([F]u → ∃!v ([G]v & [R]uv))›

AOT-define *MapsEOneToOne* :: ‹$\tau \Rightarrow \tau \Rightarrow \tau \Rightarrow \varphi$› (‹- |: - $_{1-1}$⟶E -›)
equi−*rem*:*2*:
‹R |: F $_{1-1}$⟶E G $\equiv_{df}$
R |: F ⟶E G & ∀ t∀ u∀ v (([F]t & [F]u & [G]v) → ([R]tv & [R]uv → $t =_E u$))›

AOT-define *MapsEOnto* :: ‹$\tau \Rightarrow \tau \Rightarrow \tau \Rightarrow \varphi$› (‹- |: - ⟶$_{onto}E$ -›)
equi−*rem*:*3*:
‹R |: F ⟶$_{onto}E$ G $\equiv_{df}$ R |: F ⟶E G & ∀ v ([G]v → ∃ u ([F]u & [R]uv))›

AOT-define *MapsEOneToOneOnto* :: ‹$\tau \Rightarrow \tau \Rightarrow \tau \Rightarrow \varphi$› (‹- |: - $_{1-1}$⟶$_{onto}E$ -›)
equi−*rem*:*4*:
‹R |: F $_{1-1}$⟶$_{onto}E$ G $\equiv_{df}$ R |: F $_{1-1}$⟶E G & R |: F ⟶$_{onto}E$ G›

AOT-theorem *equi*−*rem*−*thm*:

‹R |: F $_{1-1}$⟷$_E$ G ≡ R |: F $_{1-1}$⟶$_{onto}$E G›
proof –
AOT-have ‹R |: F $_{1-1}$⟷$_E$ G ≡ R |: [λx O!x & [F]x] $_{1-1}$⟷ [λx O!x & [G]x]›
proof(*safe intro*!: ≡I →I &I)
AOT-assume ‹R |: F $_{1-1}$⟷$_E$ G›
AOT-hence ‹∀ u ([F]u → ∃!v ([G]v & [R]uv))›
and ‹∀ v ([G]v → ∃!u ([F]u & [R]uv))›
using *equi*:*2*[*THEN* ≡$_{df}$*E*] &*E* **by** *blast*+
AOT-hence *a*: ‹([F]u → ∃!v ([G]v & [R]uv))›
and *b*: ‹([G]v → ∃!u ([F]u & [R]uv))› **for** *u v*
using *Ordinary*.∀ *E* **by** *fast*+
AOT-have ‹([λx [O!]x & [F]x]x → ∃!y ([λx [O!]x & [G]x]y & [R]xy))› **for** *x*
apply (*AOT-subst* ‹[λx [O!]x & [F]x]x› ‹[O!]x & [F]x›)
apply (*rule beta*−*C*−*meta*[*THEN* →*E*])
apply *cqt*:*2*[*lambda*]
apply (*AOT-subst* ‹[λx [O!]x & [G]x]x› ‹[O!]x & [G]x› **for**: *x*)
apply (*rule beta*−*C*−*meta*[*THEN* →*E*])
apply *cqt*:*2*[*lambda*]
apply (*AOT-subst* ‹O!y & [G]y & [R]xy› ‹O!y & ([G]y & [R]xy)› **for**: *y*)
apply (*meson* ≡*E*(*6*) *Associativity of* & *oth*−*class*−*taut*:*3*:*a*)
apply (*rule* →*I*) **apply** (*frule* &*E*(*1*)) **apply** (*drule* &*E*(*2*))
by (*fact a*[*unconstrain u*, *THEN* →*E*, *THEN* →*E*, *of x*])
AOT-hence *A*: ‹∀ x ([λx [O!]x & [F]x]x → ∃!y ([λx [O!]x & [G]x]y & [R]xy))›
by (*rule GEN*)
AOT-have ‹([λx [O!]x & [G]x]y → ∃!x ([λx [O!]x & [F]x]x & [R]xy))› **for** *y*
apply (*AOT-subst* ‹[λx [O!]x & [G]x]y› ‹[O!]y & [G]y›)
apply (*rule beta*−*C*−*meta*[*THEN* →*E*])
apply *cqt*:*2*[*lambda*]
apply (*AOT-subst* ‹[λx [O!]x & [F]x]x› ‹[O!]x & [F]x› **for**: *x*)
apply (*rule beta*−*C*−*meta*[*THEN* →*E*])
apply *cqt*:*2*[*lambda*]
apply (*AOT-subst* ‹O!x & [F]x & [R]xy› ‹O!x & ([F]x & [R]xy)› **for**: *x*)
apply (*meson* ≡*E*(*6*) *Associativity of* & *oth*−*class*−*taut*:*3*:*a*)
apply (*rule* →*I*) **apply** (*frule* &*E*(*1*)) **apply** (*drule* &*E*(*2*))
by (*fact b*[*unconstrain v*, *THEN* →*E*, *THEN* →*E*, *of y*])
AOT-hence *B*: ‹∀ y ([λx [O!]x & [G]x]y → ∃!x ([λx [O!]x & [F]x]x & [R]xy))›
by (*rule GEN*)
AOT-show ‹R |: [λx [O!]x & [F]x] $_{1-1}$⟷ [λx [O!]x & [G]x]›
by (*safe intro*!: *1*−*1*−*cor*[*THEN* ≡$_{df}$*I*] &*I*
cqt:*2*[*const-var*][*axiom-inst*] *A B*)
cqt:*2*[*lambda*]+
next
AOT-assume ‹R |: [λx [O!]x & [F]x] $_{1-1}$⟷ [λx [O!]x & [G]x]›
AOT-hence *a*: ‹([λx [O!]x & [F]x]x → ∃!y ([λx [O!]x & [G]x]y & [R]xy))› **and**
b: ‹([λx [O!]x & [G]x]y → ∃!x ([λx [O!]x & [F]x]x & [R]xy))› **for** *x y*
using *1*−*1*−*cor*[*THEN* ≡$_{df}$*E*] &*E* ∀ *E*(*2*) **by** *blast*+
AOT-have ‹[F]u → ∃!v ([G]v & [R]uv)› **for** *u*
proof (*safe intro*!: →*I*)
AOT-assume *fu*: ‹[F]u›
AOT-have *0*: ‹[λx [O!]x & [F]x]u›
by (*auto intro*!: *β*←*C*(*1*) *cqt*:*2 cqt*:*2*[*const-var*][*axiom-inst*]
Ordinary.*ψ fu* &*I*)
AOT-show ‹∃!v ([G]v & [R]uv)›
apply (*AOT-subst* ‹[O!]x & ([G]x & [R]ux)›
‹([O!]x & [G]x) & [R]ux› **for**: *x*)
apply (*simp add*: *Associativity of* &)
apply (*AOT-subst* (*reverse*) ‹[O!]x & [G]x›
‹[λx [O!]x & [G]x]x› **for**: *x*)
apply (*rule beta*−*C*−*meta*[*THEN* →*E*])
apply *cqt*:*2*[*lambda*]
using *a*[*THEN* →*E*, *OF 0*] **by** *blast*
qed
AOT-hence *A*: ‹∀ u ([F]u → ∃!v ([G]v & [R]uv))›

by (*rule Ordinary.GEN*)
AOT-have ‹$[G]v \rightarrow \exists!u\ ([F]u\ \&\ [R]uv)$› **for** v
proof (*safe intro*!: →*I*)
AOT-assume *gu*: ‹$[G]v$›
AOT-have *0*: ‹$[\lambda x\ [O!]x\ \&\ [G]x]v$›
by (*auto intro*!: *β*←*C*(*1*) *cqt*:*2 cqt*:*2*[*const-var*][*axiom-inst*]
Ordinary.ψ gu &*I*)
AOT-show ‹$\exists!u\ ([F]u\ \&\ [R]uv)$›
apply (*AOT-subst* ‹$[O!]x\ \&\ ([F]x\ \&\ [R]xv)$› ‹$([O!]x\ \&\ [F]x)\ \&\ [R]xv$› **for**: x)
apply (*simp add*: *Associativity of* &)
apply (*AOT-subst* (*reverse*) ‹$[O!]x\ \&\ [F]x$›‹$[\lambda x\ [O!]x\ \&\ [F]x]x$› **for**: x)
apply (*rule beta*−*C*−*meta*[*THEN* →*E*])
apply *cqt*:*2*[*lambda*]
using *b*[*THEN* →*E*, *OF 0*] **by** *blast*
qed
AOT-hence *B*: ‹$\forall v\ ([G]v \rightarrow \exists!u\ ([F]u\ \&\ [R]uv))$› **by** (*rule Ordinary.GEN*)
AOT-show ‹$R\ |{:}\ F\ {}_{1-1}{\longleftrightarrow}_E\ G$›
by (*safe intro*!: *equi*:*2*[*THEN* $\equiv_{df}$*I*] &*I A B cqt*:*2*[*const-var*][*axiom-inst*])
qed
also AOT-have ‹$\ldots \equiv R\ |{:}\ F\ {}_{1-1}{\longrightarrow}_{onto}E\ G$›
proof(*safe intro*!: ≡*I* →*I* &*I*)
AOT-assume ‹$R\ |{:}\ [\lambda x\ [O!]x\ \&\ [F]x]\ {}_{1-1}{\longleftrightarrow}\ [\lambda x\ [O!]x\ \&\ [G]x]$›
AOT-hence *a*: ‹$([\lambda x\ [O!]x\ \&\ [F]x]x \rightarrow \exists!y\ ([\lambda x\ [O!]x\ \&\ [G]x]y\ \&\ [R]xy))$› **and**
b: ‹$([\lambda x\ [O!]x\ \&\ [G]x]y \rightarrow \exists!x\ ([\lambda x\ [O!]x\ \&\ [F]x]x\ \&\ [R]xy))$› **for** $x\ y$
using *1*−*1*−*cor*[*THEN* $\equiv_{df}$*E*] &*E* ∀*E*(*2*) **by** *blast*+
AOT-show ‹$R\ |{:}\ F\ {}_{1-1}{\longrightarrow}_{onto}E\ G$›
proof (*safe intro*!: *equi*−*rem*:*4*[*THEN* $\equiv_{df}$*I*] &*I equi*−*rem*:*3*[*THEN* $\equiv_{df}$*I*]
equi−*rem*:*2*[*THEN* $\equiv_{df}$*I*] *equi*−*rem*:*1*[*THEN* $\equiv_{df}$*I*]
cqt:*2*[*const-var*][*axiom-inst*] *Ordinary.GEN* →*I*)
fix u
AOT-assume *fu*: ‹$[F]u$›
AOT-have *0*: ‹$[\lambda x\ [O!]x\ \&\ [F]x]u$›
by (*auto intro*!: *β*←*C*(*1*) *cqt*:*2 cqt*:*2*[*const-var*][*axiom-inst*]
Ordinary.ψ fu &*I*)
AOT-hence *1*: ‹$\exists!y\ ([\lambda x\ [O!]x\ \&\ [G]x]y\ \&\ [R]uy)$›
using *a*[*THEN* →*E*] **by** *blast*
AOT-show ‹$\exists!v\ ([G]v\ \&\ [R]uv)$›
apply (*AOT-subst* ‹$[O!]x\ \&\ ([G]x\ \&\ [R]ux)$› ‹$([O!]x\ \&\ [G]x)\ \&\ [R]ux$› **for**: x)
apply (*simp add*: *Associativity of* &)
apply (*AOT-subst* (*reverse*) ‹$[O!]x\ \&\ [G]x$› ‹$[\lambda x\ [O!]x\ \&\ [G]x]x$› **for**: x)
apply (*rule beta*−*C*−*meta*[*THEN* →*E*])
apply *cqt*:*2*[*lambda*]
by (*fact 1*)
next
fix $t\ u\ v$
AOT-assume ‹$[F]t\ \&\ [F]u\ \&\ [G]v$› **and** *rtv-tuv*: ‹$[R]tv\ \&\ [R]uv$›
AOT-hence *oft*: ‹$[\lambda x\ O!x\ \&\ [F]x]t$› **and**
ofu: ‹$[\lambda x\ O!x\ \&\ [F]x]u$› **and**
ogv: ‹$[\lambda x\ O!x\ \&\ [G]x]v$›
by (*auto intro*!: *β*←*C*(*1*) *cqt*:*2* &*I*
simp: *Ordinary.ψ dest*: &*E*)
AOT-hence ‹$\exists!x\ ([\lambda x\ [O!]x\ \&\ [F]x]x\ \&\ [R]xv)$›
using *b*[*THEN* →*E*] **by** *blast*
then AOT-obtain a **where**
a-prop: ‹$[\lambda x\ [O!]x\ \&\ [F]x]a\ \&\ [R]av\ \&$
$\forall x\ (([\lambda x\ [O!]x\ \&\ [F]x]x\ \&\ [R]xv) \rightarrow x = a)$›
using *uniqueness*:*1*[*THEN* $\equiv_{df}$*E*] ∃*E*[*rotated*] **by** *blast*
AOT-hence *ua*: ‹$u = a$›
using *ofu rtv-tuv*[*THEN* &*E*(*2*)] ∀*E*(*2*) →*E* &*I* &*E*(*2*) **by** *blast*
moreover AOT-have *ta*: ‹$t = a$›
using *a-prop oft rtv-tuv*[*THEN* &*E*(*1*)] ∀*E*(*2*) →*E* &*I* &*E*(*2*) **by** *blast*
ultimately AOT-have ‹$t = u$› **by** (*metis rule*=*E id-sym*)
AOT-thus ‹$t =_E u$›

```
      using rule=E id-sym ord=Eequiv:1 Ordinary.ψ ta ua →E by fast
  next
    fix u
    AOT-assume ‹[F]u›
    AOT-hence ‹[λx O!x & [F]x]u›
      by (auto intro!: β←C(1) cqt:2 &I
              simp: cqt:2[const-var][axiom-inst]  Ordinary.ψ)
    AOT-hence ‹∃!y ([λx [O!]x & [G]x]y & [R]uy)›
      using a[THEN →E] by blast
    then AOT-obtain a where
      a-prop: ‹[λx [O!]x & [G]x]a & [R]ua &
              ∀x (([λx [O!]x & [G]x]x & [R]ux) → x = a)›
      using uniqueness:1[THEN ≡df E] ∃E[rotated] by blast
    AOT-have ‹O!a & [G]a›
      by (rule β→C(1)) (auto simp: a-prop[THEN &E(1), THEN &E(1)])
    AOT-hence ‹O!a› and ‹[G]a› using &E by blast+
    moreover AOT-have ‹∀v ([G]v & [R]uv → v =E a)›
    proof(safe intro!: Ordinary.GEN →I; frule &E(1); drule &E(2))
      fix v
      AOT-assume ‹[G]v› and ruv: ‹[R]uv›
      AOT-hence ‹[λx [O!]x & [G]x]v›
        by (auto intro!: β←C(1) cqt:2 &I simp: Ordinary.ψ)
      AOT-hence ‹v = a›
        using a-prop[THEN &E(2), THEN ∀E(2), THEN →E, OF &I] ruv by blast
      AOT-thus ‹v =E a›
        using rule=E ord=Eequiv:1 Ordinary.ψ →E by fast
    qed
    ultimately AOT-have ‹O!a & ([G]a & [R]ua & ∀v' ([G]v' & [R]uv' → v' =E a))›
      using ∃I &I a-prop[THEN &E(1), THEN &E(2)] by simp
    AOT-hence ‹∃v ([G]v & [R]uv & ∀v' ([G]v' & [R]uv' → v' =E v))›
      by (rule ∃I)
    AOT-thus ‹∃!v ([G]v & [R]uv)›
      by (rule equi:1[THEN ≡E(2)])
  next
    fix v
    AOT-assume ‹[G]v›
    AOT-hence ‹[λx O!x & [G]x]v›
      by (auto intro!: β←C(1) cqt:2 &I Ordinary.ψ)
    AOT-hence ‹∃!x ([λx [O!]x & [F]x]x & [R]xv)›
      using b[THEN →E] by blast
    then AOT-obtain a where
      a-prop: ‹[λx [O!]x & [F]x]a & [R]av &
              ∀y ([λx [O!]x & [F]x]y & [R]yv → y = a)›
      using uniqueness:1[THEN ≡df E, THEN ∃E[rotated]] by blast
    AOT-have ‹O!a & [F]a›
      by (rule β→C(1)) (auto simp: a-prop[THEN &E(1), THEN &E(1)])
    AOT-hence ‹O!a & ([F]a & [R]av)›
      using a-prop[THEN &E(1), THEN &E(2)] &E &I by metis
    AOT-thus ‹∃u ([F]u & [R]uv)›
      by (rule ∃I)
  qed
next
  AOT-assume ‹R |: F 1−1⟶onto E G›
  AOT-hence 1: ‹R |: F 1−1⟶E G›
      and 2: ‹R |: F ⟶onto E G›
    using equi−rem:4[THEN ≡df E] &E by blast+
  AOT-hence 3: ‹R |: F ⟶E G›
      and A: ‹∀t ∀u ∀v ([F]t & [F]u & [G]v → ([R]tv & [R]uv → t =E u))›
    using equi−rem:2[THEN ≡df E, OF 1] &E by blast+
  AOT-hence B: ‹∀u ([F]u → ∃!v ([G]v & [R]uv))›
    using equi−rem:1[THEN ≡df E] &E by blast
  AOT-have C: ‹∀v ([G]v → ∃u ([F]u & [R]uv))›
    using equi−rem:3[THEN ≡df E, OF 2] &E by blast
```

```
AOT-show ‹R |: [λx [O!]x & [F]x] 1−1⟷ [λx [O!]x & [G]x]›
proof (rule 1−1−cor[THEN ≡df I];
       safe intro!: &I cqt:2 GEN →I)
  fix x
  AOT-assume 1: ‹[λx [O!]x & [F]x]x›
  AOT-have ‹O!x & [F]x›
    by (rule β→C(1)) (auto simp: 1)
  AOT-hence ‹∃!v ([G]v & [R]xv)›
    using B[THEN ∀E(2), THEN →E, THEN →E] &E by blast
  then AOT-obtain y where
    y-prop: ‹O!y & ([G]y & [R]xy & ∀u ([G]u & [R]xu → u =E y))›
    using equi:1[THEN ≡E(1)] ∃E[rotated] by fastforce
  AOT-hence ‹[λx O!x & [G]x]y›
    by (auto intro!: β←C(1) cqt:2 &I dest: &E)
  moreover AOT-have ‹∀z ([λx O!x & [G]x]z & [R]xz → z = y)›
  proof(safe intro!: GEN →I; frule &E(1); drule &E(2))
    fix z
    AOT-assume 1: ‹[λx [O!]x & [G]x]z›
    AOT-have 2: ‹O!z & [G]z›
      by (rule β→C(1)) (auto simp: 1)
    moreover AOT-assume ‹[R]xz›
    ultimately AOT-have ‹z =E y›
      using y-prop[THEN &E(2), THEN &E(2), THEN ∀E(2),
                 THEN →E, THEN →E, rotated, OF &I] &E
      by blast
    AOT-thus ‹z = y›
      using 2[THEN &E(1)] by (metis =E−simple:2 →E)
  qed
  ultimately AOT-have ‹[λx O!x & [G]x]y & [R]xy &
                       ∀z ([λx O!x & [G]x]z & [R]xz → z = y)›
    using y-prop[THEN &E(2), THEN &E(1), THEN &E(2)] &I by auto
  AOT-hence ‹∃y ([λx O!x & [G]x]y & [R]xy &
             ∀z ([λx O!x & [G]x]z & [R]xz → z = y))›
    by (rule ∃I)
  AOT-thus ‹∃!y ([λx [O!]x & [G]x]y & [R]xy)›
    using uniqueness:1[THEN ≡df I] by fast
next
  fix y
  AOT-assume 1: ‹[λx [O!]x & [G]x]y›
  AOT-have oy-gy: ‹O!y & [G]y›
    by (rule β→C(1)) (auto simp: 1)
  AOT-hence ‹∃u ([F]u & [R]uy)›
    using C[THEN ∀E(2), THEN →E, THEN →E] &E by blast
  then AOT-obtain x where x-prop: ‹O!x & ([F]x & [R]xy)›
    using ∃E[rotated] by blast
  AOT-hence ofx: ‹[λx O!x & [F]x]x›
    by (auto intro!: β←C(1) cqt:2 &I dest: &E)
  AOT-have ‹∃α ([λx [O!]x & [F]x]α & [R]αy &
            ∀β ([λx [O!]x & [F]x]β & [R]βy → β = α))›
  proof (safe intro!: ∃I(2)[where β=x] &I GEN →I)
    AOT-show ‹[λx O!x & [F]x]x› using ofx.
  next
    AOT-show ‹[R]xy› using x-prop[THEN &E(2), THEN &E(2)].
  next
    fix z
    AOT-assume 1: ‹[λx [O!]x & [F]x]z & [R]zy›
    AOT-have oz-fz: ‹O!z & [F]z›
      by (rule β→C(1)) (auto simp: 1[THEN &E(1)])
    AOT-have ‹z =E x›
      using A[THEN ∀E(2)[where β=z], THEN →E, THEN ∀E(2)[where β=x],
            THEN →E, THEN ∀E(2)[where β=y], THEN →E,
            THEN →E, THEN →E, OF oz-fz[THEN &E(1)],
            OF x-prop[THEN &E(1)], OF oy-gy[THEN &E(1)], OF &I, OF &I,
```

OF oz-fz[*THEN* &*E*(*2*)], *OF x-prop*[*THEN* &*E*(*2*), *THEN* &*E*(*1*)],
OF oy-gy[*THEN* &*E*(*2*)], *OF* &*I*, *OF 1*[*THEN* &*E*(*2*)],
OF x-prop[*THEN* &*E*(*2*), *THEN* &*E*(*2*)]]**.**
AOT-thus ‹$z = x$›
by (*metis* =*E*−*simple*:*2* *vdash*−*properties*:*10*)
qed
AOT-thus ‹$\exists!x$ ([λx [$O!$]x & [F]x]x & [R]xy)›
by (*rule uniqueness*:*1*[*THEN* $\equiv_{df}I$])
qed
qed
finally show *?thesis***.**
qed

AOT-theorem *empty*−*approx*:*1*: ‹($\neg\exists u$ [F]u & $\neg\exists v$ [H]v) → $F \approx_E H$›
proof(*rule* →*I*; *frule* &*E*(*1*); *drule* &*E*(*2*))
AOT-assume *0*: ‹$\neg\exists u$ [F]u› **and** *1*: ‹$\neg\exists v$ [H]v›
AOT-have ‹$\forall u$ ([F]u → $\exists!v$ ([H]v & [R]uv))› **for** R
proof(*rule Ordinary.GEN*; *rule* →*I*; *rule raa*−*cor*:*1*)
fix u
AOT-assume ‹[F]u›
AOT-hence ‹$\exists u$ [F]u› **using** *Ordinary*.∃*I* &*I* **by** *fast*
AOT-thus ‹$\exists u$ [F]u & $\neg\exists u$ [F]u› **using** &*I 0* **by** *blast*
qed
moreover AOT-have ‹$\forall v$ ([H]v → $\exists!u$ ([F]u & [R]uv))› **for** R
proof(*rule Ordinary.GEN*; *rule* →*I*; *rule raa*−*cor*:*1*)
fix v
AOT-assume ‹[H]v›
AOT-hence ‹$\exists v$ [H]v› **using** *Ordinary*.∃*I* &*I* **by** *fast*
AOT-thus ‹$\exists v$ [H]v & $\neg\exists v$ [H]v› **using** *1* &*I* **by** *blast*
qed
ultimately AOT-have ‹R |: F $_{1-1}\!\!\longleftrightarrow_E$ H› **for** R
apply (*safe intro*!: *equi*:*2*[*THEN* $\equiv_{df}I$] &*I GEN cqt*:*2*[*const-var*][*axiom-inst*])
using ∀*E* **by** *blast*+
AOT-hence ‹$\exists R$ R |: F $_{1-1}\!\!\longleftrightarrow_E$ H› **by** (*rule* ∃*I*)
AOT-thus ‹$F \approx_E H$›
by (*rule equi*:*3*[*THEN* $\equiv_{df}I$])
qed

AOT-theorem *empty*−*approx*:*2*: ‹($\exists u$ [F]u & $\neg\exists v$ [H]v) → $\neg$($F \approx_E H$)›
proof(*rule* →*I*; *frule* &*E*(*1*); *drule* &*E*(*2*); *rule raa*−*cor*:*2*)
AOT-assume *1*: ‹$\exists u$ [F]u› **and** *2*: ‹$\neg\exists v$ [H]v›
AOT-obtain b **where** *b-prop*: ‹$O!b$ & [F]b›
using *1* ∃*E*[*rotated*] **by** *blast*
AOT-assume ‹$F \approx_E H$›
AOT-hence ‹$\exists R$ R |: F $_{1-1}\!\!\longleftrightarrow_E$ H›
by (*rule equi*:*3*[*THEN* $\equiv_{df}E$])
then AOT-obtain R **where** ‹R |: F $_{1-1}\!\!\longleftrightarrow_E$ H›
using ∃*E*[*rotated*] **by** *blast*
AOT-hence ϑ: ‹$\forall u$ ([F]u → $\exists!v$ ([H]v & [R]uv))›
using *equi*:*2*[*THEN* $\equiv_{df}E$] &*E* **by** *blast*+
AOT-have ‹$\exists!v$ ([H]v & [R]bv)› **for** u
using ϑ[*THEN* ∀*E*(*2*)[**where** β=b], *THEN* →*E*, *THEN* →*E*,
OF b-prop[*THEN* &*E*(*1*)], *OF b-prop*[*THEN* &*E*(*2*)]]**.**
AOT-hence ‹$\exists v$ ([H]v & [R]bv & $\forall u$ ([H]u & [R]bu → $u =_E v$))›
by (*rule equi*:*1*[*THEN* ≡*E*(*1*)])
then AOT-obtain x **where** ‹$O!x$ & ([H]x & [R]bx & $\forall u$ ([H]u & [R]bu → $u =_E x$))›
using ∃*E*[*rotated*] **by** *blast*
AOT-hence ‹$O!x$ & [H]x› **using** &*E* &*I* **by** *blast*
AOT-hence ‹$\exists v$ [H]v› **by** (*rule* ∃*I*)
AOT-thus ‹$\exists v$ [H]v & $\neg\exists v$ [H]v› **using** *2* &*I* **by** *blast*
qed

AOT-define *FminusU* :: ‹Π ⇒ τ ⇒ Π› (‹-⁻⁻›)
  F−u: ‹$[F]^{-x}$ $=_{df}$ [λz [F]z & z $\neq_E$ x]›

Note: not explicitly in PLM.

AOT-theorem *F−u[den]*: ‹$[F]^{-x}$↓›
  by (*rule* $=_{df}$*I(1)[OF F−u,* **where** $\tau_1\tau_n$=(-,-), *simplified]*; *cqt:2[lambda]*)
AOT-theorem *F−u[equiv]*: ‹[$[F]^{-x}$]y ≡ ([F]y & y $\neq_E$ x)›
  by (*auto intro*: *F−u[THEN* $=_{df}$*I(1)*, **where** $\tau_1\tau_n$=(-,-), *simplified]*
      intro!: *cqt:2 beta−C−cor:2[THEN →E, THEN ∀ E(2)]*)

AOT-theorem *eqP′*: ‹F $\approx_E$ G & [F]u & [G]v → $[F]^{-u}$ $\approx_E$ $[G]^{-v}$›
proof (*rule →I*; *frule &E(2)*; *drule &E(1)*; *frule &E(2)*; *drule &E(1)*)
  AOT-assume ‹F $\approx_E$ G›
  AOT-hence ‹∃ R R |: F $_{1-1}$⟷$_E$ G›
    using *equi:3[THEN* $\equiv_{df}$*E]* **by** *blast*
  then AOT-obtain R **where** *R-prop*: ‹R |: F $_{1-1}$⟷$_E$ G›
    using *∃ E[rotated]* **by** *blast*
  AOT-hence *A*: ‹∀ u ([F]u → ∃!v ([G]v & [R]uv))›
      and *B*: ‹∀ v ([G]v → ∃!u ([F]u & [R]uv))›
    using *equi:2[THEN* $\equiv_{df}$*E] &E* **by** *blast+*
  AOT-have ‹R |: F $_{1-1}$⟶$_{onto}$E G›
    using *equi−rem−thm[THEN ≡E(1), OF R-prop]*.
  AOT-hence ‹R |: F $_{1-1}$⟶E G & R |: F ⟶$_{onto}$E G›
    using *equi−rem:4[THEN* $\equiv_{df}$*E]* **by** *blast*
  AOT-hence *C*: ‹∀ t∀ u∀ v (([F]t & [F]u & [G]v) → ([R]tv & [R]uv → t $=_E$ u))›
    using *equi−rem:2[THEN* $\equiv_{df}$*E] &E* **by** *blast*
  AOT-assume *fu*: ‹[F]u›
  AOT-assume *gv*: ‹[G]v›
  AOT-have ‹[λz [Π]z & z $\neq_E$ κ]↓› **for** Π κ
    by *cqt:2[lambda]*
  note Π-*minus-κI* = *rule−id−df:2:b[2][*
      where τ=‹(λ(Π, κ). «$[\Pi]^{-\kappa}$»)›, *simplified, OF F−u, simplified, OF this]*
   and Π-*minus-κE* = *rule−id−df:2:a[2][*
      where τ=‹(λ(Π, κ). «$[\Pi]^{-\kappa}$»)›, *simplified, OF F−u, simplified, OF this]*
  AOT-have Π-*minus-κ-den*: ‹$[\Pi]^{-\kappa}$↓› **for** Π κ
    by (*rule* Π-*minus-κI*) *cqt:2[lambda]+*
  {
    fix R
    AOT-assume *R-prop*: ‹R |: F $_{1-1}$⟷$_E$ G›
    AOT-hence *A*: ‹∀ u ([F]u → ∃!v ([G]v & [R]uv))›
        and *B*: ‹∀ v ([G]v → ∃!u ([F]u & [R]uv))›
      using *equi:2[THEN* $\equiv_{df}$*E] &E* **by** *blast+*
    AOT-have ‹R |: F $_{1-1}$⟶$_{onto}$E G›
      using *equi−rem−thm[THEN ≡E(1), OF R-prop]*.
    AOT-hence ‹R |: F $_{1-1}$⟶E G & R |: F ⟶$_{onto}$E G›
      using *equi−rem:4[THEN* $\equiv_{df}$*E]* **by** *blast*
    AOT-hence *C*: ‹∀ t∀ u∀ v (([F]t & [F]u & [G]v) → ([R]tv & [R]uv → t $=_E$ u))›
      using *equi−rem:2[THEN* $\equiv_{df}$*E] &E* **by** *blast*

    AOT-assume *Ruv*: ‹[R]uv›
    AOT-have ‹R |: $[F]^{-u}$ $_{1-1}$⟷$_E$ $[G]^{-v}$›
    proof(*safe intro*!: *equi:2[THEN* $\equiv_{df}$*I] &I cqt:2[const-var][axiom-inst]*
              Π-*minus-κ-den Ordinary.GEN →I*)
      fix u′
      AOT-assume ‹[$[F]^{-u}$]u′›
      AOT-hence *0*: ‹[λz [F]z & z $\neq_E$ u]u′›
        using Π-*minus-κE* **by** *fast*
      AOT-have *0*: ‹[F]u′ & u′ $\neq_E$ u›
        by (*rule β→C(1)[***where** $\kappa_1\kappa_n$=*AOT-term-of-var* (*Ordinary.Rep u′*)*]*) (*fact 0*)
      AOT-have ‹∃!v ([G]v & [R]u′v)›
        using *A[THEN Ordinary.∀ E[***where** α=u′*], THEN →E, OF 0[THEN &E(1)]]*.
      then AOT-obtain v′ **where**
        v′-prop: ‹[G]v′ & [R]u′v′ & ∀ t ([G]t & [R]u′t → t $=_E$ v′)›

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    using equi:1[THEN ≡E(1)] Ordinary.∃ E[rotated] by fastforce

  AOT-show ‹∃!v' ([[G]^-v]v' & [R]u'v')›
  proof (safe intro!: equi:1[THEN ≡E(2)] Ordinary.∃ I[where β=v']
                      &I Ordinary.GEN →I)
    AOT-show ‹[[G]^-v]v'›
    proof (rule Π-minus-κI;
           safe intro!: β←C(1) cqt:2 &I thm−neg=E[THEN ≡E(2)])
      AOT-show ‹[G]v'› using v'-prop &E by blast
    next
      AOT-show ‹¬v' =_E v›
      proof (rule raa−cor:2)
        AOT-assume ‹v' =_E v›
        AOT-hence ‹v' = v› by (metis =E−simple:2 →E)
        AOT-hence Ruv': ‹[R]uv'› using rule=E Ruv id-sym by fast
        AOT-have ‹u' =_E u›
          by (rule C[THEN Ordinary.∀ E, THEN Ordinary.∀ E,
                   THEN Ordinary.∀ E[where α=v'], THEN →E, THEN →E])
            (safe intro!: &I 0[THEN &E(1)] fu
                          v'-prop[THEN &E(1), THEN &E(1)]
                          Ruv' v'-prop[THEN &E(1), THEN &E(2)])
        moreover AOT-have ‹¬(u' =_E u)›
          using 0 &E(2) ≡E(1) thm−neg=E by blast
        ultimately AOT-show ‹u' =_E u & ¬u' =_E u› using &I by blast
      qed
    qed
  next
    AOT-show ‹[R]u'v'› using v'-prop &E by blast
  next
    fix t
    AOT-assume t-prop: ‹[[G]^-v]t & [R]u't›
    AOT-have gt-t-noteq-v: ‹[G]t & t ≠_E v›
      apply (rule β→C(1)[where κ₁κₙ=AOT-term-of-var (Ordinary.Rep t)])
      apply (rule Π-minus-κE)
      by (fact t-prop[THEN &E(1)])
    AOT-show ‹t =_E v'›
      using v'-prop[THEN &E(2), THEN Ordinary.∀ E, THEN →E,
                  OF &I, OF gt-t-noteq-v[THEN &E(1)],
                  OF t-prop[THEN &E(2)]].
  qed
next
  fix v'
  AOT-assume G-minus-v-v': ‹[[G]^-v]v'›
  AOT-have gt-t-noteq-v: ‹[G]v' & v' ≠_E v›
    apply (rule β→C(1)[where κ₁κₙ=AOT-term-of-var (Ordinary.Rep v')])
    apply (rule Π-minus-κE)
    by (fact G-minus-v-v')
  AOT-have ‹∃!u([F]u & [R]uv')›
    using B[THEN Ordinary.∀ E, THEN →E, OF gt-t-noteq-v[THEN &E(1)]].
  then AOT-obtain u' where
    u'-prop: ‹[F]u' & [R]u'v' & ∀ t ([F]t & [R]tv' → t =_E u')›
    using equi:1[THEN ≡E(1)] Ordinary.∃ E[rotated] by fastforce
  AOT-show ‹∃!u' ([[F]^-u]u' & [R]u'v')›
  proof (safe intro!: equi:1[THEN ≡E(2)] Ordinary.∃ I[where β=u'] &I
                      u'-prop[THEN &E(1), THEN &E(2)] Ordinary.GEN →I)
    AOT-show ‹[[F]^-u]u'›
    proof (rule Π-minus-κI;
           safe intro!: β←C(1) cqt:2 &I thm−neg=E[THEN ≡E(2)]
           u'-prop[THEN &E(1), THEN &E(1)]; rule raa−cor:2)
      AOT-assume u'-eq-u: ‹u' =_E u›
      AOT-hence ‹u' = u›
        using =E−simple:2 vdash−properties:10 by blast
      AOT-hence Ru'v: ‹[R]u'v› using rule=E Ruv id-sym by fast
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        AOT-have ‹v′ ≠E v›
          using &E(2) gt-t-noteq-v by blast
        AOT-hence v′-noteq-v: ‹¬(v′ =E v)› by (metis ≡E(1) thm−neg=E)
        AOT-have ‹∃ u ([G]u & [R]u′u & ∀ v ([G]v & [R]u′v → v =E u))›
          using A[THEN Ordinary.∀ E, THEN →E,
                OF u′-prop[THEN &E(1), THEN &E(1)],
                THEN equi:1[THEN ≡E(1)]].
        then AOT-obtain t where
          t-prop: ‹[G]t & [R]u′t & ∀ v ([G]v & [R]u′v → v =E t)›
          using Ordinary.∃ E[rotated] by meson
        AOT-have ‹v =E t› if ‹[G]v› and ‹[R]u′v› for v
          using t-prop[THEN &E(2), THEN Ordinary.∀ E, THEN →E,
                    OF &I, OF that].
        AOT-hence ‹v′ =E t› and ‹v =E t›
          by (auto simp: gt-t-noteq-v[THEN &E(1)] Ru′v gv
                      u′-prop[THEN &E(1), THEN &E(2)])
        AOT-hence ‹v′ =E v›
          using rule=E =E−simple:2 id-sym →E by fast
        AOT-thus ‹v′ =E v & ¬v′ =E v›
          using v′-noteq-v &I by blast
      qed
    next
      fix t
      AOT-assume 0: ‹[[F]⁻ᵘ]t & [R]tv′›
      moreover AOT-have ‹[F]t & t ≠E u›
        apply (rule β→C(1)[where κ1κn=AOT-term-of-var (Ordinary.Rep t)])
        apply (rule Π-minus-κE)
        by (fact 0[THEN &E(1)])
      ultimately AOT-show ‹t =E u′›
        using u′-prop[THEN &E(2), THEN Ordinary.∀ E, THEN →E, OF &I]
            &E by blast
    qed
  qed
  AOT-hence ‹∃ R R |: [F]⁻ᵘ 1−1⟷E [G]⁻ᵛ›
    by (rule ∃ I)
} note 1 = this
moreover {
  AOT-assume not-Ruv: ‹¬[R]uv›
  AOT-have ‹∃!v ([G]v & [R]uv)›
    using A[THEN Ordinary.∀ E, THEN →E, OF fu].
  then AOT-obtain b where
    b-prop: ‹O!b & ([G]b & [R]ub & ∀ t([G]t & [R]ut → t =E b))›
    using equi:1[THEN ≡E(1)] ∃ E[rotated] by fastforce
  AOT-hence ob: ‹O!b› and gb: ‹[G]b› and Rub: ‹[R]ub›
    using &E by blast+
  AOT-have ‹O!t → ([G]t & [R]ut → t =E b)› for t
    using b-prop &E(2) ∀ E(2) by blast
  AOT-hence b-unique: ‹t =E b› if ‹O!t› and ‹[G]t› and ‹[R]ut› for t
    by (metis Adjunction modus−tollens:1 reductio−aa:1 that)
  AOT-have not-v-eq-b: ‹¬(v =E b)›
  proof(rule raa−cor:2)
    AOT-assume ‹v =E b›
    AOT-hence 0: ‹v = b›
      by (metis =E−simple:2 →E)
    AOT-have ‹[R]uv›
      using b-prop[THEN &E(2), THEN &E(1), THEN &E(2)]
            rule=E[rotated, OF 0[symmetric]] by fast
    AOT-thus ‹[R]uv & ¬[R]uv›
      using not-Ruv &I by blast
  qed
  AOT-have not-b-eq-v: ‹¬(b =E v)›
    using modus−tollens:1 not-v-eq-b ord=Eequiv:2 by blast
  AOT-have ‹∃!u ([F]u & [R]uv)›
```

```
  using B[THEN Ordinary.∀E, THEN →E, OF gv].
then AOT-obtain a where
  a-prop: ‹O!a & ([F]a & [R]av & ∀t([F]t & [R]tv → t =E a))›
  using equi:1[THEN ≡E(1)] ∃E[rotated] by fastforce
AOT-hence Oa: ‹O!a› and fa: ‹[F]a› and Rav: ‹[R]av›
  using &E by blast+
AOT-have ‹O!t → ([F]t & [R]tv → t =E a)› for t
  using a-prop &E ∀E(2) by blast
AOT-hence a-unique: ‹t =E a› if ‹O!t› and ‹[F]t› and ‹[R]tv› for t
  by (metis Adjunction modus−tollens:1 reductio−aa:1 that)
AOT-have not-u-eq-a: ‹¬(u =E a)›
proof(rule raa−cor:2)
  AOT-assume ‹u =E a›
  AOT-hence 0: ‹u = a›
    by (metis =E−simple:2 →E)
  AOT-have ‹[R]uv›
    using a-prop[THEN &E(2), THEN &E(1), THEN &E(2)]
          rule=E[rotated, OF 0[symmetric]] by fast
  AOT-thus ‹[R]uv & ¬[R]uv›
    using not-Ruv &I by blast
qed
AOT-have not-a-eq-u: ‹¬(a =E u)›
  using modus−tollens:1 not-u-eq-a ord=Eequiv:2 by blast
let ?R = ‹«[λu′v′ (u′ ≠E u & v′ ≠E v & [R]u′v′) ∨
                 (u′ =E a & v′ =E b) ∨
                 (u′ =E u & v′ =E v)]»›
AOT-have ‹[«?R»]↓› by cqt:2[lambda]
AOT-hence ‹∃ β β = [«?R»]›
  using free−thms:1 ≡E(1) by fast
then AOT-obtain R1 where R1-def: ‹R1 = [«?R»]›
  using ∃E[rotated] by blast
AOT-have Rxy1: ‹[R]xy› if ‹[R1]xy› and ‹x ≠E u› and ‹x ≠E a› for x y
proof −
  AOT-have 0: ‹[«?R»]xy›
    by (rule rule=E[rotated, OF R1-def]) (fact that(1))
  AOT-have ‹(x ≠E u & y ≠E v & [R]xy) ∨ (x =E a & y =E b) ∨ (x =E u & y =E v)›
    using β→C(1)[OF 0] by simp
  AOT-hence ‹x ≠E u & y ≠E v & [R]xy› using that(2,3)
    by (metis ∨E(3) Conjunction Simplification(1) ≡E(1)
              modus−tollens:1 thm−neg=E)
  AOT-thus ‹[R]xy› using &E by blast+
qed
AOT-have Rxy2: ‹[R]xy›  if ‹[R1]xy› and ‹y ≠E v› and ‹y ≠E b› for x y
proof −
  AOT-have 0: ‹[«?R»]xy›
    by (rule rule=E[rotated, OF R1-def]) (fact that(1))
  AOT-have ‹(x ≠E u & y ≠E v & [R]xy) ∨ (x =E a & y =E b) ∨ (x =E u & y =E v)›
    using β→C(1)[OF 0] by simp
  AOT-hence ‹x ≠E u & y ≠E v & [R]xy›
    using that(2,3)
    by (metis ∨E(3) Conjunction Simplification(2) ≡E(1)
              modus−tollens:1 thm−neg=E)
  AOT-thus ‹[R]xy› using &E by blast+
qed
AOT-have R1xy: ‹[R1]xy› if ‹[R]xy› and ‹x ≠E u› and ‹y ≠E v› for x y
  by (rule rule=E[rotated, OF R1-def[symmetric]])
     (auto intro!: β←C(1) cqt:2
             simp: &I ex:1:a prod-denotesI rule−ui:3 that ∨I(1))
AOT-have R1ab: ‹[R1]ab›
  apply (rule rule=E[rotated, OF R1-def[symmetric]])
  apply (safe intro!: β←C(1) cqt:2 prod-denotesI &I)
  by (meson a-prop b-prop &I &E(1) ∨I(1) ∨I(2) ord=Eequiv:1 →E)
AOT-have R1uv: ‹[R1]uv›
```

```
    apply (rule rule=E[rotated, OF R₁-def[symmetric]])
    apply (safe intro!: β←C(1) cqt:2 prod-denotesI &I)
    by (meson &I ∨I(2) ord=Eequiv:1 Ordinary.ψ →E)
  moreover AOT-have ‹R₁ |: F 1−1⟷E G›
  proof (safe intro!: equi:2[THEN ≡dfI] &I cqt:2 Ordinary.GEN →I)
    fix u'
    AOT-assume fu': ‹[F]u'›
    {
      AOT-assume not-u'-eq-u: ‹¬(u' =E u)› and not-u'-eq-a: ‹¬(u' =E a)›
      AOT-hence u'-noteq-u: ‹u' ≠E u› and u'-noteq-a: ‹u' ≠E a›
        by (metis ≡E(2) thm−neg=E)+
      AOT-have ‹∃!v ([G]v & [R]u'v)›
        using A[THEN Ordinary.∀E, THEN →E, OF fu'].
      AOT-hence ‹∃v ([G]v & [R]u'v & ∀t ([G]t & [R]u't → t =E v))›
        using equi:1[THEN ≡E(1)] by simp
      then AOT-obtain v' where
        v'-prop: ‹[G]v' & [R]u'v' & ∀t ([G]t & [R]u't → t =E v')›
        using Ordinary.∃E[rotated] by meson
      AOT-hence gv': ‹[G]v'› and Ru'v': ‹[R]u'v'›
        using &E by blast+
      AOT-have not-v'-eq-v: ‹¬v' =E v›
      proof (rule raa−cor:2)
        AOT-assume ‹v' =E v›
        AOT-hence ‹v' = v›
          by (metis =E−simple:2 →E)
        AOT-hence Ru'v: ‹[R]u'v›
          using rule=E Ru'v' by fast
        AOT-have ‹u' =E a›
          using a-unique[OF Ordinary.ψ, OF fu', OF Ru'v].
        AOT-thus ‹u' =E a & ¬u' =E a›
          using not-u'-eq-a &I by blast
      qed
      AOT-hence v'-noteq-v: ‹v' ≠E v›
        using ≡E(2) thm−neg=E by blast
      AOT-have ‹∀t ([G]t & [R]u't → t =E v')›
        using v'-prop &E by blast
      AOT-hence ‹[G]t & [R]u't → t =E v'› for t
        using Ordinary.∀E by meson
      AOT-hence v'-unique: ‹t =E v'› if ‹[G]t› and ‹[R]u't› for t
        by (metis &I that →E)

      AOT-have ‹[G]v' & [R₁]u'v' & ∀t ([G]t & [R₁]u't → t =E v')›
      proof (safe intro!: &I gv' R₁xy Ru'v' u'-noteq-u u'-noteq-a →I
                          Ordinary.GEN thm−neg=E[THEN ≡E(2)] not-v'-eq-v)
        fix t
        AOT-assume 1: ‹[G]t & [R₁]u't›
        AOT-have ‹[R]u't›
          using Rxy1[OF 1[THEN &E(2)], OF u'-noteq-u, OF u'-noteq-a].
        AOT-thus ‹t =E v'›
          using v'-unique 1[THEN &E(1)] by blast
      qed
      AOT-hence ‹∃v ([G]v & [R₁]u'v & ∀t ([G]t & [R₁]u't → t =E v))›
        by (rule Ordinary.∃I)
      AOT-hence ‹∃!v ([G]v & [R₁]u'v)›
        by (rule equi:1[THEN ≡E(2)])
    }
    moreover {
      AOT-assume 0: ‹u' =E u›
      AOT-hence u'-eq-u: ‹u' = u›
        using =E−simple:2 →E by blast
      AOT-have ‹∃!v ([G]v & [R₁]u'v)›
      proof (safe intro!: equi:1[THEN ≡E(2)] Ordinary.∃I[where β=v]
                         &I Ordinary.GEN →I gv)
```

```
    AOT-show ‹[R₁]u'v›
      apply (rule rule=E[rotated, OF R₁-def[symmetric]])
      apply (safe intro!: β←C(1) cqt:2 &I prod-denotesI)
      by (safe intro!: ∨I(2) &I 0 ord=Eequiv:1[THEN →E, OF Ordinary.ψ])
  next
    fix v'
    AOT-assume ‹[G]v' & [R₁]u'v'›
    AOT-hence 0: ‹[R₁]uv'›
      using rule=E[rotated, OF u'-eq-u] &E(2) by fast
    AOT-have 1: ‹[«?R»]uv'›
      by (rule rule=E[rotated, OF R₁-def]) (fact 0)
    AOT-have 2: ‹(u ≠E u & v' ≠E v & [R]uv') ∨
                 (u =E a & v' =E b) ∨
                 (u =E u & v' =E v)›
      using β→C(1)[OF 1] by simp
    AOT-have ‹¬u ≠E u›
      using ≡E(4) modus−tollens:1 ord=Eequiv:1 Ordinary.ψ
            reductio−aa:2 thm−neg=E by blast
    AOT-hence ‹¬((u ≠E u & v' ≠E v & [R]uv') ∨ (u =E a & v' =E b))›
      using not-u-eq-a
      by (metis ∨E(2) Conjunction Simplification(1)
               modus−tollens:1 reductio−aa:1)
    AOT-hence ‹(u =E u & v' =E v)›
      using 2 by (metis ∨E(2))
    AOT-thus ‹v' =E v›
      using &E by blast
  qed
}
moreover {
  AOT-assume 0: ‹u' =E a›
  AOT-hence u'-eq-a: ‹u' = a›
    using =E−simple:2 →E by blast
  AOT-have ‹∃!v ([G]v & [R₁]u'v)›
  proof (safe intro!: equi:1[THEN ≡E(2)] ∃I(2)[where β=b] &I
                      Ordinary.GEN →I b-prop[THEN &E(1)]
                      b-prop[THEN &E(2), THEN &E(1), THEN &E(1)])
    AOT-show ‹[R₁]u'b›
      apply (rule rule=E[rotated, OF R₁-def[symmetric]])
      apply (safe intro!: β←C(1) cqt:2 &I prod-denotesI)
      apply (rule ∨I(1); rule ∨I(2); rule &I)
       apply (fact 0)
      using b-prop &E(1) ord=Eequiv:1 →E by blast
  next
    fix v'
    AOT-assume gv'-R1u'v': ‹[G]v' & [R₁]u'v'›
    AOT-hence 0: ‹[R₁]av'›
      using u'-eq-a by (meson rule=E &E(2))
    AOT-have 1: ‹[«?R»]av'›
      by (rule rule=E[rotated, OF R₁-def]) (fact 0)
    AOT-have ‹(a ≠E u & v' ≠E v & [R]av') ∨
              (a =E a & v' =E b) ∨
              (a =E u & v' =E v)›
      using β→C(1)[OF 1] by simp
    moreover {
      AOT-assume 0: ‹a ≠E u & v' ≠E v & [R]av'›
      AOT-have ‹∃!v ([G]v & [R]u'v)›
        using A[THEN Ordinary.∀E, THEN →E, OF fu'].
      AOT-hence ‹∃!v ([G]v & [R]av)›
        using u'-eq-a rule=E by fast
      AOT-hence ‹∃v ([G]v & [R]av & ∀t ([G]t & [R]at → t =E v))›
        using equi:1[THEN ≡E(1)] by fast
      then AOT-obtain s where
        s-prop: ‹[G]s & [R]as & ∀t ([G]t & [R]at → t =E s)›
```

```
          using Ordinary.∃E[rotated] by meson
        AOT-have ‹v' =E s›
          using s-prop[THEN &E(2), THEN Ordinary.∀E]
                gv'-R1u'v'[THEN &E(1)] 0[THEN &E(2)]
          by (metis &I vdash-properties:10)
        moreover AOT-have ‹v =E s›
          using s-prop[THEN &E(2), THEN Ordinary.∀E] gv Rav
          by (metis &I →E)
        ultimately AOT-have ‹v' =E v›
          by (metis &I ord=Eequiv:2 ord=Eequiv:3 →E)
        moreover AOT-have ‹¬(v' =E v)›
          using 0[THEN &E(1), THEN &E(2)]
          by (metis ≡E(1) thm-neg=E)
        ultimately AOT-have ‹v' =E b›
          by (metis raa-cor:3)
      }
      moreover {
        AOT-assume ‹a =E u & v' =E v›
        AOT-hence ‹v' =E b›
          by (metis &E(1) not-a-eq-u reductio-aa:1)
      }
      ultimately AOT-show ‹v' =E b›
        by (metis &E(2) ∨E(3) reductio-aa:1)
    qed
  }
  ultimately AOT-show ‹∃!v ([G]v & [R1]u'v)›
    by (metis raa-cor:1)
next
  fix v'
  AOT-assume gv': ‹[G]v'›
  {
    AOT-assume not-v'-eq-v: ‹¬(v' =E v)›
           and not-v'-eq-b: ‹¬(v' =E b)›
    AOT-hence v'-noteq-v: ‹v' ≠E v›
          and v'-noteq-b: ‹v' ≠E b›
      by (metis ≡E(2) thm-neg=E)+
    AOT-have ‹∃!u ([F]u & [R]uv')›
      using B[THEN Ordinary.∀E, THEN →E, OF gv'].
    AOT-hence ‹∃u ([F]u & [R]uv' & ∀t ([F]t & [R]tv' → t =E u))›
      using equi:1[THEN ≡E(1)] by simp
    then AOT-obtain u' where
      u'-prop: ‹[F]u' & [R]u'v' & ∀t ([F]t & [R]tv' → t =E u')›
      using Ordinary.∃E[rotated] by meson
    AOT-hence fu': ‹[F]u'› and Ru'v': ‹[R]u'v'›
      using &E by blast+
    AOT-have not-u'-eq-u: ‹¬u' =E u›
    proof (rule raa-cor:2)
      AOT-assume ‹u' =E u›
      AOT-hence ‹u' = u›
        by (metis =E-simple:2 →E)
      AOT-hence Ruv': ‹[R]uv'›
        using rule=E Ru'v' by fast
      AOT-have ‹v' =E b›
        using b-unique[OF Ordinary.ψ, OF gv', OF Ruv'].
      AOT-thus ‹v' =E b & ¬v' =E b›
        using not-v'-eq-b &I by blast
    qed
    AOT-hence u'-noteq-u: ‹u' ≠E u›
      using ≡E(2) thm-neg=E by blast
    AOT-have ‹∀t ([F]t & [R]tv' → t =E u')›
      using u'-prop &E by blast
    AOT-hence ‹[F]t & [R]tv' → t =E u'› for t
      using Ordinary.∀E by meson
```

```
    AOT-hence u'-unique: ‹t =E u'› if ‹[F]t› and ‹[R]tv'› for t
      by (metis &I that →E)

    AOT-have ‹[F]u' & [R1]u'v' & ∀ t ([F]t & [R1]tv' → t =E u')›
    proof (safe intro!: &I gv' R1xy Ru'v' u'-noteq-u Ordinary.GEN →I
                        thm−neg=E[THEN ≡E(2)] not-v'-eq-v fu')
      fix t
      AOT-assume 1: ‹[F]t & [R1]tv'›
      AOT-have ‹[R]tv'›
        using Rxy2[OF 1[THEN &E(2)], OF v'-noteq-v, OF v'-noteq-b].
      AOT-thus ‹t =E u'›
        using u'-unique 1[THEN &E(1)] by blast
    qed
    AOT-hence ‹∃ u ([F]u & [R1]uv' & ∀ t ([F]t & [R1]tv' → t =E u))›
      by (rule Ordinary.∃ I)
    AOT-hence ‹∃!u ([F]u & [R1]uv')›
      by (rule equi:1[THEN ≡E(2)])
  }
  moreover {
    AOT-assume 0: ‹v' =E v›
    AOT-hence u'-eq-u: ‹v' = v›
      using =E−simple:2 →E by blast
    AOT-have ‹∃!u ([F]u & [R1]uv')›
    proof (safe intro!: equi:1[THEN ≡E(2)] Ordinary.∃ I[where β=u]
                        &I Ordinary.GEN →I fu)
      AOT-show ‹[R1]uv'›
        by (rule rule=E[rotated, OF R1-def[symmetric]])
           (safe intro!: β←C(1) cqt:2 &I prod-denotesI Ordinary.ψ
                         ∨I(2) 0 ord=Eequiv:1[THEN →E])
    next
      fix u'
      AOT-assume ‹[F]u' & [R1]u'v'›
      AOT-hence 0: ‹[R1]u'v›
        using rule=E[rotated, OF u'-eq-u] &E(2) by fast
      AOT-have 1: ‹[«?R»]u'v›
        by (rule rule=E[rotated, OF R1-def]) (fact 0)
      AOT-have 2: ‹(u' ≠E u & v ≠E v & [R]u'v) ∨
                   (u' =E a & v =E b) ∨
                   (u' =E u & v =E v)›
        using β→C(1)[OF 1, simplified] by simp
      AOT-have ‹¬v ≠E v›
        using ≡E(4) modus−tollens:1 ord=Eequiv:1 Ordinary.ψ
              reductio−aa:2 thm−neg=E by blast
      AOT-hence ‹¬((u' ≠E u & v ≠E v & [R]u'v) ∨ (u' =E a & v =E b))›
        by (metis &E(1) &E(2) ∨E(3) not-v-eq-b raa−cor:3)
      AOT-hence ‹(u' =E u & v =E v)›
        using 2 by (metis ∨E(2))
      AOT-thus ‹u' =E u›
        using &E by blast
    qed
  }
  moreover {
    AOT-assume 0: ‹v' =E b›
    AOT-hence v'-eq-b: ‹v' = b›
      using =E−simple:2 →E by blast
    AOT-have ‹∃!u ([F]u & [R1]uv')›
    proof (safe intro!: equi:1[THEN ≡E(2)] ∃ I(2)[where β=a] &I
                        Ordinary.GEN →I b-prop[THEN &E(1)] Oa fa
                        b-prop[THEN &E(2), THEN &E(1), THEN &E(1)])
      AOT-show ‹[R1]av'›
        apply (rule rule=E[rotated, OF R1-def[symmetric]])
        apply (safe intro!: β←C(1) cqt:2 &I prod-denotesI)
        apply (rule ∨I(1); rule ∨I(2); rule &I)
```

using *Oa ord=Eequiv:1* $\rightarrow$*E* **apply** *blast*
using *0* **by** *blast*
next
fix u'
AOT-assume *fu′-R1u′v′*: ‹$[F]u'$ & $[R_1]u'v'$›
AOT-hence *0*: ‹$[R_1]u'b$›
using *v′-eq-b* **by** (*meson rule=E* &*E*(*2*))
AOT-have *1*: ‹[«*?R*»]$u'b$›
by (*rule rule=E*[*rotated*, *OF* R_1*-def*]) (*fact 0*)
AOT-have ‹$(u' \neq_E u$ & $b \neq_E v$ & $[R]u'b) \vee$
$(u' =_E a$ & $b =_E b) \vee$
$(u' =_E u$ & $b =_E v)$›
using $\beta{\rightarrow}C(1)$[*OF 1*, *simplified*] **by** *simp*
moreover {
AOT-assume *0*: ‹$u' \neq_E u$ & $b \neq_E v$ & $[R]u'b$›
AOT-have ‹$\exists!u$ ($[F]u$ & $[R]uv'$)›
using *B*[*THEN Ordinary.*∀*E*, *THEN* →*E*, *OF gv′*].
AOT-hence ‹$\exists!u$ ($[F]u$ & $[R]ub$)›
using *v′-eq-b rule=E* **by** *fast*
AOT-hence ‹$\exists u$ ($[F]u$ & $[R]ub$ & $\forall t$ ($[F]t$ & $[R]tb \rightarrow t =_E u$))›
using *equi:1*[*THEN* ≡*E*(*1*)] **by** *fast*
then AOT-obtain *s* **where**
s-prop: ‹$[F]s$ & $[R]sb$ & $\forall t$ ($[F]t$ & $[R]tb \rightarrow t =_E s$)›
using *Ordinary.*∃*E*[*rotated*] **by** *meson*
AOT-have ‹$u' =_E s$›
using *s-prop*[*THEN* &*E*(*2*), *THEN Ordinary.*∀*E*]
fu′-R1u′v′[*THEN* &*E*(*1*)] *0*[*THEN* &*E*(*2*)]
by (*metis* &*I* →*E*)
moreover AOT-have ‹$u =_E s$›
using *s-prop*[*THEN* &*E*(*2*), *THEN Ordinary.*∀*E*] *fu Rub*
by (*metis* &*I* →*E*)
ultimately AOT-have ‹$u' =_E u$›
by (*metis* &*I ord=Eequiv:2 ord=Eequiv:3* →*E*)
moreover AOT-have ‹$\neg(u' =_E u)$›
using *0*[*THEN* &*E*(*1*), *THEN* &*E*(*1*)] **by** (*metis* ≡*E*(*1*) *thm−neg=E*)
ultimately AOT-have ‹$u' =_E a$›
by (*metis raa−cor:3*)
}
moreover {
AOT-assume ‹$u' =_E u$ & $b =_E v$›
AOT-hence ‹$u' =_E a$›
by (*metis* &*E*(*2*) *not-b-eq-v reductio−aa:1*)
}
ultimately AOT-show ‹$u' =_E a$›
by (*metis* &*E*(*1*) ∨*E*(*3*) *reductio−aa:1*)
qed
}
ultimately AOT-show ‹$\exists!u$ ($[F]u$ & $[R_1]uv'$)›
by (*metis raa−cor:1*)
qed
ultimately AOT-have ‹$\exists R\ R$ |: $[F]^{-u}\ {}_{1-1}{\longleftrightarrow}_E\ [G]^{-v}$›
using *1* **by** *blast*
}
ultimately AOT-have ‹$\exists R\ R$ |: $[F]^{-u}\ {}_{1-1}{\longleftrightarrow}_E\ [G]^{-v}$›
using *R-prop* **by** (*metis reductio−aa:2*)
AOT-thus ‹$[F]^{-u} \approx_E [G]^{-v}$›
by (*rule equi:3*[*THEN* $\equiv_{df}I$])
qed

AOT-theorem *P′−eq*: ‹$[F]^{-u} \approx_E [G]^{-v}$ & $[F]u$ & $[G]v \rightarrow F \approx_E G$›
proof(*safe intro*!: →*I*; *frule* &*E*(*1*); *drule* &*E*(*2*);
frule &*E*(*1*); *drule* &*E*(*2*))

```
AOT-have ‹[λz [Π]z & z ≠E κ]↓› for Π κ by cqt:2[lambda]
note Π-minus-κI = rule−id−df:2:b[2][
    where τ=‹(λ(Π, κ). «[Π]−κ»)›, simplified, OF F−u, simplified, OF this]
 and Π-minus-κE = rule−id−df:2:a[2][
 where τ=‹(λ(Π, κ). «[Π]−κ»)›, simplified, OF F−u, simplified, OF this]
AOT-have Π-minus-κ-den: ‹[Π]−κ↓› for Π κ
  by (rule Π-minus-κI) cqt:2[lambda]+

AOT-have Π-minus-κE1: ‹[Π]κ'›
     and Π-minus-κE2: ‹κ' ≠E κ› if ‹[[Π]−κ]κ'› for Π κ κ'
proof −
  AOT-have ‹[λz [Π]z & z ≠E κ]κ'›
    using Π-minus-κE that by fast
  AOT-hence ‹[Π]κ' & κ' ≠E κ›
    by (rule β→C(1))
  AOT-thus ‹[Π]κ'› and ‹κ' ≠E κ›
    using &E by blast+
qed
AOT-have Π-minus-κI': ‹[[Π]−κ]κ'› if ‹[Π]κ'› and ‹κ' ≠E κ› for Π κ κ'
proof −
  AOT-have κ'-den: ‹κ'↓›
    by (metis russell−axiom[exe,1].ψ-denotes-asm that(1))
  AOT-have ‹[λz [Π]z & z ≠E κ]κ'›
    by (safe intro!: β←C(1) cqt:2 κ'-den &I that)
  AOT-thus ‹[[Π]−κ]κ'›
    using Π-minus-κI by fast
qed

AOT-assume Gv: ‹[G]v›
AOT-assume Fu: ‹[F]u›
AOT-assume ‹[F]−u ≈E [G]−v›
AOT-hence ‹∃R R |: [F]−u 1−1⟷E [G]−v›
  using equi:3[THEN ≡dfE] by blast
then AOT-obtain R where R-prop: ‹R |: [F]−u 1−1⟷E [G]−v›
  using ∃E[rotated] by blast
AOT-hence Fact1: ‹∀r([[F]−u]r → ∃!s ([[G]−v]s & [R]rs))›
      and Fact1': ‹∀s([[G]−v]s → ∃!r ([[F]−u]r & [R]rs))›
  using equi:2[THEN ≡dfE] &E by blast+
AOT-have ‹R |: [F]−u 1−1⟶ontoE [G]−v›
  using equi−rem−thm[unvarify F G, OF Π-minus-κ-den, OF Π-minus-κ-den,
                    THEN ≡E(1), OF R-prop].
AOT-hence ‹R |: [F]−u 1−1⟶E [G]−v & R |: [F]−u ⟶ontoE [G]−v›
  using equi−rem:4[THEN ≡dfE] by blast
AOT-hence Fact2:
  ‹∀r∀s∀t(([[F]−u]r & [[F]−u]s & [[G]−v]t) → ([R]rt & [R]st → r =E s))›
  using equi−rem:2[THEN ≡dfE] &E by blast

let ?R = ‹«[λxy ([[F]−u]x & [[G]−v]y & [R]xy) ∨ (x =E u & y =E v)]»›
AOT-have R-den: ‹«?R»↓› by cqt:2[lambda]

AOT-show ‹F ≈E G›
proof(safe intro!: equi:3[THEN ≡dfI] ∃I(1)[where τ=?R] R-den
                   equi:2[THEN ≡dfI] &I cqt:2 Ordinary.GEN →I)
  fix r
  AOT-assume Fr: ‹[F]r›
  {
    AOT-assume not-r-eq-u: ‹¬(r =E u)›
    AOT-hence r-noteq-u: ‹r ≠E u›
      using ≡E(2) thm−neg=E by blast
    AOT-have ‹[[F]−u]r›
      by(rule Π-minus-κI; safe intro!: β←C(1) cqt:2 &I Fr r-noteq-u)
    AOT-hence ‹∃!s ([[G]−v]s & [R]rs)›
      using Fact1[THEN ∀E(2)] →E Ordinary.ψ by blast
```

```
    AOT-hence ‹∃ s ([[G]⁻ᵛ]s & [R]rs & ∀ t ([[G]⁻ᵛ]t & [R]rt → t =E s))›
      using equi:1[THEN ≡E(1)] by simp
    then AOT-obtain s where s-prop: ‹[[G]⁻ᵛ]s & [R]rs & ∀ t ([[G]⁻ᵛ]t & [R]rt → t =E s)›
      using Ordinary.∃ E[rotated] by meson
    AOT-hence G-minus-v-s: ‹[[G]⁻ᵛ]s› and Rrs: ‹[R]rs›
      using &E by blast+
    AOT-have s-unique: ‹t =E s› if ‹[[G]⁻ᵛ]t› and ‹[R]rt› for t
      using s-prop[THEN &E(2), THEN Ordinary.∀ E, THEN →E, OF &I, OF that].
    AOT-have Gs: ‹[G]s›
      using Π-minus-κE1[OF G-minus-v-s].
    AOT-have s-noteq-v: ‹s ≠E v›
      using Π-minus-κE2[OF G-minus-v-s].
    AOT-have ‹∃ s ([G]s & [«?R»]rs & (∀ t ([G]t & [«?R»]rt → t =E s)))›
    proof(safe intro!: Ordinary.∃ I[where β=s] &I Gs Ordinary.GEN →I)
      AOT-show ‹[«?R»]rs›
        by (auto intro!: β←C(1) cqt:2 &I ∨I(1) Π-minus-κI′ Fr Gs
                         s-noteq-v Rrs r-noteq-u
                 simp: &I ex:1:a prod-denotesI rule−ui:3)
    next
      fix t
      AOT-assume 0: ‹[G]t & [«?R»]rt›
      AOT-hence ‹([[F]⁻ᵘ]r & [[G]⁻ᵛ]t & [R]rt) ∨ (r =E u & t =E v)›
        using β→C(1)[OF 0[THEN &E(2)], simplified] by blast
      AOT-hence 1: ‹[[F]⁻ᵘ]r & [[G]⁻ᵛ]t & [R]rt›
        using not-r-eq-u by (metis &E(1) ∨E(3) reductio−aa:1)
      AOT-show ‹t =E s› using s-unique 1 &E by blast
    qed
  }
  moreover {
    AOT-assume r-eq-u: ‹r =E u›
    AOT-have ‹∃ s ([G]s & [«?R»]rs & (∀ t ([G]t & [«?R»]rt → t =E s)))›
    proof(safe intro!: Ordinary.∃ I[where β=v] &I Gv Ordinary.GEN →I)
      AOT-show ‹[«?R»]rv›
        by (auto intro!: β←C(1) cqt:2 &I ∨I(2) Π-minus-κI′ Fr r-eq-u
                         ord=Eequiv:1[THEN →E] Ordinary.ψ
                 simp: &I ex:1:a prod-denotesI rule−ui:3)
    next
      fix t
      AOT-assume 0: ‹[G]t & [«?R»]rt›
      AOT-hence ‹([[F]⁻ᵘ]r & [[G]⁻ᵛ]t & [R]rt) ∨ (r =E u & t =E v)›
        using β→C(1)[OF 0[THEN &E(2)], simplified] by blast
      AOT-hence ‹r =E u & t =E v›
        using r-eq-u Π-minus-κE2
        by (metis &E(1) ∨E(2) ≡E(1) reductio−aa:1 thm−neg=E)
      AOT-thus ‹t =E v› using &E by blast
    qed
  }
  ultimately AOT-show ‹∃!s ([G]s & [«?R»]rs)›
    using reductio−aa:2 equi:1[THEN ≡E(2)] by fast
next
  fix s
  AOT-assume Gs: ‹[G]s›

  {
    AOT-assume not-s-eq-v: ‹¬(s =E v)›
    AOT-hence s-noteq-v: ‹s ≠E v›
      using ≡E(2) thm−neg=E by blast
    AOT-have ‹[[G]⁻ᵛ]s›
      by (rule Π-minus-κI; auto intro!: β←C(1) cqt:2 &I Gs s-noteq-v)
    AOT-hence ‹∃!r ([[F]⁻ᵘ]r & [R]rs)›
      using Fact1′[THEN Ordinary.∀ E] →E by blast
    AOT-hence ‹∃ r ([[F]⁻ᵘ]r & [R]rs & ∀ t ([[F]⁻ᵘ]t & [R]ts → t =E r))›
      using equi:1[THEN ≡E(1)] by simp
```

```
      then AOT-obtain r where
        r-prop: ‹[[F]⁻ᵘ]r & [R]rs & ∀ t ([[F]⁻ᵘ]t & [R]ts → t =E r)›
        using Ordinary.∃ E[rotated] by meson
      AOT-hence F-minus-u-r: ‹[[F]⁻ᵘ]r› and Rrs: ‹[R]rs›
        using &E by blast+
      AOT-have r-unique: ‹t =E r› if ‹[[F]⁻ᵘ]t› and ‹[R]ts› for t
        using r-prop[THEN &E(2), THEN Ordinary.∀ E,
                THEN →E, OF &I, OF that].
      AOT-have Fr: ‹[F]r›
        using Π-minus-κE1[OF F-minus-u-r].
      AOT-have r-noteq-u: ‹r ≠E u›
        using Π-minus-κE2[OF F-minus-u-r].
      AOT-have ‹∃ r ([F]r & [«?R»]rs & (∀ t ([F]t & [«?R»]ts → t =E r)))›
      proof(safe intro!: Ordinary.∃ I[where β=r] &I Fr Ordinary.GEN →I)
        AOT-show ‹[«?R»]rs›
          by (auto intro!: β←C(1) cqt:2 &I ∨I(1) Π-minus-κI′ Fr
                      Gs s-noteq-v Rrs r-noteq-u
                simp: &I ex:1:a prod-denotesI rule−ui:3)
      next
        fix t
        AOT-assume 0: ‹[F]t & [«?R»]ts›
        AOT-hence ‹([[F]⁻ᵘ]t & [[G]⁻ᵛ]s & [R]ts) ∨ (t =E u & s =E v)›
          using β→C(1)[OF 0[THEN &E(2)], simplified] by blast
        AOT-hence 1: ‹[[F]⁻ᵘ]t & [[G]⁻ᵛ]s & [R]ts›
          using not-s-eq-v by (metis &E(2) ∨E(3) reductio−aa:1)
        AOT-show ‹t =E r› using r-unique 1 &E by blast
      qed
    }
    moreover {
      AOT-assume s-eq-v: ‹s =E v›
      AOT-have ‹∃ r ([F]r & [«?R»]rs & (∀ t ([F]t & [«?R»]ts → t =E r)))›
      proof(safe intro!: Ordinary.∃ I[where β=u] &I Fu Ordinary.GEN →I)
        AOT-show ‹[«?R»]us›
          by (auto intro!: β←C(1) cqt:2 &I prod-denotesI ∨I(2)
                      Π-minus-κI′ Gs s-eq-v Ordinary.ψ
                      ord=Eequiv:1[THEN →E])
      next
        fix t
        AOT-assume 0: ‹[F]t & [«?R»]ts›
        AOT-hence 1: ‹([[F]⁻ᵘ]t & [[G]⁻ᵛ]s & [R]ts) ∨ (t =E u & s =E v)›
          using β→C(1)[OF 0[THEN &E(2)], simplified] by blast
        moreover AOT-have ‹¬([[F]⁻ᵘ]t & [[G]⁻ᵛ]s & [R]ts)›
        proof (rule raa−cor:2)
          AOT-assume ‹([[F]⁻ᵘ]t & [[G]⁻ᵛ]s & [R]ts)›
          AOT-hence ‹[[G]⁻ᵛ]s› using &E by blast
          AOT-thus ‹s =E v & ¬(s =E v)›
            by (metis Π-minus-κE2 ≡E(4) reductio−aa:1 s-eq-v thm−neg=E)
        qed
        ultimately AOT-have ‹t =E u & s =E v›
          by (metis ∨E(2))
        AOT-thus ‹t =E u› using &E by blast
      qed
    }
    ultimately AOT-show ‹∃!r ([F]r & [«?R»]rs)›
      using ≡E(2) equi:1 reductio−aa:2 by fast
  qed
qed


AOT-theorem approx−cont:1: ‹∃ F∃ G ◊(F ≈E G & ◊¬F ≈E G)›
proof −
  let ?P = ‹«[λx E!x & ¬𝓐E!x]»›
  AOT-have ‹◊q₀ & ◊¬q₀› by (metis q₀-prop)
```

```
AOT-hence 1: ‹◇∃ x(E!x & ¬𝓐E!x) & ◇¬∃ x(E!x & ¬𝓐E!x)›
  by (rule q0-def[THEN =df E(2), rotated])
     (simp add: log−prop−prop:2)
AOT-have ϑ: ‹◇∃ x [«?P»]x & ◇¬∃ x [«?P»]x›
  apply (AOT-subst ‹[«?P»]x› ‹E!x & ¬𝓐E!x› for: x)
   apply (rule beta−C−meta[THEN →E]; cqt:2[lambda])
  by (fact 1)
show ?thesis
proof (rule ∃ I(1))+
  AOT-have ‹◇[L]⁻ ≈E [«?P»] & ◇¬[L]⁻ ≈E [«?P»]›
  proof (rule &I; rule RM◇[THEN →E]; (rule →I)?)
    AOT-modally-strict {
      AOT-assume A: ‹¬∃ x [«?P»]x›
      AOT-show ‹[L]⁻ ≈E [«?P»]›
      proof (safe intro!: empty−approx:1[unvarify F H, THEN →E]
                          rel−neg−T:3 &I)
        AOT-show ‹[«?P»]↓› by cqt:2[lambda]
      next
        AOT-show ‹¬∃ u [L⁻]u›
        proof (rule raa−cor:2)
          AOT-assume ‹∃ u [L⁻]u›
          then AOT-obtain u where ‹[L⁻]u›
            using Ordinary.∃ E[rotated] by blast
          moreover AOT-have ‹¬[L⁻]u›
            using thm−noncont−e−e:2[THEN contingent−properties:2[THEN ≡df E],
                                   THEN &E(2)]
            by (metis qml:2[axiom-inst] rule−ui:3 →E)
          ultimately AOT-show ‹p & ¬p› for p
            by (metis  raa−cor:3)
        qed
      next
        AOT-show ‹¬∃ v [«?P»]v›
        proof (rule raa−cor:2)
          AOT-assume ‹∃ v [«?P»]v›
          then AOT-obtain u where ‹[«?P»]u›
            using Ordinary.∃ E[rotated] by blast
          AOT-hence ‹[«?P»]u›
            using &E by blast
          AOT-hence ‹∃ x [«?P»]x›
            by (rule ∃ I)
          AOT-thus ‹∃ x [«?P»]x & ¬∃ x [«?P»]x›
            using A &I by blast
        qed
      qed
    }
  next
    AOT-show ‹◇¬∃ x [«?P»]x›
      using ϑ &E by blast
  next
    AOT-modally-strict {
      AOT-assume A: ‹∃ x [«?P»]x›
      AOT-have B: ‹¬[«?P»] ≈E [L]⁻›
      proof (safe intro!: empty−approx:2[unvarify F H, THEN →E]
                          rel−neg−T:3 &I)
        AOT-show ‹[«?P»]↓›
          by cqt:2[lambda]
      next
        AOT-obtain x where Px: ‹[«?P»]x›
          using A ∃ E by blast
        AOT-hence ‹E!x & ¬𝓐E!x›
          by (rule β→C(1))
        AOT-hence 1: ‹◇E!x›
          by (metis T◇ &E(1) vdash−properties:10)
```

```
          AOT-have ‹[λx ◇E!x]x›
            by (auto intro!: β←C(1) cqt:2 1)
          AOT-hence ‹O!x›
            by (rule AOT-ordinary[THEN =df I(2), rotated]) cqt:2[lambda]
          AOT-hence ‹O!x & [«?P»]x›
            using Px &I by blast
          AOT-thus ‹∃ u [«?P»]u›
            by (rule ∃ I)
        next
          AOT-show ‹¬∃ u [L⁻]u›
          proof (rule raa−cor:2)
            AOT-assume ‹∃ u [L⁻]u›
            then AOT-obtain u where ‹[L⁻]u›
              using Ordinary.∃ E[rotated] by blast
            moreover AOT-have ‹¬[L⁻]u›
              using thm−noncont−e−e:2[THEN contingent−properties:2[THEN ≡df E]]
              by (metis qml:2[axiom-inst] rule−ui:3 →E &E(2))
            ultimately AOT-show ‹p & ¬p› for p
              by (metis raa−cor:3)
          qed
        qed
        AOT-show ‹¬[L]⁻ ≈E [«?P»]›
        proof (rule raa−cor:2)
          AOT-assume ‹[L]⁻ ≈E [«?P»]›
          AOT-hence ‹[«?P»] ≈E [L]⁻›
            apply (rule eq−part:2[unvarify F G, THEN →E, rotated 2])
             apply cqt:2[lambda]
            by (simp add: rel−neg−T:3)
          AOT-thus ‹[«?P»] ≈E [L]⁻ & ¬[«?P»] ≈E [L]⁻›
            using B &I by blast
        qed
      }
    next
      AOT-show ‹◇∃ x [«?P»]x›
        using ϑ &E by blast
    qed
    AOT-thus ‹◇([L]⁻ ≈E [«?P»] & ◇¬[L]⁻ ≈E [«?P»])›
      using S5Basic:11 ≡E(2) by blast
  next
    AOT-show ‹[λx [E!]x & ¬𝓐[E!]x]↓›
      by cqt:2
  next
    AOT-show ‹[L]⁻↓›
      by (simp add: rel−neg−T:3)
  qed
qed


AOT-theorem approx−cont:2:
  ‹∃ F∃ G ◇([λz 𝓐[F]z] ≈E G & ◇¬[λz 𝓐[F]z] ≈E G)›
proof −
  let ?P = ‹«[λx E!x & ¬𝓐E!x]»›
  AOT-have ‹◇q0 & ◇¬q0› by (metis q0-prop)
  AOT-hence 1: ‹◇∃ x(E!x & ¬𝓐E!x) & ◇¬∃ x(E!x & ¬𝓐E!x)›
    by (rule q0-def[THEN =df E(2), rotated])
       (simp add: log−prop−prop:2)
  AOT-have ϑ: ‹◇∃ x [«?P»]x & ◇¬∃ x [«?P»]x›
    apply (AOT-subst ‹[«?P»]x› ‹E!x & ¬𝓐E!x› for: x)
     apply (rule beta−C−meta[THEN →E]; cqt:2)
    by (fact 1)
  show ?thesis
  proof (rule ∃ I(1))+
    AOT-have ‹◇[λz 𝓐[L⁻]z] ≈E [«?P»] & ◇¬[λz 𝓐[L⁻]z] ≈E [«?P»]›
```

```
proof (rule &I; rule RM◇[THEN →E]; (rule →I)?)
  AOT-modally-strict {
    AOT-assume A: ‹¬∃ x [«?P»]x›
    AOT-show ‹[λz 𝓐[L⁻]z] ≈E [«?P»]›
    proof (safe intro!: empty−approx:1[unvarify F H, THEN →E]
                        rel−neg−T:3 &I)
      AOT-show ‹[«?P»]↓› by cqt:2
    next
      AOT-show ‹¬∃ u [λz 𝓐[L⁻]z]u›
      proof (rule raa−cor:2)
        AOT-assume ‹∃ u [λz 𝓐[L⁻]z]u›
        then AOT-obtain u where ‹[λz 𝓐[L⁻]z]u›
          using Ordinary.∃ E[rotated] by blast
        AOT-hence ‹𝓐[L⁻]u›
          using β→C(1) &E by blast
        moreover AOT-have ‹□¬[L⁻]u›
          using thm−noncont−e−e:2[THEN contingent−properties:2[THEN ≡df E]]
          by (metis RN qml:2[axiom-inst] rule−ui:3 →E &E(2))
        ultimately AOT-show ‹p & ¬p› for p
          by (metis Act−Sub:3 KBasic2:1 ≡E(1) raa−cor:3 →E)
      qed
    next
      AOT-show ‹¬∃ v [«?P»]v›
      proof (rule raa−cor:2)
        AOT-assume ‹∃ v [«?P»]v›
        then AOT-obtain u where ‹[«?P»]u›
          using Ordinary.∃ E[rotated] by blast
        AOT-hence ‹[«?P»]u›
          using &E by blast
        AOT-hence ‹∃ x [«?P»]x›
          by (rule ∃ I)
        AOT-thus ‹∃ x [«?P»]x & ¬∃ x [«?P»]x›
          using A &I by blast
      qed
    next
      AOT-show ‹[λz 𝓐[L⁻]z]↓› by cqt:2
    qed
  }
next
  AOT-show ‹◇¬∃ x [«?P»]x› using ϑ &E by blast
next
  AOT-modally-strict {
    AOT-assume A: ‹∃ x [«?P»]x›
    AOT-have B: ‹¬[«?P»] ≈E [λz 𝓐[L⁻]z]›
    proof (safe intro!: empty−approx:2[unvarify F H, THEN →E]
                        rel−neg−T:3 &I)
      AOT-show ‹[«?P»]↓› by cqt:2
    next
      AOT-obtain x where Px: ‹[«?P»]x›
        using A ∃ E by blast
      AOT-hence ‹E!x & ¬𝓐E!x›
        by (rule β→C(1))
      AOT-hence ‹◇E!x›
        by (metis T◇ &E(1) →E)
      AOT-hence ‹[λx ◇E!x]x›
        by (auto intro!: β←C(1) cqt:2)
      AOT-hence ‹O!x›
        by (rule AOT-ordinary[THEN =df I(2), rotated]) cqt:2
      AOT-hence ‹O!x & [«?P»]x›
        using Px &I by blast
      AOT-thus ‹∃ u [«?P»]u›
        by (rule ∃ I)
    next
```

```
          AOT-show ‹¬∃ u [λz 𝓐[L⁻]z]u›
          proof (rule raa−cor:2)
            AOT-assume ‹∃ u [λz 𝓐[L⁻]z]u›
            then AOT-obtain u where ‹[λz 𝓐[L⁻]z]u›
              using Ordinary.∃ E[rotated] by blast
            AOT-hence ‹𝓐[L⁻]u›
              using β→C(1) &E by blast
            moreover AOT-have ‹□¬[L⁻]u›
              using thm−noncont−e−e:2[THEN contingent−properties:2[THEN ≡df E]]
              by (metis RN qml:2[axiom-inst] rule−ui:3 →E &E(2))
            ultimately AOT-show ‹p & ¬p› for p
              by (metis Act−Sub:3 KBasic2:1 ≡E(1) raa−cor:3 →E)
          qed
        next
          AOT-show ‹[λz 𝓐[L⁻]z]↓› by cqt:2
        qed
        AOT-show ‹¬[λz 𝓐[L⁻]z] ≈E [«?P»]›
        proof (rule raa−cor:2)
          AOT-assume ‹[λz 𝓐[L⁻]z] ≈E [«?P»]›
          AOT-hence ‹[«?P»] ≈E [λz 𝓐[L⁻]z]›
            by (rule eq−part:2[unvarify F G, THEN →E, rotated 2])
               cqt:2+
          AOT-thus ‹[«?P»] ≈E [λz 𝓐[L⁻]z] & ¬[«?P»] ≈E [λz 𝓐[L⁻]z]›
            using B &I by blast
        qed
      }
    next
      AOT-show ‹◇∃ x [«?P»]x›
        using ϑ &E by blast
    qed
    AOT-thus ‹◇([λz 𝓐[L⁻]z] ≈E [«?P»] & ◇¬[λz 𝓐[L⁻]z] ≈E [«?P»])›
      using S5Basic:11 ≡E(2) by blast
  next
    AOT-show ‹[λx [E!]x & ¬𝓐[E!]x]↓› by cqt:2
  next
    AOT-show ‹[L]⁻↓›
      by (simp add: rel−neg−T:3)
  qed
qed

notepad
begin
```

We already have defined being equivalent on the ordinary objects in the Extended Relation Comprehension theory.

```
  AOT-have ‹F ≡E G ≡df F↓ & G↓ & ∀ u ([F]u ≡ [G]u)› for F G
    using eqE by blast
end

AOT-theorem apE−eqE:1: ‹F ≡E G → F ≈E G›
proof(rule →I)
  AOT-assume 0: ‹F ≡E G›
  AOT-have ‹∃ R R |: F 1−1⟷E G›
  proof (safe intro!: ∃ I(1)[where τ=«(=E)»] equi:2[THEN ≡df I] &I
                    =E[denotes] cqt:2[const-var][axiom-inst] Ordinary.GEN
                    →I equi:1[THEN ≡E(2)])
    fix u
    AOT-assume Fu: ‹[F]u›
    AOT-hence Gu: ‹[G]u›
      using ≡df E[OF eqE, OF 0, THEN &E(2),
                THEN Ordinary.∀ E[where α=u], THEN ≡E(1)]
            Ordinary.ψ Fu by blast
    AOT-show ‹∃ v ([G]v & u =E v & ∀ v′ ([G]v′ & u =E v′ → v′ =E v))›
```

by (*safe intro*!: *Ordinary*.∃ *I*[**where** β=*u*] &*I GEN* →*I Ordinary*.ψ *Gu*
ord=*Eequiv*:*1*[*THEN* →*E*, *OF Ordinary*.ψ]
ord=*Eequiv*:*2*[*THEN* →*E*] *dest*!: &*E*(*2*))
next
fix *v*
AOT-assume *Gv*: ‹[*G*]*v*›
AOT-hence *Fv*: ‹[*F*]*v*›
using $\equiv_{df}E$[*OF eqE*, *OF 0*, *THEN* &*E*(*2*),
THEN Ordinary.∀ *E*[**where** α=*v*], *THEN* ≡*E*(*2*)]
Ordinary.ψ *Gv* **by** *blast*
AOT-show ‹∃ *u* ([*F*]*u* & $u =_E v$ & ∀ *v*′ ([*F*]*v*′ & $v' =_E v$ → $v' =_E u$))›
by (*safe intro*!: *Ordinary*.∃ *I*[**where** β=*v*] &*I GEN* →*I Ordinary*.ψ *Fv*
ord=*Eequiv*:*1*[*THEN* →*E*, *OF Ordinary*.ψ]
ord=*Eequiv*:*2*[*THEN* →*E*] *dest*!: &*E*(*2*))
qed
AOT-thus ‹$F \approx_E G$›
by (*rule equi*:*3*[*THEN* $\equiv_{df}I$])
qed

AOT-theorem *apE*−*eqE*:*2*: ‹($F \approx_E G$ & $G \equiv_E H$) → $F \approx_E H$›
proof(*rule* →*I*)
AOT-assume ‹$F \approx_E G$ & $G \equiv_E H$›
AOT-hence ‹$F \approx_E G$› **and** ‹$G \approx_E H$›
using *apE*−*eqE*:*1*[*THEN* →*E*] &*E* **by** *blast*+
AOT-thus ‹$F \approx_E H$›
by (*metis Adjunction eq*−*part*:*3 vdash*−*properties*:*10*)
qed

AOT-act-theorem *eq*−*part*−*act*:*1*: ‹[λ*z* 𝓐[*F*]*z*] $\equiv_E$ *F*›
proof (*safe intro*!: *eqE*[*THEN* $\equiv_{df}I$] &*I cqt*:*2 Ordinary*.*GEN* →*I*)
fix *u*
AOT-have ‹[λ*z* 𝓐[*F*]*z*]*u* ≡ 𝓐[*F*]*u*›
by (*rule beta*−*C*−*meta*[*THEN* →*E*]) *cqt*:*2*[*lambda*]
also AOT-have ‹. . . ≡ [*F*]*u*›
using *act*−*conj*−*act*:*4 logic*−*actual*[*act-axiom-inst*, *THEN* →*E*] **by** *blast*
finally AOT-show ‹[λ*z* 𝓐[*F*]*z*]*u* ≡ [*F*]*u*›.
qed

AOT-act-theorem *eq*−*part*−*act*:*2*: ‹[λ*z* 𝓐[*F*]*z*] $\approx_E$ *F*›
by (*safe intro*!: *apE*−*eqE*:*1*[*unvarify F*, *THEN* →*E*] *eq*−*part*−*act*:*1*) *cqt*:*2*

AOT-theorem *actuallyF*:*1*: ‹𝓐(*F* $\approx_E$ [λ*z* 𝓐[*F*]*z*])›
proof −
AOT-have *1*: ‹𝓐([*F*]*x* ≡ 𝓐[*F*]*x*)› **for** *x*
by (*meson Act*−*Basic*:*5 act*−*conj*−*act*:*4* ≡*E*(*2*) *Commutativity of* ≡)
AOT-have ‹𝓐([*F*]*x* ≡ [λ*z* 𝓐[*F*]*z*]*x*)› **for** *x*
apply (*AOT-subst* ‹[λ*z* 𝓐[*F*]*z*]*x*› ‹𝓐[*F*]*x*›)
apply (*rule beta*−*C*−*meta*[*THEN* →*E*])
apply *cqt*:*2*[*lambda*]
by (*fact 1*)
AOT-hence ‹*O*!*x* → 𝓐([*F*]*x* ≡ [λ*z* 𝓐[*F*]*z*]*x*)› **for** *x*
by (*metis* →*I*)
AOT-hence ‹∀ *u* 𝓐([*F*]*u* ≡ [λ*z* 𝓐[*F*]*z*]*u*)›
using ∀ *I* **by** *fast*
AOT-hence *1*: ‹𝓐∀ *u* ([*F*]*u* ≡ [λ*z* 𝓐[*F*]*z*]*u*)›
by (*metis Ordinary*.*res*−*var*−*bound*−*reas*[*2*] →*E*)
AOT-modally-strict {
AOT-have ‹[λ*z* 𝓐[*F*]*z*]↓› **by** *cqt*:*2*
} **note** *2* = *this*
AOT-have ‹𝓐(*F* $\equiv_E$ [λ*z* 𝓐[*F*]*z*])›
apply (*AOT-subst* ‹*F* $\equiv_E$ [λ*z* 𝓐[*F*]*z*]› ‹∀ *u* ([*F*]*u* ≡ [λ*z* 𝓐[*F*]*z*]*u*)›)

using *eqE*[*THEN* ≡*Df*, *THEN* ≡*S*(*1*), *OF* &*I*,
OF cqt:*2*[*const-var*][*axiom-inst*], *OF 2*]
by (*auto simp*: *1*)
moreover AOT-have ‹$\mathcal{A}$(*F* $\equiv_E$ [λ*z* $\mathcal{A}$[*F*]*z*] → *F* $\approx_E$ [λ*z* $\mathcal{A}$[*F*]*z*])›
using *apE*−*eqE*:*1*[*unvarify G*, *THEN RA*[*2*], *OF 2*] **by** *metis*
ultimately AOT-show ‹$\mathcal{A}$*F* $\approx_E$ [λ*z* $\mathcal{A}$[*F*]*z*]›
by (*metis act*−*cond* →*E*)
qed

AOT-theorem *actuallyF*:*2*: ‹*Rigid*([λ*z* $\mathcal{A}$[*F*]*z*])›
proof(*safe intro*!: *GEN* →*I df*−*rigid*−*rel*:*1*[*THEN* $\equiv_{df}$*I*] &*I*)
AOT-show ‹[λ*z* $\mathcal{A}$[*F*]*z*]↓› **by** *cqt*:*2*
next
AOT-show ‹□∀*x* ([λ*z* $\mathcal{A}$[*F*]*z*]*x* → □[λ*z* $\mathcal{A}$[*F*]*z*]*x*)›
proof(*rule RN*; *rule GEN*; *rule* →*I*)
AOT-modally-strict {
fix *x*
AOT-assume ‹[λ*z* $\mathcal{A}$[*F*]*z*]*x*›
AOT-hence ‹$\mathcal{A}$[*F*]*x*›
by (*rule β*→*C*(*1*))
AOT-hence *1*: ‹□$\mathcal{A}$[*F*]*x*› **by** (*metis Act*−*Basic*:*6* ≡*E*(*1*))
AOT-show ‹□[λ*z* $\mathcal{A}$[*F*]*z*]*x*›
apply (*AOT-subst* ‹[λ*z* $\mathcal{A}$[*F*]*z*]*x*› ‹$\mathcal{A}$[*F*]*x*›)
apply (*rule beta*−*C*−*meta*[*THEN* →*E*])
apply *cqt*:*2*[*lambda*]
by (*fact 1*)
}
qed
qed

AOT-theorem *approx*−*nec*:*1*: ‹*Rigid*(*F*) → *F* $\approx_E$ [λ*z* $\mathcal{A}$[*F*]*z*]›
proof(*rule* →*I*)
AOT-assume ‹*Rigid*([*F*])›
AOT-hence *A*: ‹□∀*x* ([*F*]*x* → □[*F*]*x*)›
using *df*−*rigid*−*rel*:*1*[*THEN* $\equiv_{df}$*E*, *THEN* &*E*(*2*)] **by** *blast*
AOT-hence *0*: ‹∀*x* □([*F*]*x* → □[*F*]*x*)›
using *CBF*[*THEN* →*E*] **by** *blast*
AOT-hence *1*: ‹∀*x* ([*F*]*x* → □[*F*]*x*)›
using *A qml*:*2*[*axiom-inst*, *THEN* →*E*] **by** *blast*
AOT-have *act-F-den*: ‹[λ*z* $\mathcal{A}$[*F*]*z*]↓›
by *cqt*:*2*
AOT-show ‹*F* $\approx_E$ [λ*z* $\mathcal{A}$[*F*]*z*]›
proof (*safe intro*!: *apE*−*eqE*:*1*[*unvarify G*, *THEN* →*E*] *eqE*[*THEN* $\equiv_{df}$*I*] &*I*
cqt:*2 act-F-den Ordinary*.*GEN* →*I* ≡*I*)
fix *u*
AOT-assume ‹[*F*]*u*›
AOT-hence ‹□[*F*]*u*›
using *1*[*THEN* ∀*E*(*2*), *THEN* →*E*] **by** *blast*
AOT-hence *act-F-u*: ‹$\mathcal{A}$[*F*]*u*›
by (*metis nec*−*imp*−*act* →*E*)
AOT-show ‹[λ*z* $\mathcal{A}$[*F*]*z*]*u*›
by (*auto intro*!: *β*←*C*(*1*) *cqt*:*2 act-F-u*)
next
fix *u*
AOT-assume ‹[λ*z* $\mathcal{A}$[*F*]*z*]*u*›
AOT-hence ‹$\mathcal{A}$[*F*]*u*›
by (*rule β*→*C*(*1*))
AOT-thus ‹[*F*]*u*›
using *0*[*THEN* ∀*E*(*2*)]
by (*metis* ≡*E*(*1*) *sc*−*eq*−*fur*:*2* →*E*)
qed
qed

```
AOT-theorem approx−nec:2:
  ‹F ≈E G ≡ ∀ H ([λz 𝓐[H]z] ≈E F ≡ [λz 𝓐[H]z] ≈E G)›
proof(rule ≡I; rule →I)
  AOT-assume 0: ‹F ≈E G›
  AOT-assume 0: ‹F ≈E G›
  AOT-hence ‹∀ H (H ≈E F ≡ H ≈E G)›
    using eq−part:4[THEN ≡E(1), OF 0] by blast
  AOT-have ‹[λz 𝓐[H]z] ≈E F ≡ [λz 𝓐[H]z] ≈E G› for H
    by (rule ∀ E(1)[OF eq−part:4[THEN ≡E(1), OF 0]]) cqt:2
  AOT-thus ‹∀ H ([λz 𝓐[H]z] ≈E F ≡ [λz 𝓐[H]z] ≈E G)›
    by (rule GEN)
next
  AOT-assume 0: ‹∀ H ([λz 𝓐[H]z] ≈E F ≡ [λz 𝓐[H]z] ≈E G)›
  AOT-obtain H where ‹Rigidifies(H,F)›
    using rigid−der:3 ∃ E by metis
  AOT-hence H: ‹Rigid(H) & ∀ x ([H]x ≡ [F]x)›
    using df−rigid−rel:2[THEN ≡dfE] by blast
  AOT-have H-rigid: ‹□∀ x ([H]x → □[H]x)›
    using H[THEN &E(1), THEN df−rigid−rel:1[THEN ≡dfE], THEN &E(2)].
  AOT-hence ‹∀ x □([H]x → □[H]x)›
    using CBF vdash−properties:10 by blast
  AOT-hence ‹□([H]x → □[H]x)› for x using ∀ E(2) by blast
  AOT-hence rigid: ‹[H]x ≡ 𝓐[H]x› for x
     by (metis ≡E(6) oth−class−taut:3:a sc−eq−fur:2 →E)
  AOT-have ‹H ≡E F›
  proof (safe intro!: eqE[THEN ≡dfI] &I cqt:2 Ordinary.GEN →I)
    AOT-show ‹[H]u ≡ [F]u› for u using H[THEN &E(2)] ∀ E(2) by fast
  qed
  AOT-hence ‹H ≈E F›
    by (rule apE−eqE:2[THEN →E, OF &I, rotated])
       (simp add: eq−part:1)
  AOT-hence F-approx-H: ‹F ≈E H›
    by (metis eq−part:2 →E)
  moreover AOT-have H-eq-act-H: ‹H ≡E [λz 𝓐[H]z]›
  proof (safe intro!: eqE[THEN ≡dfI] &I cqt:2 Ordinary.GEN →I)
    AOT-show ‹[H]u ≡ [λz 𝓐[H]z]u› for u
      apply (AOT-subst ‹[λz 𝓐[H]z]u› ‹𝓐[H]u›)
       apply (rule beta−C−meta[THEN →E])
       apply cqt:2[lambda]
      using rigid by blast
  qed
  AOT-have a: ‹F ≈E [λz 𝓐[H]z]›
    apply (rule apE−eqE:2[unvarify H, THEN →E])
     apply cqt:2[lambda]
    using F-approx-H H-eq-act-H &I by blast
  AOT-hence ‹[λz 𝓐[H]z] ≈E F›
    apply (rule eq−part:2[unvarify G, THEN →E, rotated])
    by cqt:2[lambda]
  AOT-hence b: ‹[λz 𝓐[H]z] ≈E G›
    by (rule 0[THEN ∀ E(1), THEN ≡E(1), rotated]) cqt:2
  AOT-show ‹F ≈E G›
    by (rule eq−part:3[unvarify G, THEN →E, rotated, OF &I, OF a, OF b])
       cqt:2
qed

AOT-theorem approx−nec:3:
  ‹(Rigid(F) & Rigid(G)) → □(F ≈E G → □F ≈E G)›
proof (rule →I)
  AOT-assume ‹Rigid(F) & Rigid(G)›
  AOT-hence ‹□∀ x([F]x → □[F]x)› and ‹□∀ x([G]x → □[G]x)›
    using df−rigid−rel:1[THEN ≡dfE, THEN &E(2)] &E by blast+
  AOT-hence ‹□(□∀ x([F]x → □[F]x) & □∀ x([G]x → □[G]x))›
```

```
  using KBasic:3 4 &I ≡E(2) vdash−properties:10 by meson
moreover AOT-have ‹□(□∀x([F]x → □[F]x) & □∀x([G]x → □[G]x)) →
                  □(F ≈E G → □F ≈E G)›
proof(rule RM; rule →I; rule →I)
  AOT-modally-strict {
    AOT-assume ‹□∀x([F]x → □[F]x) & □∀x([G]x → □[G]x)›
    AOT-hence ‹□∀x([F]x → □[F]x)› and ‹□∀x([G]x → □[G]x)›
      using &E by blast+
    AOT-hence ‹∀x□([F]x → □[F]x)› and ‹∀x□([G]x → □[G]x)›
      using CBF[THEN →E] by blast+
    AOT-hence F-nec: ‹□([F]x → □[F]x)›
          and G-nec: ‹□([G]x → □[G]x)› for x
      using ∀E(2) by blast+
    AOT-assume ‹F ≈E G›
    AOT-hence ‹∃R R |: F 1−1⟷E G›
      by (metis ≡dfE equi:3)
    then AOT-obtain R where ‹R |: F 1−1⟷E G›
      using ∃E[rotated] by blast
    AOT-hence C1: ‹∀u ([F]u → ∃!v ([G]v & [R]uv))›
          and C2: ‹∀v ([G]v → ∃!u ([F]u & [R]uv))›
      using equi:2[THEN ≡dfE] &E by blast+
    AOT-obtain R' where ‹Rigidifies(R', R)›
      using rigid−der:3 ∃E[rotated] by blast
    AOT-hence 1: ‹Rigid(R') & ∀x1...∀xn ([R']x1...xn ≡ [R]x1...xn)›
      using df−rigid−rel:2[THEN ≡dfE] by blast
    AOT-hence ‹□∀x1...∀xn ([R']x1...xn → □[R']x1...xn)›
      using df−rigid−rel:1[THEN ≡dfE] &E by blast
    AOT-hence ‹∀x1...∀xn (◇[R']x1...xn → □[R']x1...xn)›
      using ≡E(1) rigid−rel−thms:1 by blast
    AOT-hence D: ‹∀x1∀x2 (◇[R']x1x2 → □[R']x1x2)›
      using tuple-forall[THEN ≡dfE] by blast
    AOT-have E: ‹∀x1∀x2 ([R']x1x2 ≡ [R]x1x2)›
      using tuple-forall[THEN ≡dfE, OF 1[THEN &E(2)]] by blast
    AOT-have ‹∀u □([F]u → ∃!v ([G]v & [R']uv))›
         and ‹∀v □([G]v → ∃!u ([F]u & [R']uv))›
    proof (safe intro!: Ordinary.GEN →I)
      fix u
      AOT-show ‹□([F]u → ∃!v ([G]v & [R']uv))›
      proof (rule raa−cor:1)
        AOT-assume ‹¬□([F]u → ∃!v ([G]v & [R']uv))›
        AOT-hence 1: ‹◇¬([F]u → ∃!v ([G]v & [R']uv))›
          using KBasic:11 ≡E(1) by blast
        AOT-have ‹◇([F]u & ¬∃!v ([G]v & [R']uv))›
          apply (AOT-subst ‹[F]u & ¬∃!v ([G]v & [R']uv)›
                           ‹¬([F]u → ∃!v ([G]v & [R']uv))›)
           apply (meson ≡E(6) oth−class−taut:1:b oth−class−taut:3:a)
          by (fact 1)
        AOT-hence A: ‹◇[F]u & ◇¬∃!v ([G]v & [R']uv)›
          using KBasic2:3 →E by blast
        AOT-hence ‹□[F]u›
          using F-nec &E(1) ≡E(1) sc−eq−box−box:1 →E by blast
        AOT-hence ‹[F]u›
          by (metis qml:2[axiom-inst] →E)
        AOT-hence ‹∃!v ([G]v & [R]uv)›
          using C1[THEN Ordinary.∀E, THEN →E] by blast
        AOT-hence ‹∃v ([G]v & [R]uv & ∀v' ([G]v' & [R]uv' → v' =E v))›
          using equi:1[THEN ≡E(1)] by auto
        then AOT-obtain a where
          a-prop: ‹O!a & ([G]a & [R]ua & ∀v' ([G]v' & [R]uv' → v' =E a))›
          using ∃E[rotated] by blast
        AOT-have ‹∃v □([G]v & [R']uv & ∀v' ([G]v' & [R']uv' → v' =E v))›
        proof(safe intro!: ∃I(2)[where β=a] &I a-prop[THEN &E(1)]
                           KBasic:3[THEN ≡E(2)])
```

```
      AOT-show ‹□[G]a›
        using a-prop[THEN &E(2), THEN &E(1), THEN &E(1)]
        by (metis G-nec qml:2[axiom-inst] →E)
    next
      AOT-show ‹□[R']ua›
        using D[THEN ∀E(2), THEN ∀E(2), THEN →E]
              E[THEN ∀E(2), THEN ∀E(2), THEN ≡E(2),
                OF a-prop[THEN &E(2), THEN &E(1), THEN &E(2)]]
        by (metis T◊ →E)
    next
      AOT-have ‹∀v' □([G]v' & [R']uv' → v' =E a)›
      proof (rule Ordinary.GEN; rule raa−cor:1)
        fix v'
        AOT-assume ‹¬□([G]v' & [R']uv' → v' =E a)›
        AOT-hence ‹◊¬([G]v' & [R']uv' → v' =E a)›
          by (metis KBasic:11 ≡E(1))
        AOT-hence ‹◊([G]v' & [R']uv' & ¬v' =E a)›
          by (AOT-subst ‹[G]v' & [R']uv' & ¬v' =E a›
                        ‹¬([G]v' & [R']uv' → v' =E a)›)
             (meson ≡E(6) oth−class−taut:1:b oth−class−taut:3:a)
        AOT-hence 1: ‹◊[G]v'› and 2: ‹◊[R']uv'› and 3: ‹◊¬v' =E a›
          using KBasic2:3[THEN →E, THEN &E(1)]
                KBasic2:3[THEN →E, THEN &E(2)] by blast+
        AOT-have Gv': ‹[G]v'› using G-nec 1
          by (meson B◊ KBasic:13 →E)
        AOT-have ‹□[R']uv'›
          using 2 D[THEN ∀E(2), THEN ∀E(2), THEN →E] by blast
        AOT-hence R'uv': ‹[R']uv'›
          by (metis B◊ T◊ →E)
        AOT-hence ‹[R]uv'›
          using E[THEN ∀E(2), THEN ∀E(2), THEN ≡E(1)] by blast
        AOT-hence ‹v' =E a›
          using a-prop[THEN &E(2), THEN &E(2), THEN Ordinary.∀E,
                       THEN →E, OF &I, OF Gv'] by blast
        AOT-hence ‹□(v' =E a)›
          by (metis id−nec3:1 ≡E(4) raa−cor:3)
        moreover AOT-have ‹¬□(v' =E a)›
          using 3 KBasic:11 ≡E(2) by blast
        ultimately AOT-show ‹□(v' =E a) & ¬□(v' =E a)›
          using &I by blast
      qed
      AOT-thus ‹□∀v'([G]v' & [R']uv' → v' =E a)›
        using Ordinary.res−var−bound−reas[BF] →E by fast
    qed
    AOT-hence ‹□∃v ([G]v & [R']uv & ∀v' ([G]v' & [R']uv' → v' =E v))›
      using Ordinary.res−var−bound−reas[Buridan] →E by fast
    AOT-hence ‹□∃!v ([G]v & [R']uv)›
      by (AOT-subst-thm equi:1)
    moreover AOT-have ‹¬□∃!v ([G]v & [R']uv)›
      using A[THEN &E(2)] KBasic:11[THEN ≡E(2)] by blast
    ultimately AOT-show ‹□∃!v ([G]v & [R']uv) & ¬□∃!v ([G]v & [R']uv)›
      by (rule &I)
  qed
next
  fix v
  AOT-show ‹□([G]v → ∃!u ([F]u & [R']uv))›
  proof (rule raa−cor:1)
    AOT-assume ‹¬□([G]v → ∃!u ([F]u & [R']uv))›
    AOT-hence 1: ‹◊¬([G]v → ∃!u ([F]u & [R']uv))›
      using KBasic:11 ≡E(1) by blast
    AOT-hence ‹◊([G]v & ¬∃!u ([F]u & [R']uv))›
      by (AOT-subst ‹[G]v & ¬∃!u ([F]u & [R']uv)›
                    ‹¬([G]v → ∃!u ([F]u & [R']uv))›)
```

```
    (meson ≡E(6) oth−class−taut:1:b oth−class−taut:3:a)
AOT-hence A: ‹◇[G]v & ◇¬∃!u ([F]u & [R']uv)›
  using KBasic2:3 →E by blast
AOT-hence ‹□[G]v›
  using G-nec &E(1) ≡E(1) sc−eq−box−box:1 →E by blast
AOT-hence ‹[G]v› by (metis qml:2[axiom-inst] →E)
AOT-hence ‹∃!u ([F]u & [R]uv)›
  using C2[THEN Ordinary.∀E, THEN →E] by blast
AOT-hence ‹∃u ([F]u & [R]uv & ∀u' ([F]u' & [R]u'v → u' =E u))›
  using equi:1[THEN ≡E(1)] by auto
then AOT-obtain a where
    a-prop: ‹O!a & ([F]a & [R]av & ∀u' ([F]u' & [R]u'v → u' =E a))›
  using ∃E[rotated] by blast
AOT-have ‹∃u □([F]u & [R']uv & ∀u' ([F]u' & [R']u'v → u' =E u))›
proof(safe intro!: ∃I(2)[where β=a] &I a-prop[THEN &E(1)]
                  KBasic:3[THEN ≡E(2)])
  AOT-show ‹□[F]a›
    using a-prop[THEN &E(2), THEN &E(1), THEN &E(1)]
    by (metis F-nec qml:2[axiom-inst] →E)
next
  AOT-show ‹□[R']av›
    using D[THEN ∀E(2), THEN ∀E(2), THEN →E]
          E[THEN ∀E(2), THEN ∀E(2), THEN ≡E(2),
            OF a-prop[THEN &E(2), THEN &E(1), THEN &E(2)]]
    by (metis T◇ →E)
next
  AOT-have ‹∀u' □([F]u' & [R']u'v → u' =E a)›
  proof (rule Ordinary.GEN; rule raa−cor:1)
    fix u'
    AOT-assume ‹¬□([F]u' & [R']u'v → u' =E a)›
    AOT-hence ‹◇¬([F]u' & [R']u'v → u' =E a)›
      by (metis KBasic:11 ≡E(1))
    AOT-hence ‹◇([F]u' & [R']u'v & ¬u' =E a)›
      by (AOT-subst ‹[F]u' & [R']u'v & ¬u' =E a›
                   ‹¬([F]u' & [R']u'v → u' =E a)›)
         (meson ≡E(6) oth−class−taut:1:b oth−class−taut:3:a)
    AOT-hence 1: ‹◇[F]u'› and 2: ‹◇[R']u'v› and 3: ‹◇¬u' =E a›
      using KBasic2:3[THEN →E, THEN &E(1)]
            KBasic2:3[THEN →E, THEN &E(2)] by blast+
    AOT-have Fu': ‹[F]u'› using F-nec 1
      by (meson B◇ KBasic:13 →E)
    AOT-have ‹□[R']u'v›
      using 2 D[THEN ∀E(2), THEN ∀E(2), THEN →E] by blast
    AOT-hence R'u'v: ‹[R']u'v›
      by (metis B◇ T◇ →E)
    AOT-hence ‹[R]u'v›
      using E[THEN ∀E(2), THEN ∀E(2), THEN ≡E(1)] by blast
    AOT-hence ‹u' =E a›
      using a-prop[THEN &E(2), THEN &E(2), THEN Ordinary.∀E,
                   THEN →E, OF &I, OF Fu'] by blast
    AOT-hence ‹□(u' =E a)›
      by (metis id−nec3:1 ≡E(4) raa−cor:3)
    moreover AOT-have ‹¬□(u' =E a)›
      using 3 KBasic:11 ≡E(2) by blast
    ultimately AOT-show ‹□(u' =E a) & ¬□(u' =E a)›
      using &I by blast
  qed
  AOT-thus ‹□∀u'([F]u' & [R']u'v → u' =E a)›
    using Ordinary.res−var−bound−reas[BF] →E by fast
qed
AOT-hence 1: ‹□∃u ([F]u & [R']uv & ∀u' ([F]u' & [R']u'v → u' =E u))›
  using Ordinary.res−var−bound−reas[Buridan] →E by fast
AOT-hence ‹□∃!u ([F]u & [R']uv)›
```

by (*AOT-subst-thm equi*:*1*)
moreover AOT-have ‹¬□∃!*u* ([*F*]*u* & [*R'*]*uv*)›
using *A*[*THEN* &*E*(*2*)] *KBasic*:*11*[*THEN* ≡*E*(*2*)] **by** *blast*
ultimately AOT-show ‹□∃!*u* ([*F*]*u* & [*R'*]*uv*) & ¬□∃!*u* ([*F*]*u* & [*R'*]*uv*)›
by (*rule* &*I*)
qed
qed
AOT-hence ‹□∀ *u* ([*F*]*u* → ∃!*v* ([*G*]*v* & [*R'*]*uv*))›
and ‹□∀ *v* ([*G*]*v* → ∃!*u* ([*F*]*u* & [*R'*]*uv*))›
using *Ordinary.res−var−bound−reas*[*BF*][*THEN* →*E*] **by** *auto*
moreover AOT-have ‹□[*R'*]↓› **and** ‹□[*F*]↓› **and** ‹□[*G*]↓›
by (*simp-all add*: *ex*:*2*:*a*)
ultimately AOT-have ‹□([*R'*]↓ & [*F*]↓ & [*G*]↓ & ∀ *u* ([*F*]*u* → ∃!*v* ([*G*]*v* & [*R'*]*uv*)) &
∀ *v* ([*G*]*v* → ∃!*u* ([*F*]*u* & [*R'*]*uv*)))›
using *KBasic*:*3* &*I* ≡*E*(*2*) **by** *meson*
AOT-hence ‹□*R'* |: *F* $_{1-1}$⟷$_E$ *G*›
by (*AOT-subst-def equi*:*2*)
AOT-hence ‹∃ *R* □*R* |: *F* $_{1-1}$⟷$_E$ *G*›
by (*rule* ∃ *I*(*2*))
AOT-hence ‹□∃ *R R* |: *F* $_{1-1}$⟷$_E$ *G*›
by (*metis Buridan* →*E*)
AOT-thus ‹□*F* ≈$_E$ *G*›
by (*AOT-subst-def equi*:*3*)
}
qed
ultimately AOT-show ‹□(*F* ≈$_E$ *G* → □*F* ≈$_E$ *G*)›
using →*E* **by** *blast*
qed

AOT-define *numbers* :: ‹τ ⇒ τ ⇒ φ› (‹*Numbers'*(-,-')›)
‹*Numbers*(*x*,*G*) ≡$_{df}$ *A*!*x* & *G*↓ & ∀ *F*(*x*[*F*] ≡ [λ*z* 𝓐[*F*]*z*] ≈$_E$ *G*)›

AOT-theorem *numbers*[*den*]:
‹Π↓ → (*Numbers*(κ, Π) ≡ *A*!κ & ∀ *F*(κ[*F*] ≡ [λ*z* 𝓐[*F*]*z*] ≈$_E$ Π))›
apply (*safe intro*!: *numbers*[*THEN* ≡$_{df}$*I*] &*I* ≡*I* →*I cqt*:*2*
dest!: *numbers*[*THEN* ≡$_{df}$*E*])
using &*E* **by** *blast*+

AOT-theorem *num−tran*:*1*:
‹*G* ≈$_E$ *H* → (*Numbers*(*x*, *G*) ≡ *Numbers*(*x*, *H*))›
proof (*safe intro*!: →*I* ≡*I*)
AOT-assume *0*: ‹*G* ≈$_E$ *H*›
AOT-assume ‹*Numbers*(*x*, *G*)›
AOT-hence *Ax*: ‹*A*!*x*› **and** ϑ: ‹∀ *F* (*x*[*F*] ≡ [λ*z* 𝓐[*F*]*z*] ≈$_E$ *G*)›
using *numbers*[*THEN* ≡$_{df}$*E*] &*E* **by** *blast*+
AOT-show ‹*Numbers*(*x*, *H*)›
proof(*safe intro*!: *numbers*[*THEN* ≡$_{df}$*I*] &*I Ax cqt*:*2 GEN*)
fix *F*
AOT-have ‹*x*[*F*] ≡ [λ*z* 𝓐[*F*]*z*] ≈$_E$ *G*›
using ϑ[*THEN* ∀ *E*(*2*)]**.**
also AOT-have ‹. . . ≡ [λ*z* 𝓐[*F*]*z*] ≈$_E$ *H*›
using *0 approx−nec*:*2*[*THEN* ≡*E*(*1*), *THEN* ∀ *E*(*2*)] **by** *metis*
finally AOT-show ‹*x*[*F*] ≡ [λ*z* 𝓐[*F*]*z*] ≈$_E$ *H*›**.**
qed
next
AOT-assume ‹*G* ≈$_E$ *H*›
AOT-hence *0*: ‹*H* ≈$_E$ *G*›
by (*metis eq−part*:*2* →*E*)
AOT-assume ‹*Numbers*(*x*, *H*)›
AOT-hence *Ax*: ‹*A*!*x*› **and** ϑ: ‹∀ *F* (*x*[*F*] ≡ [λ*z* 𝓐[*F*]*z*] ≈$_E$ *H*)›
using *numbers*[*THEN* ≡$_{df}$*E*] &*E* **by** *blast*+
AOT-show ‹*Numbers*(*x*, *G*)›

```
  proof(safe intro!: numbers[THEN ≡df I] &I Ax cqt:2  GEN)
    fix F
    AOT-have ‹x[F] ≡ [λz 𝓐[F]z] ≈E  H›
      using ϑ[THEN ∀E(2)].
    also AOT-have ‹... ≡ [λz 𝓐[F]z] ≈E  G›
      using 0 approx−nec:2[THEN ≡E(1), THEN ∀E(2)] by metis
    finally AOT-show ‹x[F] ≡ [λz 𝓐[F]z] ≈E  G›.
  qed
qed

AOT-theorem num−tran:2:
  ‹(Numbers(x, G) & Numbers(x,H)) → G ≈E  H›
proof (rule →I; frule &E(1); drule &E(2))
  AOT-assume ‹Numbers(x,G)›
  AOT-hence ‹∀F (x[F] ≡ [λz 𝓐[F]z] ≈E  G)›
    using numbers[THEN ≡df E] &E by blast
  AOT-hence 1: ‹x[F] ≡ [λz 𝓐[F]z] ≈E  G› for F
    using ∀E(2) by blast
  AOT-assume ‹Numbers(x,H)›
  AOT-hence ‹∀F (x[F] ≡ [λz 𝓐[F]z] ≈E  H)›
    using numbers[THEN ≡df E] &E by blast
  AOT-hence ‹x[F] ≡ [λz 𝓐[F]z] ≈E  H› for F
    using ∀E(2) by blast
  AOT-hence ‹[λz 𝓐[F]z] ≈E  G ≡ [λz 𝓐[F]z] ≈E  H› for F
    by (metis 1 ≡E(6))
  AOT-thus ‹G ≈E  H›
    using approx−nec:2[THEN ≡E(2), OF GEN] by blast
qed

AOT-theorem num−tran:3:
  ‹G ≡E  H → (Numbers(x, G) ≡ Numbers(x, H))›
  using apE−eqE:1 Hypothetical Syllogism num−tran:1 by blast

AOT-theorem pre−Hume:
  ‹(Numbers(x,G) & Numbers(y,H)) → (x = y ≡ G ≈E  H)›
proof(safe intro!: →I ≡I; frule &E(1); drule &E(2))
  AOT-assume ‹Numbers(x, G)›
  moreover AOT-assume ‹x = y›
  ultimately AOT-have ‹Numbers(y, G)› by (rule rule=E)
  moreover AOT-assume ‹Numbers(y, H)›
  ultimately AOT-show ‹G ≈E  H› using num−tran:2 →E &I by blast
next
  AOT-assume ‹Numbers(x, G)›
  AOT-hence Ax: ‹A!x› and xF: ‹∀F (x[F] ≡ [λz 𝓐[F]z] ≈E  G)›
    using numbers[THEN ≡df E] &E by blast+
  AOT-assume ‹Numbers(y, H)›
  AOT-hence Ay: ‹A!y› and yF: ‹∀F (y[F] ≡ [λz 𝓐[F]z] ≈E  H)›
    using numbers[THEN ≡df E] &E by blast+
  AOT-assume G-approx-H: ‹G ≈E  H›
  AOT-show ‹x = y›
  proof(rule ab−obey:1[THEN →E, THEN →E, OF &I, OF Ax, OF Ay]; rule GEN)
    fix F
    AOT-have ‹x[F] ≡ [λz 𝓐[F]z] ≈E  G›
      using xF[THEN ∀E(2)].
    also AOT-have ‹... ≡ [λz 𝓐[F]z] ≈E  H›
      using approx−nec:2[THEN ≡E(1), OF G-approx-H, THEN ∀E(2)].
    also AOT-have ‹... ≡ y[F]›
      using yF[THEN ∀E(2), symmetric].
    finally AOT-show ‹x[F] ≡ y[F]›.
  qed
qed

AOT-theorem two−num−not:
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  ‹∃ u∃ v(u ≠ v) → ∃ x∃ G∃ H(Numbers(x,G) & Numbers(x, H) & ¬G ≡E H)›
proof (rule →I)
  AOT-have eqE-den: ‹[λx x =E y]↓› for y by cqt:2
  AOT-assume ‹∃ u∃ v(u ≠ v)›
  then AOT-obtain c where Oc: ‹O!c› and ‹∃ v (c ≠ v)›
    using &E ∃ E[rotated] by blast
  then AOT-obtain d where Od: ‹O!d› and c-noteq-d: ‹c ≠ d›
    using &E ∃ E[rotated] by blast
  AOT-hence c-noteqE-d: ‹c ≠E d›
    using =E−simple:2[THEN →E] =E−simple:2 ≡E(2) modus−tollens:1
          =−infix ≡df E thm−neg=E by fast
  AOT-hence not-c-eqE-d: ‹¬c =E d›
    using ≡E(1) thm−neg=E by blast
  AOT-have ‹∃ x (A!x & ∀ F (x[F] ≡ [λz 𝓐[F]z] ≈E [λx x =E c]))›
    by (simp add: A−objects[axiom-inst])
  then AOT-obtain a where a-prop: ‹A!a & ∀ F (a[F] ≡ [λz 𝓐[F]z] ≈E [λx x =E c])›
    using ∃ E[rotated] by blast
  AOT-have ‹∃ x (A!x & ∀ F (x[F] ≡ [λz 𝓐[F]z] ≈E [λx x =E d]))›
    by (simp add: A−objects vdash−properties:1[2])
  then AOT-obtain b where b-prop: ‹A!b & ∀ F (b[F] ≡ [λz 𝓐[F]z] ≈E [λx x =E d])›
    using ∃ E[rotated] by blast
  AOT-have num-a-eq-c: ‹Numbers(a, [λx x =E c])›
    by (safe intro!: numbers[THEN ≡df I] &I a-prop[THEN &E(1)]
                  a-prop[THEN &E(2)]) cqt:2
  moreover AOT-have num-b-eq-d: ‹Numbers(b, [λx x =E d])›
    by (safe intro!: numbers[THEN ≡df I] &I b-prop[THEN &E(1)]
                  b-prop[THEN &E(2)]) cqt:2
  moreover AOT-have ‹[λx x =E c] ≈E [λx x =E d]›
  proof (rule equi:3[THEN ≡df I])
    let ?R = ‹«[λxy (x =E c & y =E d)]»›
    AOT-have Rcd: ‹[«?R»]cd›
      by (auto intro!: β←C(1) cqt:2 &I prod-denotesI
                  ord=Eequiv:1[THEN →E] Od Oc)
    AOT-show ‹∃ R R |: [λx x =E c] 1−1⟷E [λx x =E d]›
    proof (safe intro!: ∃ I(1)[where τ=‹?R›] equi:2[THEN ≡df I] &I
                  eqE-den Ordinary.GEN →I)
      AOT-show ‹«?R»↓› by cqt:2
    next
      fix u
      AOT-assume ‹[λx x =E c]u›
      AOT-hence ‹u =E c›
        by (metis β→C(1))
      AOT-hence u-is-c: ‹u = c›
        by (metis =E−simple:2 →E)
      AOT-show ‹∃ !v ([λx x =E d]v & [«?R»]uv)›
      proof (safe intro!: equi:1[THEN ≡E(2)] ∃ I(2)[where β=d] &I
                  Od Ordinary.GEN →I)
        AOT-show ‹[λx x =E d]d›
          by (auto intro!: β←C(1) cqt:2 ord=Eequiv:1[THEN →E, OF Od])
      next
        AOT-show ‹[«?R»]ud›
          using u-is-c[symmetric] Rcd rule=E by fast
      next
        fix v
        AOT-assume ‹[λx x =E d]v & [«?R»]uv›
        AOT-thus ‹v =E d›
          by (metis β→C(1) &E(1))
      qed
    next
      fix v
      AOT-assume ‹[λx x =E d]v›
      AOT-hence ‹v =E d›
        by (metis β→C(1))
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      AOT-hence v-is-d: ‹v = d›
        by (metis =E−simple:2 →E)
      AOT-show ‹∃!u ([λx x =E c]u & [«?R»]uv)›
      proof (safe intro!: equi:1[THEN ≡E(2)] ∃I(2)[where β=c] &I
                          Oc Ordinary.GEN →I)
        AOT-show ‹[λx x =E c]c›
          by (auto intro!: β←C(1) cqt:2 ord=Eequiv:1[THEN →E, OF Oc])
      next
        AOT-show ‹[«?R»]cv›
          using v-is-d[symmetric] Rcd rule=E by fast
      next
        fix u
        AOT-assume ‹[λx x =E c]u & [«?R»]uv›
        AOT-thus ‹u =E c›
          by (metis β→C(1) &E(1))
      qed
    next
      AOT-show ‹«?R»↓›
        by cqt:2
    qed
  qed
  ultimately AOT-have ‹a = b›
    using pre−Hume[unvarify G H, OF eqE-den, OF eqE-den, THEN →E,
                   OF &I, THEN ≡E(2)] by blast
  AOT-hence num-a-eq-d: ‹Numbers(a, [λx x =E d])›
    using num-b-eq-d rule=E id-sym by fast
  AOT-have not-equiv: ‹¬[λx x =E c] ≡E [λx x =E d]›
  proof (rule raa−cor:2)
    AOT-assume ‹[λx x =E c] ≡E [λx x =E d]›
    AOT-hence ‹[λx x =E c]c ≡ [λx x =E d]c›
      using eqE[THEN ≡df E, THEN &E(2), THEN ∀E(2), THEN →E] Oc by blast
    moreover AOT-have ‹[λx x =E c]c›
      by (auto intro!: β←C(1) cqt:2 ord=Eequiv:1[THEN →E, OF Oc])
    ultimately AOT-have ‹[λx x =E d]c›
      using ≡E(1) by blast
    AOT-hence ‹c =E d›
      by (rule β→C(1))
    AOT-thus ‹c =E d & ¬c =E d›
      using not-c-eqE-d &I by blast
  qed
  AOT-show ‹∃x ∃G ∃H (Numbers(x,G) & Numbers(x,H) & ¬G ≡E H)›
    apply (rule ∃I(2)[where β=a])
    apply (rule ∃I(1)[where τ=‹«[λx x =E c]»›])
     apply (rule ∃I(1)[where τ=‹«[λx x =E d]»›])
    by (safe intro!: eqE-den &I num-a-eq-c num-a-eq-d not-equiv)
qed

AOT-theorem num:1: ‹∃x Numbers(x,G)›
  by (AOT-subst ‹Numbers(x,G)› ‹[A!]x & ∀F (x[F] ≡ [λz 𝓐[F]z] ≈E G)› for: x)
     (auto simp: numbers[den][THEN →E, OF cqt:2[const-var][axiom-inst]]
                 A−objects[axiom-inst])

AOT-theorem num:2: ‹∃!x Numbers(x,G)›
  by (AOT-subst ‹Numbers(x,G)› ‹[A!]x & ∀F (x[F] ≡ [λz 𝓐[F]z] ≈E G)› for: x)
     (auto simp: numbers[den][THEN →E, OF cqt:2[const-var][axiom-inst]]
                 A−objects!)

AOT-theorem num−cont:1:
  ‹∃x∃G(Numbers(x, G) & ¬□Numbers(x, G))›
proof −
  AOT-have ‹∃F∃G ◇([λz 𝓐[F]z] ≈E G & ◇¬[λz 𝓐[F]z] ≈E G)›
    using approx−cont:2.
  then AOT-obtain F where ‹∃G ◇([λz 𝓐[F]z] ≈E G & ◇¬[λz 𝓐[F]z] ≈E G)›
```

using $\exists E[rotated]$ **by** *blast*
then AOT-obtain G **where** $‹\Diamond([\lambda z\ \boldsymbol{\mathcal{A}}[F]z] \approx_E G\ \&\ \Diamond\neg[\lambda z\ \boldsymbol{\mathcal{A}}[F]z] \approx_E G)›$
using $\exists E[rotated]$ **by** *blast*
AOT-hence ϑ: $‹\Diamond[\lambda z\ \boldsymbol{\mathcal{A}}[F]z] \approx_E G›$ **and** ζ: $‹\Diamond\neg[\lambda z\ \boldsymbol{\mathcal{A}}[F]z] \approx_E G›$
using $KBasic2{:}3[THEN \rightarrow E]\ \&E\ 4\Diamond[THEN \rightarrow E]$ **by** *blast+*
AOT-obtain a **where** $‹Numbers(a,\ G)›$
using $num{:}1\ \exists E[rotated]$ **by** *blast*
moreover AOT-have $‹\neg\Box Numbers(a,\ G)›$
proof $(rule\ raa{-}cor{:}2)$
AOT-assume $‹\Box Numbers(a,\ G)›$
AOT-hence $‹\Box([A!]a\ \&\ G{\downarrow}\ \&\ \forall F\ (a[F] \equiv [\lambda z\ \boldsymbol{\mathcal{A}}[F]z] \approx_E G))›$
by $(AOT\text{-}subst\text{-}def\ (reverse)\ numbers)$
AOT-hence $‹\Box A!a›$ **and** $‹\Box\forall F\ (a[F] \equiv [\lambda z\ \boldsymbol{\mathcal{A}}[F]z] \approx_E G)›$
using $KBasic{:}3[THEN \equiv E(1)]\ \&E$ **by** *blast+*
AOT-hence $‹\forall F\ \Box(a[F] \equiv [\lambda z\ \boldsymbol{\mathcal{A}}[F]z] \approx_E G)›$
using $CBF[THEN \rightarrow E]$ **by** *blast*
AOT-hence $‹\Box(a[F] \equiv [\lambda z\ \boldsymbol{\mathcal{A}}[F]z] \approx_E G)›$
using $\forall E(2)$ **by** *blast*
AOT-hence A: $‹\Box(a[F] \rightarrow [\lambda z\ \boldsymbol{\mathcal{A}}[F]z] \approx_E G)›$
and B: $‹\Box([\lambda z\ \boldsymbol{\mathcal{A}}[F]z] \approx_E G \rightarrow a[F])›$
using $KBasic{:}4[THEN \equiv E(1)]\ \&E$ **by** *blast+*
AOT-have $‹\Box(\neg[\lambda z\ \boldsymbol{\mathcal{A}}[F]z] \approx_E G \rightarrow \neg a[F])›$
apply $(AOT\text{-}subst\ ‹\neg[\lambda z\ \boldsymbol{\mathcal{A}}[F]z] \approx_E G \rightarrow \neg a[F]›\ ‹a[F] \rightarrow [\lambda z\ \boldsymbol{\mathcal{A}}[F]z] \approx_E G›)$
using $\equiv I\ useful{-}tautologies{:}4\ useful{-}tautologies{:}5$ **apply** *presburger*
by $(fact\ A)$
AOT-hence $‹\Diamond\neg a[F]›$
by $(metis\ KBasic{:}13\ \zeta \rightarrow E)$
AOT-hence $‹\neg a[F]›$
by $(metis\ KBasic{:}11\ en{-}eq{:}2[1] \equiv E(2) \equiv E(4))$
AOT-hence $‹\neg\Diamond a[F]›$
by $(metis\ en{-}eq{:}3[1] \equiv E(4))$
moreover AOT-have $‹\Diamond a[F]›$
by $(meson\ B\ \vartheta\ KBasic{:}13 \rightarrow E)$
ultimately AOT-show $‹\Diamond a[F]\ \&\ \neg\Diamond a[F]›$
using $\&I$ **by** *blast*
qed

ultimately AOT-have $‹Numbers(a,\ G)\ \&\ \neg\Box Numbers(a,\ G)›$
using $\&I$ **by** *blast*
AOT-hence $‹\exists G\ (Numbers(a,\ G)\ \&\ \neg\Box Numbers(a,\ G))›$
by $(rule\ \exists I)$
AOT-thus $‹\exists x \exists G\ (Numbers(x,\ G)\ \&\ \neg\Box Numbers(x,\ G))›$
by $(rule\ \exists I)$
qed

AOT-theorem $num{-}cont{:}2$:
$‹Rigid(G) \rightarrow \Box\forall x(Numbers(x,G) \rightarrow \Box Numbers(x,G))›$
proof$(rule \rightarrow I)$
AOT-assume $‹Rigid(G)›$
AOT-hence $‹\Box\forall z([G]z \rightarrow \Box[G]z)›$
using $df{-}rigid{-}rel{:}1[THEN \equiv_{df} E,\ THEN\ \&E(2)]$ **by** *blast*
AOT-hence $‹\Box\Box\forall z([G]z \rightarrow \Box[G]z)›$ **by** $(metis\ S5Basic{:}6 \equiv E(1))$
moreover AOT-have $‹\Box\Box\forall z([G]z \rightarrow \Box[G]z) \rightarrow \Box\forall x(Numbers(x,G) \rightarrow \Box Numbers(x,G))›$
proof$(rule\ RM;\ safe\ intro!{:} \rightarrow I\ GEN)$
AOT-modally-strict {
AOT-have *act-den*: $‹[\lambda z\ \boldsymbol{\mathcal{A}}[F]z]{\downarrow}›$ **for** F **by** $cqt{:}2[lambda]$
fix x
AOT-assume *G-nec*: $‹\Box\forall z([G]z \rightarrow \Box[G]z)›$
AOT-hence *G-rigid*: $‹Rigid(G)›$
using $df{-}rigid{-}rel{:}1[THEN \equiv_{df} I,\ OF\ \&I]\ cqt{:}2$
by *blast*
AOT-assume $‹Numbers(x,\ G)›$
AOT-hence $‹[A!]x\ \&\ G{\downarrow}\ \&\ \forall F\ (x[F] \equiv [\lambda z\ \boldsymbol{\mathcal{A}}[F]z] \approx_E G)›$

```
      using numbers[THEN ≡df E] by blast
    AOT-hence Ax: ‹[A!]x› and ‹∀ F (x[F] ≡ [λz 𝓐[F]z] ≈E G)›
      using &E by blast+
    AOT-hence ‹x[F] ≡ [λz 𝓐[F]z] ≈E G› for F
      using ∀ E(2) by blast
    moreover AOT-have ‹□([λz 𝓐[F]z] ≈E G → □[λz 𝓐[F]z] ≈E G)› for F
      using approx−nec:3[unvarify F, OF act-den, THEN →E, OF &I,
                         OF actuallyF:2, OF G-rigid].
    moreover AOT-have ‹□(x[F] → □x[F])› for F
      by (simp add: RN pre−en−eq:1[1])
    ultimately AOT-have ‹□(x[F] ≡ [λz 𝓐[F]z] ≈E G)› for F
      using sc−eq−box−box:5 →E qml:2[axiom-inst] &I by meson
    AOT-hence ‹∀ F □(x[F] ≡ [λz 𝓐[F]z] ≈E G)›
      by (rule ∀ I)
    AOT-hence 1: ‹□∀ F (x[F] ≡ [λz 𝓐[F]z] ≈E G)›
      using BF[THEN →E] by fast
    AOT-have ‹□G↓›
      by (simp add: ex:2:a)
    moreover AOT-have ‹□[A!]x›
      using Ax oa−facts:2 →E by blast
    ultimately AOT-have ‹□(A!x & G↓)›
      by (metis KBasic:3 &I ≡E(2))
    AOT-hence ‹□(A!x & G↓ & ∀ F (x[F] ≡ [λz 𝓐[F]z] ≈E G))›
      using 1 KBasic:3 &I ≡E(2) by fast
    AOT-thus ‹□Numbers(x, G)›
      by (AOT-subst-def numbers)
  }
  qed
  ultimately AOT-show ‹□∀ x(Numbers(x,G) → □Numbers(x,G))›
    using →E by blast
qed

AOT-theorem num−cont:3:
  ‹□∀ x(Numbers(x, [λz 𝓐[G]z]) → □Numbers(x, [λz 𝓐[G]z]))›
  by (rule num−cont:2[unvarify G, THEN →E];
      (cqt:2[lambda] | rule actuallyF:2))

AOT-theorem num−uniq: ‹ιx Numbers(x, G)↓›
  using ≡E(2) A−Exists:2 RA[2] num:2 by blast

AOT-define num :: ‹τ ⇒ κs› (‹#-› [100] 100)
  num−def:1: ‹#G =df ιx Numbers(x, G)›

AOT-theorem num−def:2: ‹#G↓›
  using num−def:1[THEN =df I(1)] num−uniq by simp

AOT-theorem num−can:1:
  ‹#G = ιx(A!x & ∀ F (x[F] ≡ [λz 𝓐[F]z] ≈E G))›
proof −
  AOT-have ‹□∀ x(Numbers(x,G) ≡ [A!]x & ∀ F (x[F] ≡ [λz 𝓐[F]z] ≈E G))›
    by (safe intro!: RN GEN numbers[den][THEN →E] cqt:2)
  AOT-hence ‹ιx Numbers(x, G) = ιx([A!]x & ∀ F (x[F] ≡ [λz 𝓐[F]z] ≈E G))›
    using num−uniq equiv−desc−eq:3[THEN →E, OF &I] by auto
  thus ?thesis
    by (rule =df I(1)[OF num−def:1, OF num−uniq])
qed

AOT-theorem num−can:2: ‹#G = ιx(A!x & ∀ F (x[F] ≡ F ≈E G))›
proof (rule id-trans[OF num−can:1]; rule equiv−desc−eq:2[THEN →E];
       safe intro!: &I A−descriptions GEN Act−Basic:5[THEN ≡E(2)]
                    logic−actual−nec:3[axiom-inst, THEN ≡E(2)])
  AOT-have act-den: ‹⊢□ [λz 𝓐[F]z]↓› for F
    by cqt:2
```

```
  AOT-have eq−part:3[terms]: ‹⊢□ F ≈E G & F ≈E H → G ≈E H› for F G H
    by (metis &I eq−part:2 eq−part:3 →I &E →E)
  fix x
  {
    fix F
    AOT-have ‹𝒜(F ≈E [λz 𝒜[F]z])›
      by (simp add: actuallyF:1)
    moreover AOT-have ‹𝒜((F ≈E [λz 𝒜[F]z]) → ([λz 𝒜[F]z] ≈E G ≡ F ≈E G))›
      by (auto intro!: RA[2] →I ≡I
              simp: eq−part:3[unvarify G, OF act-den, THEN →E, OF &I]
                    eq−part:3[terms][unvarify G, OF act-den, THEN →E, OF &I])
    ultimately AOT-have ‹𝒜([λz 𝒜[F]z] ≈E G ≡ F ≈E G)›
      using logic−actual−nec:2[axiom-inst, THEN ≡E(1), THEN →E] by blast

    AOT-hence ‹𝒜[λz 𝒜[F]z] ≈E G ≡ 𝒜F ≈E G›
      by (metis Act−Basic:5 ≡E(1))
    AOT-hence 0: ‹(𝒜x[F] ≡ 𝒜[λz 𝒜[F]z] ≈E G) ≡ (𝒜x[F] ≡ 𝒜F ≈E G)›
      by (auto intro!: ≡I →I elim: ≡E)
    AOT-have ‹𝒜(x[F] ≡ [λz 𝒜[F]z] ≈E G) ≡ (𝒜x[F] ≡ 𝒜[λz 𝒜[F]z] ≈E G)›
      by (simp add: Act−Basic:5)
    also AOT-have ‹... ≡ (𝒜x[F] ≡ 𝒜F ≈E G)› using 0.
    also AOT-have ‹... ≡ 𝒜((x[F] ≡ F ≈E G))›
      by (meson Act−Basic:5 ≡E(6) oth−class−taut:3:a)
    finally AOT-have 0: ‹𝒜(x[F] ≡ [λz 𝒜[F]z] ≈E G) ≡ 𝒜((x[F] ≡ F ≈E G))›.
  } note 0 = this
  AOT-have ‹𝒜∀F (x[F] ≡ [λz 𝒜[F]z] ≈E G) ≡ ∀F 𝒜(x[F] ≡ [λz 𝒜[F]z] ≈E G)›
    using logic−actual−nec:3 vdash−properties:1[2] by blast
  also AOT-have ‹... ≡  ∀F 𝒜((x[F] ≡ F ≈E G))›
    apply (safe intro!: ≡I →I GEN)
    using 0 ≡E(1) ≡E(2) rule−ui:3 by blast+
  also AOT-have ‹... ≡ 𝒜(∀F (x[F] ≡ F ≈E G))›
    using ≡E(6) logic−actual−nec:3[axiom-inst] oth−class−taut:3:a by fast
  finally AOT-have 0: ‹𝒜∀F (x[F] ≡ [λz 𝒜[F]z] ≈E G) ≡ 𝒜(∀F (x[F] ≡ F ≈E G))›.
  AOT-have ‹𝒜([A!]x & ∀F (x[F] ≡ [λz 𝒜[F]z] ≈E G)) ≡
          (𝒜A!x & 𝒜∀F (x[F] ≡ [λz 𝒜[F]z] ≈E G))›
    by (simp add: Act−Basic:2)
  also AOT-have ‹... ≡ 𝒜[A!]x & 𝒜(∀F (x[F] ≡ F ≈E G))›
    using 0 oth−class−taut:4:f →E by blast
  also AOT-have ‹... ≡ 𝒜(A!x & ∀F (x[F] ≡ F ≈E G))›
    using Act−Basic:2 ≡E(6) oth−class−taut:3:a by blast
  finally AOT-show ‹𝒜([A!]x & ∀F (x[F] ≡ [λz 𝒜[F]z] ≈E G)) ≡
                  𝒜([A!]x & ∀F (x[F] ≡ F ≈E G))›.
qed

AOT-define NaturalCardinal :: ‹τ ⇒ φ› (‹NaturalCardinal′(-′)›)
  card: ‹NaturalCardinal(x) ≡df ∃G(x = #G)›

AOT-theorem natcard−nec: ‹NaturalCardinal(x) → □NaturalCardinal(x)›
proof(rule →I)
  AOT-assume ‹NaturalCardinal(x)›
  AOT-hence ‹∃G(x = #G)› using card[THEN ≡dfE] by blast
  then AOT-obtain G where ‹x = #G› using ∃E[rotated] by blast
  AOT-hence ‹□x = #G› by (metis id−nec:2 →E)
  AOT-hence ‹∃G □x = #G› by (rule ∃I)
  AOT-hence ‹□∃G x = #G› by (metis Buridan →E)
  AOT-thus ‹□NaturalCardinal(x)›
    by (AOT-subst-def card)
qed

AOT-act-theorem hume:1: ‹Numbers(#G, G)›
  apply (rule =dfI(1)[OF num−def:1])
  apply (simp add: num−uniq)
  using num−uniq vdash−properties:10 y−in:3 by blast
```

```
AOT-act-theorem hume:2: ‹#F = #G ≡ F ≈E G›
  by (safe intro!: pre−Hume[unvarify x y, OF num−def:2,
                              OF num−def:2, THEN →E] &I hume:1)

AOT-act-theorem hume:3: ‹#F = #G ≡ ∃ R (R |: F 1−1⟶ontoE G)›
  using equi−rem−thm
  apply (AOT-subst (reverse) ‹R |: F 1−1⟶ontoE G›
                              ‹R |: F 1−1⟷E G› for: R :: ‹<κ×κ>›)
  using equi:3 hume:2 ≡E(5) ≡Df by blast

AOT-act-theorem hume:4: ‹F ≡E G → #F = #G›
  by (metis apE−eqE:1 deduction−theorem hume:2 ≡E(2) →E)

AOT-theorem hume−strict:1:
  ‹∃ x (Numbers(x, F) & Numbers(x, G)) ≡ F ≈E G›
proof(safe intro!: ≡I →I)
  AOT-assume ‹∃ x (Numbers(x, F) & Numbers(x, G))›
  then AOT-obtain a where ‹Numbers(a, F) & Numbers(a, G)›
    using ∃ E[rotated] by blast
  AOT-thus ‹F ≈E G›
    using num−tran:2 →E by blast
next
  AOT-assume 0: ‹F ≈E G›
  moreover AOT-obtain b where num-b-F: ‹Numbers(b, F)›
    by (metis instantiation num:1)
  moreover AOT-have num-b-G: ‹Numbers(b, G)›
    using calculation num−tran:1[THEN →E, THEN ≡E(1)] by blast
  ultimately AOT-have ‹Numbers(b, F) & Numbers(b, G)›
    by (safe intro!: &I)
  AOT-thus ‹∃ x (Numbers(x, F) & Numbers(x, G))›
    by (rule ∃ I)
qed

AOT-theorem hume−strict:2:
  ‹∃ x∃ y (Numbers(x, F) &
         ∀ z(Numbers(z,F) → z = x) &
         Numbers(y, G) &
         ∀ z (Numbers(z, G) → z = y) &
         x = y) ≡
   F ≈E G›
proof(safe intro!: ≡I →I)
  AOT-assume ‹∃ x∃ y (Numbers(x, F) & ∀ z(Numbers(z,F) → z = x) &
                   Numbers(y, G) & ∀ z (Numbers(z, G) → z = y) & x = y)›
  then AOT-obtain x where
    ‹∃ y (Numbers(x, F) & ∀ z(Numbers(z,F) → z = x) & Numbers(y, G) &
         ∀ z (Numbers(z, G) → z = y) & x = y)›
    using ∃ E[rotated] by blast
  then AOT-obtain y where
    ‹Numbers(x, F) & ∀ z(Numbers(z,F) → z = x) & Numbers(y, G) &
     ∀ z (Numbers(z, G) → z = y) & x = y›
    using ∃ E[rotated] by blast
  AOT-hence ‹Numbers(x, F)› and ‹Numbers(y,G)› and ‹x = y›
    using &E by blast+
  AOT-hence ‹Numbers(y, F) & Numbers(y, G)›
    using &I rule=E by fast
  AOT-hence ‹∃ y (Numbers(y, F) & Numbers(y, G))›
    by (rule ∃ I)
  AOT-thus ‹F ≈E G›
    using hume−strict:1[THEN ≡E(1)] by blast
next
  AOT-assume ‹F ≈E G›
  AOT-hence ‹∃ x (Numbers(x, F) & Numbers(x, G))›
```

```
    using hume−strict:1[THEN ≡E(2)] by blast
  then AOT-obtain x where ‹Numbers(x, F) & Numbers(x, G)›
    using ∃E[rotated] by blast
  moreover AOT-have ‹∀z (Numbers(z, F) → z = x)›
              and ‹∀z (Numbers(z, G) → z = x)›
    using calculation
    by (auto intro!: GEN →I pre−Hume[THEN →E, OF &I, THEN ≡E(2),
                              rotated 2, OF eq−part:1] dest: &E)
  ultimately AOT-have ‹Numbers(x, F) & ∀z(Numbers(z,F) → z = x) &
                  Numbers(x, G) & ∀z (Numbers(z, G) → z = x) & x = x›
    by (auto intro!: &I id−eq:1 dest: &E)
  AOT-thus ‹∃x∃y (Numbers(x, F) & ∀z(Numbers(z,F) → z = x) & Numbers(y, G) &
              ∀z (Numbers(z, G) → z = y) & x = y)›
    by (auto intro!: ∃I)
qed

AOT-theorem unotEu: ‹¬∃y[λx O!x & x ≠E x]y›
proof(rule raa−cor:2)
  AOT-assume ‹∃y[λx O!x & x ≠E x]y›
  then AOT-obtain y where ‹[λx O!x & x ≠E x]y›
    using ∃E[rotated] by blast
  AOT-hence 0: ‹O!y & y ≠E y›
    by (rule β→C(1))
  AOT-hence ‹¬(y =E y)›
    using &E(2) ≡E(1) thm−neg=E by blast
  moreover AOT-have ‹y =E y›
    by (metis 0[THEN &E(1)] ord=Eequiv:1 →E)
  ultimately AOT-show ‹p & ¬p› for p
    by (metis raa−cor:3)
qed

AOT-define zero :: ‹κs› (‹0›)
  zero:1: ‹0 =df #[λx O!x & x ≠E x]›

AOT-theorem zero:2: ‹0↓›
  by (rule =dfI(2)[OF zero:1]; rule num−def:2[unvarify G]; cqt:2)

AOT-theorem zero−card: ‹NaturalCardinal(0)›
  apply (rule =dfI(2)[OF zero:1])
   apply (rule num−def:2[unvarify G]; cqt:2)
  apply (rule card[THEN ≡dfI])
  apply (rule ∃I(1)[where τ=‹«[λx [O!]x & x ≠E x]»›])
   apply (rule rule=I:1; rule num−def:2[unvarify G]; cqt:2)
  by cqt:2

AOT-theorem eq−num:1:
  ‹𝓐Numbers(x, G) ≡ Numbers(x,[λz 𝓐[G]z])›
proof −
  AOT-have act-den: ‹⊢□ [λz 𝓐[F]z]↓› for F by cqt:2
  AOT-have ‹□(∃x(Numbers(x, G) & Numbers(x,[λz 𝓐[G]z])) ≡ G ≈E [λz 𝓐[G]z])›
    using hume−strict:1[unvarify G, OF act-den, THEN RN].
  AOT-hence ‹𝓐(∃x(Numbers(x, G) & Numbers(x,[λz 𝓐[G]z])) ≡ G ≈E [λz 𝓐[G]z])›
    using nec−imp−act[THEN →E] by fast
  AOT-hence ‹𝓐(∃x(Numbers(x, G) & Numbers(x,[λz 𝓐[G]z])))›
    using actuallyF:1 Act−Basic:5 ≡E(1) ≡E(2) by fast
  AOT-hence ‹∃x 𝓐((Numbers(x, G) & Numbers(x,[λz 𝓐[G]z])))›
    by (metis Act−Basic:10 intro−elim:3:a)
  then AOT-obtain a where ‹𝓐(Numbers(a, G) & Numbers(a,[λz 𝓐[G]z]))›
    using ∃E[rotated] by blast
  AOT-hence act-a-num-G: ‹𝓐Numbers(a, G)›
     and act-a-num-actG: ‹𝓐Numbers(a,[λz 𝓐[G]z])›
    using Act−Basic:2 &E ≡E(1) by blast+
  AOT-hence num-a-act-g: ‹Numbers(a, [λz 𝓐[G]z])›
```

using *num−cont*:*2*[*unvarify G*, *OF act-den*, *THEN* →*E*, *OF actuallyF*:*2*,
THEN CBF[*THEN* →*E*], *THEN* ∀ *E*(*2*)]
by (*metis* ≡*E*(*1*) *sc−eq−fur*:*2 vdash−properties*:*6*)
AOT-have *0*: ‹$\vdash_\Box$ *Numbers*(*x*, *G*) & *Numbers*(*y*, *G*) → *x* = *y*› **for** *y*
using *pre−Hume*[*THEN* →*E*, *THEN* ≡*E*(*2*), *rotated*, *OF eq−part*:*1*]
→*I* **by** *blast*
show *?thesis*
proof(*safe intro*!: ≡*I* →*I*)
AOT-assume ‹**𝒜***Numbers*(*x*, *G*)›
AOT-hence ‹**𝒜***x* = *a*›
using *0*[*THEN RA*[*2*], *THEN act−cond*[*THEN* →*E*], *THEN* →*E*,
OF Act−Basic:*2*[*THEN* ≡*E*(*2*)], *OF* &*I*]
act-a-num-G **by** *blast*
AOT-hence ‹*x* = *a*› **by** (*metis id−act*:*1* ≡*E*(*2*))
AOT-hence ‹*a* = *x*› **using** *id-sym* **by** *auto*
AOT-thus ‹*Numbers*(*x*, [λ*z* **𝒜**[*G*]*z*])›
using *rule*=*E num-a-act-g* **by** *fast*
next
AOT-assume ‹*Numbers*(*x*, [λ*z* **𝒜**[*G*]*z*])›
AOT-hence ‹*a* = *x*›
using *pre−Hume*[*unvarify G H*, *THEN* →*E*, *OF act-den*, *OF act-den*, *OF* &*I*,
OF num-a-act-g, *THEN* ≡*E*(*2*)]
eq−part:*1*[*unvarify F*, *OF act-den*] **by** *blast*
AOT-thus ‹**𝒜***Numbers*(*x*, *G*)›
using *act-a-num-G rule*=*E* **by** *fast*
qed
qed

AOT-theorem *eq−num*:*2*: ‹*Numbers*(*x*,[λ*z* **𝒜**[*G*]*z*]) ≡ *x* = #*G*›
proof −
AOT-have *0*: ‹$\vdash_\Box$ *x* = ι*x Numbers*(*x*, *G*) ≡ ∀ *y* (*Numbers*(*y*, [λ*z* **𝒜**[*G*]*z*]) ≡ *y* = *x*)› **for** *x*
by (*AOT-subst* (*reverse*) ‹*Numbers*(*x*, [λ*z* **𝒜**[*G*]*z*])› ‹**𝒜***Numbers*(*x*, *G*)› **for**: *x*)
(*auto simp*: *eq−num*:*1 descriptions*[*axiom-inst*])
AOT-have ‹#*G* = ι*x Numbers*(*x*, *G*) ≡ ∀ *y* (*Numbers*(*y*, [λ*z* **𝒜**[*G*]*z*]) ≡ *y* = #*G*)›
using *0*[*unvarify x*, *OF num−def*:*2*]**.**
moreover AOT-have ‹#*G* = ι*x Numbers*(*x*, *G*)›
using *num−def*:*1 num−uniq rule−id−df*:*1* **by** *blast*
ultimately AOT-have ‹∀ *y* (*Numbers*(*y*, [λ*z* **𝒜**[*G*]*z*]) ≡ *y* = #*G*)›
using ≡*E* **by** *blast*
thus *?thesis* **using** ∀ *E*(*2*) **by** *blast*
qed

AOT-theorem *eq−num*:*3*: ‹*Numbers*(#*G*, [λ*y* **𝒜**[*G*]*y*])›
proof −
AOT-have ‹#*G* = #*G*›
by (*simp add*: *rule*=*I*:*1 num−def*:*2*)
thus *?thesis*
using *eq−num*:*2*[*unvarify x*, *OF num−def*:*2*, *THEN* ≡*E*(*2*)] **by** *blast*
qed

AOT-theorem *eq−num*:*4*:
‹*A*!#*G* & ∀ *F* (#*G*[*F*] ≡ [λ*z* **𝒜**[*F*]*z*] $\approx_E$ [λ*z* **𝒜**[*G*]*z*])›
by (*auto intro*!: &*I eq−num*:*3*[*THEN numbers*[*THEN* $\equiv_{df}$*E*],
THEN &*E*(*1*), *THEN* &*E*(*1*)]
eq−num:*3*[*THEN numbers*[*THEN* $\equiv_{df}$*E*], *THEN* &*E*(*2*)])

AOT-theorem *eq−num*:*5*: ‹#*G*[*G*]›
by (*auto intro*!: *eq−num*:*4*[*THEN* &*E*(*2*), *THEN* ∀ *E*(*2*), *THEN* ≡*E*(*2*)]
eq−part:*1*[*unvarify F*] *simp*: *cqt*:*2*)

AOT-theorem *eq−num*:*6*: ‹*Numbers*(*x*, *G*) → *NaturalCardinal*(*x*)›
proof(*rule* →*I*)
AOT-have *act-den*: ‹$\vdash_\Box$ [λ*z* **𝒜**[*F*]*z*]↓› **for** *F*

by *cqt*:*2*
AOT-obtain *F* **where** ‹*Rigidifies*(*F*, *G*)›
by (*metis instantiation rigid−der*:*3*)
AOT-hence *ϑ*: ‹*Rigid*(*F*)› **and** ‹∀ *x*([*F*]*x* ≡ [*G*]*x*)›
using *df−rigid−rel*:*2*[*THEN* $\equiv_{df}$*E*, *THEN* &*E*(*2*)]
df−rigid−rel:*2*[*THEN* $\equiv_{df}$*E*, *THEN* &*E*(*1*)]
by *blast*+
AOT-hence ‹*F* $\equiv_E$ *G*›
by (*auto intro*!: *eqE*[*THEN* $\equiv_{df}$*I*] &*I cqt*:*2 GEN* →*I elim*: ∀ *E*(*2*))
moreover AOT-assume ‹*Numbers*(*x*, *G*)›
ultimately AOT-have ‹*Numbers*(*x*, *F*)›
using *num−tran*:*3*[*THEN* →*E*, *THEN* ≡*E*(*2*)] **by** *blast*
moreover AOT-have ‹*F* $\approx_E$ [λ*z* ***𝒜***[*F*]*z*]›
using *ϑ approx−nec*:*1* →*E* **by** *blast*
ultimately AOT-have ‹*Numbers*(*x*, [λ*z* ***𝒜***[*F*]*z*])›
using *num−tran*:*1*[*unvarify H*, *OF act-den*, *THEN* →*E*, *THEN* ≡*E*(*1*)] **by** *blast*
AOT-hence ‹*x* = #*F*›
using *eq−num*:*2*[*THEN* ≡*E*(*1*)] **by** *blast*
AOT-hence ‹∃ *F x* = #*F*›
by (*rule* ∃ *I*)
AOT-thus ‹*NaturalCardinal*(*x*)›
using *card*[*THEN* $\equiv_{df}$*I*] **by** *blast*
qed

AOT-theorem *eq−df−num*: ‹∃ *G* (*x* = #*G*) ≡ ∃ *G* (*Numbers*(*x*,*G*))›
proof(*safe intro*!: ≡*I* →*I*)
AOT-assume ‹∃ *G* (*x* = #*G*)›
then AOT-obtain *P* **where** ‹*x* = #*P*›
using ∃ *E*[*rotated*] **by** *blast*
AOT-hence ‹*Numbers*(*x*,[λ*z* ***𝒜***[*P*]*z*])›
using *eq−num*:*2*[*THEN* ≡*E*(*2*)] **by** *blast*
moreover AOT-have ‹[λ*z* ***𝒜***[*P*]*z*]↓› **by** *cqt*:*2*
ultimately AOT-show ‹∃ *G*(*Numbers*(*x*,*G*))› **by** (*rule* ∃ *I*)
next
AOT-assume ‹∃ *G* (*Numbers*(*x*,*G*))›
then AOT-obtain *Q* **where** ‹*Numbers*(*x*,*Q*)›
using ∃ *E*[*rotated*] **by** *blast*
AOT-hence ‹*NaturalCardinal*(*x*)›
using *eq−num*:*6*[*THEN* →*E*] **by** *blast*
AOT-thus ‹∃ *G* (*x* = #*G*)›
using *card*[*THEN* $\equiv_{df}$*E*] **by** *blast*
qed

AOT-theorem *card−en*: ‹*NaturalCardinal*(*x*) → ∀ *F*(*x*[*F*] ≡ *x* = #*F*)›
proof(*rule* →*I*; *rule GEN*)
AOT-have *act-den*: ‹$\vdash_\Box$ [λ*z* ***𝒜***[*F*]*z*]↓› **for** *F* **by** *cqt*:*2*
fix *F*
AOT-assume ‹*NaturalCardinal*(*x*)›
AOT-hence ‹∃ *F x* = #*F*›
using *card*[*THEN* $\equiv_{df}$*E*] **by** *blast*
then AOT-obtain *P* **where** *x-def*: ‹*x* = #*P*›
using ∃ *E*[*rotated*] **by** *blast*
AOT-hence *num-x-act-P*: ‹*Numbers*(*x*,[λ*z* ***𝒜***[*P*]*z*])›
using *eq−num*:*2*[*THEN* ≡*E*(*2*)] **by** *blast*
AOT-have ‹#*P*[*F*] ≡ [λ*z* ***𝒜***[*F*]*z*] $\approx_E$ [λ*z* ***𝒜***[*P*]*z*]›
using *eq−num*:*4*[*THEN* &*E*(*2*), *THEN* ∀ *E*(*2*)] **by** *blast*
AOT-hence ‹*x*[*F*] ≡ [λ*z* ***𝒜***[*F*]*z*] $\approx_E$ [λ*z* ***𝒜***[*P*]*z*]›
using *x-def*[*symmetric*] *rule*=*E* **by** *fast*
also AOT-have ‹. . . ≡ *Numbers*(*x*, [λ*z* ***𝒜***[*F*]*z*])›
using *num−tran*:*1*[*unvarify G H*, *OF act-den*, *OF act-den*]
using *num−tran*:*2*[*unvarify G H*, *OF act-den*, *OF act-den*]
by (*metis* &*I deduction−theorem* ≡*I* ≡*E*(*2*) *num-x-act-P*)
also AOT-have ‹. . . ≡ *x* = #*F*›

```
    using eq−num:2 by blast
  finally AOT-show ‹x[F] ≡ x = #F›.
qed

AOT-theorem 0F:1: ‹¬∃ u [F]u ≡ Numbers(0, F)›
proof −
  AOT-have unotEu-act-ord: ‹¬∃ v[λx O!x & 𝓐x ≠E x]v›
  proof(rule raa−cor:2)
    AOT-assume ‹∃ v[λx O!x & 𝓐x ≠E x]v›
    then AOT-obtain y where ‹[λx O!x & 𝓐x ≠E x]y›
      using ∃ E[rotated] &E by blast
    AOT-hence 0: ‹O!y & 𝓐y ≠E y›
      by (rule β→C(1))
    AOT-have ‹𝓐¬(y =E y)›
      apply (AOT-subst  ‹¬(y =E y)› ‹y ≠E y›)
       apply (meson ≡E(2) Commutativity of ≡ thm−neg=E)
      by (fact 0[THEN &E(2)])
    AOT-hence ‹¬(y =E y)›
      by (metis ¬¬I Act−Sub:1 id−act2:1 ≡E(4))
    moreover AOT-have ‹y =E y›
      by (metis 0[THEN &E(1)] ord=Eequiv:1 →E)
    ultimately AOT-show ‹p & ¬p› for p
      by (metis raa−cor:3)
  qed
  AOT-have ‹Numbers(0, [λy 𝓐[λx O!x & x ≠E x]y])›
    apply (rule =df I(2)[OF zero:1])
     apply (rule num−def:2[unvarify G]; cqt:2)
    apply (rule eq−num:3[unvarify G])
    by cqt:2[lambda]
  AOT-hence numbers0: ‹Numbers(0, [λx [O!]x & 𝓐x ≠E x])›
  proof (rule num−tran:3[unvarify x G H, THEN →E, THEN ≡E(1), rotated 4])
    AOT-show ‹[λy 𝓐[λx O!x & x ≠E x]y] ≡E [λx [O!]x & 𝓐x ≠E x]›
    proof (safe intro!: eqE[THEN ≡df I] &I Ordinary.GEN →I cqt:2)
      fix u
      AOT-have ‹[λy 𝓐[λx O!x & x ≠E x]y]u ≡ 𝓐[λx O!x & x ≠E x]u›
        by (rule beta−C−meta[THEN →E]; cqt:2[lambda])
      also AOT-have ‹. . . ≡ 𝓐(O!u & u ≠E u)›
        apply (AOT-subst ‹[λx O!x & x ≠E x]u› ‹O!u & u ≠E u›)
         apply (rule beta−C−meta[THEN →E]; cqt:2[lambda])
        by (simp add: oth−class−taut:3:a)
      also AOT-have ‹. . . ≡ (𝓐O!u & 𝓐u ≠E u)›
        by (simp add: Act−Basic:2)
      also AOT-have ‹. . . ≡ (O!u & 𝓐u ≠E u)›
        by (metis Ordinary.ψ &I &E(2) →I ≡I ≡E(1) oa−facts:7)
      also AOT-have ‹. . . ≡ [λx [O!]x & 𝓐x ≠E x]u›
        by (rule beta−C−meta[THEN →E, symmetric]; cqt:2[lambda])
      finally AOT-show ‹[λy 𝓐[λx O!x & x ≠E x]y]u ≡ [λx [O!]x & 𝓐x ≠E x]u›.
    qed
  qed(fact zero:2 | cqt:2)+
  show ?thesis
  proof(safe intro!: ≡I →I)
    AOT-assume ‹¬∃ u [F]u›
    moreover AOT-have ‹¬∃ v [λx [O!]x & 𝓐x ≠E x]v›
      using unotEu-act-ord.
    ultimately AOT-have 0: ‹F ≈E [λx [O!]x & 𝓐x ≠E x]›
      by (rule empty−approx:1[unvarify H, THEN →E, rotated, OF &I]) cqt:2
    AOT-thus ‹Numbers(0, F)›
      by (rule num−tran:1[unvarify x H, THEN →E,
                          THEN ≡E(2), rotated, rotated])
         (fact zero:2 numbers0 | cqt:2[lambda])+
  next
    AOT-assume ‹Numbers(0, F)›
    AOT-hence 1: ‹F ≈E [λx [O!]x & 𝓐x ≠E x]›
```

by (*rule num−tran*:*2*[*unvarify x H*, *THEN* →*E*, *rotated 2*, *OF* &*I*])
(*fact numbers0 zero*:*2* | *cqt*:*2*[*lambda*])+
AOT-show ‹¬∃ *u* [*F*]*u*›
proof(*rule raa−cor*:*2*)
AOT-have *0*: ‹[λ*x* [*O*!]*x* & 𝓐*x* ≠*E* *x*]↓› **by** *cqt*:*2*[*lambda*]
AOT-assume ‹∃ *u* [*F*]*u*›
AOT-hence ‹¬(*F* ≈*E* [λ*x* [*O*!]*x* & 𝓐*x* ≠*E* *x*])›
by (*rule empty−approx*:*2*[*unvarify H*, *OF 0*, *THEN* →*E*, *OF* &*I*])
(*rule unotEu-act-ord*)
AOT-thus ‹*F* ≈*E* [λ*x* [*O*!]*x* & 𝓐*x* ≠*E* *x*] & ¬(*F* ≈*E* [λ*x* [*O*!]*x* & 𝓐*x* ≠*E* *x*])›
using *1* &*I* **by** *blast*
qed
qed
qed

AOT-theorem *0F*:*2*: ‹¬∃ *u* 𝓐[*F*]*u* ≡ #*F* = *0*›
proof(*rule* ≡*I*; *rule* →*I*)
AOT-assume *0*: ‹¬∃ *u* 𝓐[*F*]*u*›
AOT-have ‹¬∃ *u* [λ*z* 𝓐[*F*]*z*]*u*›
proof(*rule raa−cor*:*2*)
AOT-assume ‹∃ *u* [λ*z* 𝓐[*F*]*z*]*u*›
then AOT-obtain *u* **where** ‹[λ*z* 𝓐[*F*]*z*]*u*›
using *Ordinary*.∃ *E*[*rotated*] **by** *blast*
AOT-hence ‹𝓐[*F*]*u*›
by (*metis betaC*:*1*:*a*)
AOT-hence ‹∃ *u* 𝓐[*F*]*u*›
by (*rule Ordinary*.∃ *I*)
AOT-thus ‹∃ *u* 𝓐[*F*]*u* & ¬∃ *u* 𝓐[*F*]*u*›
using *0* &*I* **by** *blast*
qed
AOT-hence ‹*Numbers*(*0*,[λ*z* 𝓐[*F*]*z*])›
by (*safe intro*!: *0F*:*1*[*unvarify F*, *THEN* ≡*E*(*1*)]) *cqt*:*2*
AOT-hence ‹*0* = #*F*›
by (*rule eq−num*:*2*[*unvarify x*, *OF zero*:*2*, *THEN* ≡*E*(*1*)])
AOT-thus ‹#*F* = *0*› **using** *id-sym* **by** *blast*
next
AOT-assume ‹#*F* = *0*›
AOT-hence ‹*0* = #*F*› **using** *id-sym* **by** *blast*
AOT-hence ‹*Numbers*(*0*,[λ*z* 𝓐[*F*]*z*])›
by (*rule eq−num*:*2*[*unvarify x*, *OF zero*:*2*, *THEN* ≡*E*(*2*)])
AOT-hence *0*: ‹¬∃ *u* [λ*z* 𝓐[*F*]*z*]*u*›
by (*safe intro*!: *0F*:*1*[*unvarify F*, *THEN* ≡*E*(*2*)]) *cqt*:*2*
AOT-show ‹¬∃ *u* 𝓐[*F*]*u*›
proof(*rule raa−cor*:*2*)
AOT-assume ‹∃ *u* 𝓐[*F*]*u*›
then AOT-obtain *u* **where** ‹𝓐[*F*]*u*›
using *Ordinary*.∃ *E*[*rotated*] **by** *meson*
AOT-hence ‹[λ*z* 𝓐[*F*]*z*]*u*›
by (*auto intro*!: β←*C cqt*:*2*)
AOT-hence ‹∃ *u* [λ*z* 𝓐[*F*]*z*]*u*›
using *Ordinary*.∃ *I* **by** *blast*
AOT-thus ‹∃ *u* [λ*z* 𝓐[*F*]*z*]*u* & ¬∃ *u* [λ*z* 𝓐[*F*]*z*]*u*›
using &*I 0* **by** *blast*
qed
qed

AOT-theorem *0F*:*3*: ‹□¬∃ *u* [*F*]*u* → #*F* = *0*›
proof(*rule* →*I*)
AOT-assume ‹□¬∃ *u* [*F*]*u*›
AOT-hence *0*: ‹¬◇∃ *u* [*F*]*u*›
using *KBasic2*:*1* ≡*E*(*1*) **by** *blast*
AOT-have ‹¬∃ *u* [λ*z* 𝓐[*F*]*z*]*u*›
proof(*rule raa−cor*:*2*)

AOT-assume ‹∃ *u* [*λz* **𝒜**[*F*]*z*]*u*›
then AOT-obtain *u* **where** ‹[*λz* **𝒜**[*F*]*z*]*u*›
using *Ordinary.∃ E[rotated]* **by** *blast*
AOT-hence ‹**𝒜**[*F*]*u*›
by (*metis betaC:1:a*)
AOT-hence ‹◊[*F*]*u*›
by (*metis Act−Sub:3 →E*)
AOT-hence ‹∃ *u* ◊[*F*]*u*›
by (*rule Ordinary.∃ I*)
AOT-hence ‹◊∃ *u* [*F*]*u*›
using *Ordinary.res−var−bound−reas[CBF◊][THEN →E]* **by** *blast*
AOT-thus ‹◊∃ *u* [*F*]*u* & ¬◊∃ *u* [*F*]*u*›
using *0 &I* **by** *blast*
qed
AOT-hence ‹*Numbers*(*0*,[*λz* **𝒜**[*F*]*z*])›
by (*safe intro*!: *0F:1[unvarify F, THEN ≡E(1)]*) *cqt:2*
AOT-hence ‹*0* = #*F*›
by (*rule eq−num:2[unvarify x, OF zero:2, THEN ≡E(1)]*)
AOT-thus ‹#*F* = *0*› **using** *id-sym* **by** *blast*
qed

AOT-theorem *0F:4*: ‹*w* ⊨ ¬∃ *u* [*F*]*u* ≡ #[*F*]$_w$ = *0*›
proof (*rule rule−id−df:2:b[OF w−index,* **where** $\tau_1\tau_n$=(-,-), *simplified*])
AOT-show ‹[*λx*$_1$...*x*$_n$ *w* ⊨ [*F*]*x*$_1$...*x*$_n$]↓›
by (*simp add: w−rel:3*)
next
AOT-show ‹*w* ⊨ ¬∃ *u* [*F*]*u* ≡ #[*λx w* ⊨ [*F*]*x*] = *0*›
proof (*rule ≡I*; *rule →I*)
AOT-assume ‹*w* ⊨ ¬∃ *u* [*F*]*u*›
AOT-hence *0*: ‹¬*w* ⊨ ∃ *u* [*F*]*u*›
using *coherent:1[unvarify p, OF log−prop−prop:2, THEN ≡E(1)]* **by** *blast*
AOT-have ‹¬∃ *u* **𝒜**[*λx w* ⊨ [*F*]*x*]*u*›
proof(*rule raa−cor:2*)
AOT-assume ‹∃ *u* **𝒜**[*λx w* ⊨ [*F*]*x*]*u*›
then AOT-obtain *u* **where** ‹**𝒜**[*λx w* ⊨ [*F*]*x*]*u*›
using *Ordinary.∃ E[rotated]* **by** *meson*
AOT-hence ‹**𝒜***w* ⊨ [*F*]*u*›
by (*AOT-subst* (*reverse*) ‹*w* ⊨ [*F*]*u*› ‹[*λx w* ⊨ [*F*]*x*]*u*›;
safe intro!: *beta−C−meta[THEN →E] w−rel:1[THEN →E]*)
cqt:2
AOT-hence *1*: ‹*w* ⊨ [*F*]*u*›
using *rigid−truth−at:4[unvarify p, OF log−prop−prop:2, THEN ≡E(1)]*
by *blast*
AOT-have ‹□([*F*]*u* → ∃ *u* [*F*]*u*)›
using *Ordinary.∃ I →I RN* **by** *simp*
AOT-hence ‹*w* ⊨ ([*F*]*u* → ∃ *u* [*F*]*u*)›
using *fund:2[unvarify p, OF log−prop−prop:2, THEN ≡E(1)]*
PossibleWorld.∀ E **by** *fast*
AOT-hence ‹*w* ⊨ ∃ *u* [*F*]*u*›
using *1 conj−dist−w:2[unvarify p q, OF log−prop−prop:2,*
OF log−prop−prop:2, THEN ≡E(1),
THEN →E] **by** *blast*
AOT-thus ‹*w* ⊨ ∃ *u* [*F*]*u* & ¬*w* ⊨ ∃ *u* [*F*]*u*›
using *0 &I* **by** *blast*
qed
AOT-thus ‹#[*λx w* ⊨ [*F*]*x*] = *0*›
by (*safe intro*!: *0F:2[unvarify F, THEN ≡E(1)] w−rel:1[THEN →E]*)
cqt:2
next
AOT-assume ‹#[*λx w* ⊨ [*F*]*x*] = *0*›
AOT-hence *0*: ‹¬∃ *u* **𝒜**[*λx w* ⊨ [*F*]*x*]*u*›
by (*safe intro*!: *0F:2[unvarify F, THEN ≡E(2)] w−rel:1[THEN →E]*)
cqt:2

```
    AOT-have ‹¬w ⊨ ∃ u [F]u›
    proof (rule raa−cor:2)
      AOT-assume ‹w ⊨ ∃ u [F]u›
      AOT-hence ‹∃ x w ⊨ (O!x & [F]x)›
        using conj−dist−w:6[THEN ≡E(1)] by fast
      then AOT-obtain x where ‹w ⊨ (O!x & [F]x)›
        using ∃ E[rotated] by blast
      AOT-hence ‹w ⊨ O!x› and Fx-in-w: ‹w ⊨ [F]x›
        using conj−dist−w:1[unvarify p q] ≡E(1) log−prop−prop:2
              &E by blast+
      AOT-hence ‹◇O!x›
        using fund:1[unvarify p, OF log−prop−prop:2, THEN ≡E(2)]
              PossibleWorld.∃ I by simp
      AOT-hence ord-x: ‹O!x›
        using oa−facts:3[THEN →E] by blast
      AOT-have ‹𝓐w ⊨ [F]x›
        using rigid−truth−at:4[unvarify p, OF log−prop−prop:2, THEN ≡E(2)]
              Fx-in-w by blast
      AOT-hence ‹𝓐[λx w ⊨ [F]x]x›
        by (AOT-subst ‹[λx w ⊨ [F]x]x› ‹w ⊨ [F]x›;
            safe intro!: beta−C−meta[THEN →E] w−rel:1[THEN →E]) cqt:2
      AOT-hence ‹O!x & 𝓐[λx w ⊨ [F]x]x›
        using ord-x &I by blast
      AOT-hence ‹∃ x (O!x & 𝓐[λx w ⊨ [F]x]x)›
        using ∃ I by fast
      AOT-thus ‹∃ u (𝓐[λx w ⊨ [F]x]u) & ¬∃ u 𝓐[λx w ⊨ [F]x]u›
        using 0 &I by blast
    qed
    AOT-thus ‹w ⊨ ¬∃ u[F]u›
      using coherent:1[unvarify p, OF log−prop−prop:2, THEN ≡E(2)] by blast
  qed
qed

AOT-act-theorem zero=:1:
  ‹NaturalCardinal(x) → ∀ F (x[F] ≡ Numbers(x, F))›
proof(safe intro!: →I GEN)
  fix F
  AOT-assume ‹NaturalCardinal(x)›
  AOT-hence ‹∀ F (x[F] ≡ x = #F)›
    by (metis card−en →E)
  AOT-hence 1: ‹x[F] ≡ x = #F›
    using ∀ E(2) by blast
  AOT-have 2: ‹x[F] ≡ x = ιy(Numbers(y, F))›
    by (rule num−def:1[THEN =df E(1)])
       (auto simp: 1 num−uniq)
  AOT-have ‹x = ιy(Numbers(y, F)) → Numbers(x, F)›
    using y−in:1 by blast
  moreover AOT-have ‹Numbers(x, F) → x = ιy(Numbers(y, F))›
  proof(rule →I)
    AOT-assume 1: ‹Numbers(x, F)›
    moreover AOT-obtain z where z-prop: ‹∀ y (Numbers(y, F) → y = z)›
      using num:2[THEN uniqueness:1[THEN ≡df E]] ∃ E[rotated] &E by blast
    ultimately AOT-have ‹x = z›
      using ∀ E(2) →E by blast
    AOT-hence ‹∀ y (Numbers(y, F) → y = x)›
      using z-prop rule=E id-sym by fast
    AOT-thus ‹x = ιy(Numbers(y,F))›
      by (rule hintikka[THEN ≡E(2), OF &I, rotated])
         (fact 1)
  qed
  ultimately AOT-have ‹x = ιy(Numbers(y, F)) ≡ Numbers(x, F)›
    by (metis ≡I)
  AOT-thus ‹x[F] ≡ Numbers(x, F)›
```

using *2* **by** (*metis* ≡*E*(*5*))
qed

AOT-act-theorem *zero*=:*2*: ‹*0*[*F*] ≡ ¬∃ *u*[*F*]*u*›
proof −
AOT-have ‹*0*[*F*] ≡ *Numbers*(*0*, *F*)›
using *zero*=:*1*[*unvarify x*, *OF zero*:*2*, *THEN* →*E*,
OF zero−*card*, *THEN* ∀ *E*(*2*)]**.**
also AOT-have ‹. . . ≡ ¬∃ *u*[*F*]*u*›
using *0F*:*1*[*symmetric*]**.**
finally show *?thesis***.**
qed

AOT-act-theorem *zero*=:*3*: ‹¬∃ *u*[*F*]*u* ≡ #*F* = *0*›
proof −
AOT-have ‹¬∃ *u*[*F*]*u* ≡ *0*[*F*]› **using** *zero*=:*2*[*symmetric*]**.**
also AOT-have ‹. . . ≡ *0* = #*F*›
using *card*−*en*[*unvarify x*, *OF zero*:*2*, *THEN* →*E*,
OF zero−*card*, *THEN* ∀ *E*(*2*)]**.**
also AOT-have ‹. . . ≡ #*F* = *0*›
by (*simp add*: *deduction*−*theorem id-sym* ≡*I*)
finally show *?thesis***.**
qed

AOT-define *Hereditary* :: ‹τ ⇒ τ ⇒ φ› (‹*Hereditary*′(-,-′)›)
hered:*1*:
‹*Hereditary*(*F*, *R*) $\equiv_{df}$ *R*↓ & *F*↓ & ∀ *x*∀ *y*([*R*]*xy* → ([*F*]*x* → [*F*]*y*))›

AOT-theorem *hered*:*2*:
‹[λ*xy* ∀ *F*((∀ *z*([*R*]*xz* → [*F*]*z*) & *Hereditary*(*F*,*R*)) → [*F*]*y*)]↓›
by *cqt*:*2*[*lambda*]

AOT-define *StrongAncestral* :: ‹τ ⇒ Π› (‹-*›)
ances−*df*:
‹$R^* =_{df}$ [λ*xy* ∀ *F*((∀ *z*([*R*]*xz* → [*F*]*z*) & *Hereditary*(*F*,*R*)) → [*F*]*y*)]›

AOT-theorem *ances*:
‹[R^*]*xy* ≡ ∀ *F*((∀ *z*([*R*]*xz* → [*F*]*z*) & *Hereditary*(*F*,*R*)) → [*F*]*y*)›
apply (*rule* $=_{df}$*I*(*1*)[*OF ances*−*df*])
apply *cqt*:*2*[*lambda*]
apply (*rule beta*−*C*−*meta*[*THEN* →*E*, *OF hered*:*2*, *unvarify* $\nu_1\nu_n$,
where τ=‹(-,-)›, *simplified*])
by (*simp add*: &*I ex*:*1*:*a prod-denotesI rule*−*ui*:*3*)

AOT-theorem *anc*−*her*:*1*:
‹[*R*]*xy* → [R^*]*xy*›
proof (*safe intro*!: →*I ances*[*THEN* ≡*E*(*2*)] *GEN*)
fix *F*
AOT-assume ‹∀ *z* ([*R*]*xz* → [*F*]*z*) & *Hereditary*(*F*, *R*)›
AOT-hence ‹[*R*]*xy* → [*F*]*y*›
using ∀ *E*(*2*) &*E* **by** *blast*
moreover AOT-assume ‹[*R*]*xy*›
ultimately AOT-show ‹[*F*]*y*›
using →*E* **by** *blast*
qed

AOT-theorem *anc*−*her*:*2*:
‹([R^*]*xy* & ∀ *z*([*R*]*xz* → [*F*]*z*) & *Hereditary*(*F*,*R*)) → [*F*]*y*›
proof(*rule* →*I*; (*frule* &*E*(*1*); *drule* &*E*(*2*))+)
AOT-assume ‹[R^*]*xy*›
AOT-hence ‹(∀ *z*([*R*]*xz* → [*F*]*z*) & *Hereditary*(*F*,*R*)) → [*F*]*y*›
using *ances*[*THEN* ≡*E*(*1*)] ∀ *E*(*2*) **by** *blast*
moreover AOT-assume ‹∀ *z*([*R*]*xz* → [*F*]*z*)›

```
  moreover AOT-assume ‹Hereditary(F,R)›
  ultimately AOT-show ‹[F]y›
    using →E &I by blast
qed

AOT-theorem anc−her:3:
  ‹([F]x & [R*]xy & Hereditary(F, R)) → [F]y›
proof(rule →I; (frule &E(1); drule &E(2))+)
  AOT-assume 1: ‹[F]x›
  AOT-assume 2: ‹Hereditary(F, R)›
  AOT-hence 3: ‹∀ x ∀ y ([R]xy → ([F]x → [F]y))›
    using hered:1[THEN ≡df E] &E by blast
  AOT-have ‹∀ z ([R]xz → [F]z)›
  proof (rule GEN; rule →I)
    fix z
    AOT-assume ‹[R]xz›
    moreover AOT-have ‹[R]xz → ([F]x → [F]z)›
      using 3 ∀ E(2) by blast
    ultimately AOT-show ‹[F]z›
      using 1 →E by blast
  qed
  moreover AOT-assume ‹[R*]xy›
  ultimately AOT-show ‹[F]y›
    by (auto intro!: 2 anc−her:2[THEN →E] &I)
qed

AOT-theorem anc−her:4: ‹([R]xy & [R*]yz) → [R*]xz›
proof(rule →I; frule &E(1); drule &E(2))
  AOT-assume 0: ‹[R*]yz› and 1: ‹[R]xy›
  AOT-show ‹[R*]xz›
  proof(safe intro!: ances[THEN ≡E(2)] GEN &I →I;
                  frule &E(1); drule &E(2))
    fix F
    AOT-assume ‹∀ z ([R]xz → [F]z)›
    AOT-hence 1: ‹[F]y›
      using 1 ∀ E(2) →E by blast
    AOT-assume 2: ‹Hereditary(F,R)›
    AOT-show ‹[F]z›
      by (rule anc−her:3[THEN →E]; auto intro!: &I 1 2 0)
  qed
qed

AOT-theorem anc−her:5: ‹[R*]xy → ∃ z [R]zy›
proof (rule →I)
  AOT-have 0: ‹[λy ∃ x [R]xy]↓› by cqt:2
  AOT-assume 1: ‹[R*]xy›
  AOT-have ‹[λy∃ x [R]xy]y›
  proof(rule anc−her:2[unvarify F, OF 0, THEN →E];
        safe intro!: &I GEN →I hered:1[THEN ≡df I] cqt:2 0)
    AOT-show ‹[R*]xy› using 1.
  next
    fix z
    AOT-assume ‹[R]xz›
    AOT-hence ‹∃ x [R]xz› by (rule ∃ I)
    AOT-thus ‹[λy∃ x [R]xy]z›
      by (auto intro!: β←C(1) cqt:2)
  next
    fix x y
    AOT-assume ‹[R]xy›
    AOT-hence ‹∃ x [R]xy› by (rule ∃ I)
    AOT-thus ‹[λy ∃ x [R]xy]y›
      by (auto intro!: β←C(1) cqt:2)
  qed
```

```
  AOT-thus ‹∃ z [R]zy›
    by (rule β→C(1))
qed

AOT-theorem anc−her:6: ‹([R*]xy & [R*]yz) → [R*]xz›
proof (rule →I; frule &E(1); drule &E(2))
  AOT-assume ‹[R*]xy›
  AOT-hence ϑ: ‹∀ z ([R]xz → [F]z) & Hereditary(F,R) → [F]y› for F
    using ∀ E(2)  ances[THEN ≡E(1)] by blast
  AOT-assume ‹[R*]yz›
  AOT-hence ξ: ‹∀ z ([R]yz → [F]z) & Hereditary(F,R) → [F]z› for F
    using ∀ E(2) ances[THEN ≡E(1)] by blast
  AOT-show ‹[R*]xz›
  proof (rule ances[THEN ≡E(2)]; safe intro!: GEN →I)
    fix F
    AOT-assume ζ: ‹∀ z ([R]xz → [F]z) & Hereditary(F,R)›
    AOT-show ‹[F]z›
    proof (rule ξ[THEN →E, OF &I])
      AOT-show ‹Hereditary(F,R)›
        using ζ[THEN &E(2)].
    next
      AOT-show ‹∀ z ([R]yz → [F]z)›
      proof(rule GEN; rule →I)
        fix z
        AOT-assume ‹[R]yz›
        moreover AOT-have ‹[F]y›
          using ϑ[THEN →E, OF ζ].
        ultimately AOT-show ‹[F]z›
          using ζ[THEN &E(2), THEN hered:1[THEN ≡df E],
                THEN &E(2), THEN ∀ E(2), THEN ∀ E(2),
                THEN →E, THEN →E]
          by blast
      qed
    qed
  qed
qed

AOT-define OneToOne :: ‹τ ⇒ φ› (‹1−1′(-′)›)
  df−1−1:1: ‹1−1(R) ≡df R↓ & ∀ x∀ y∀ z([R]xz & [R]yz → x = y)›

AOT-define RigidOneToOne :: ‹τ ⇒ φ› (‹Rigid_1−1′(-′)›)
  df−1−1:2: ‹Rigid_1−1(R) ≡df 1−1(R) & Rigid(R)›

AOT-theorem df−1−1:3: ‹Rigid_1−1(R) → □1−1(R)›
proof(rule →I)
  AOT-assume ‹Rigid_1−1(R)›
  AOT-hence ‹1−1(R)› and RigidR: ‹Rigid(R)›
    using df−1−1:2[THEN ≡df E] &E by blast+
  AOT-hence 1: ‹[R]xz & [R]yz → x = y› for x y z
    using df−1−1:1[THEN ≡df E] &E(2) ∀ E(2) by blast
  AOT-have 1: ‹[R]xz & [R]yz → □x = y› for x y z
    by (AOT-subst (reverse) ‹□x = y›  ‹x = y›)
      (auto simp: 1 id−nec:2 ≡I qml:2[axiom-inst])
  AOT-have ‹□∀ x1...∀ xn ([R]x1...xn → □[R]x1...xn)›
    using df−rigid−rel:1[THEN ≡df E, OF RigidR] &E by blast
  AOT-hence ‹∀ x1...∀ xn □([R]x1...xn → □[R]x1...xn)›
    using CBF[THEN →E] by fast
  AOT-hence ‹∀ x1∀ x2 □([R]x1x2 → □[R]x1x2)›
    using tuple-forall[THEN ≡df E] by blast
  AOT-hence ‹□([R]xy → □[R]xy)› for x y
    using ∀ E(2) by blast
  AOT-hence ‹□(([R]xz → □[R]xz) & ([R]yz → □[R]yz))› for x y z
    by (metis KBasic:3 &I ≡E(3) raa−cor:3)
```

```
  moreover AOT-have ‹□(([R]xz → □[R]xz) & ([R]yz → □[R]yz)) →
                  □(([R]xz & [R]yz) → □([R]xz & [R]yz))› for x y z
    by (rule RM) (metis →I KBasic:3 &I &E(1) &E(2) ≡E(2) →E)
  ultimately AOT-have 2: ‹□(([R]xz & [R]yz) → □([R]xz & [R]yz))› for x y z
    using →E by blast
  AOT-hence 3: ‹□([R]xz & [R]yz → x = y)› for x y z
    using sc−eq−box−box:6[THEN →E, THEN →E, OF 2, OF 1] by blast
  AOT-hence 4: ‹□∀ x∀ y∀ z([R]xz & [R]yz → x = y)›
    by (safe intro!: GEN BF[THEN →E] 3)
  AOT-thus ‹□1−1(R)›
    by (AOT-subst-thm df−1−1:1[THEN ≡Df, THEN ≡S(1),
                        OF cqt:2[const-var][axiom-inst]])
qed

AOT-theorem df−1−1:4: ‹∀ R(Rigid1−1(R) → □Rigid1−1(R))›
proof(rule GEN;rule →I)
AOT-modally-strict {
  fix R
      AOT-assume 0: ‹Rigid1−1(R)›
      AOT-hence 1: ‹R↓›
        by (meson ≡df E &E(1) df−1−1:1 df−1−1:2)
      AOT-hence 2: ‹□R↓›
        using exist−nec →E by blast
      AOT-have 4: ‹□1−1(R)›
        using df−1−1:3[unvarify R, OF 1, THEN →E, OF 0].
      AOT-have ‹Rigid(R)›
        using 0 ≡df E[OF df−1−1:2] &E by blast
      AOT-hence ‹□∀ x1...∀ xn ([R]x1...xn → □[R]x1...xn)›
        using  df−rigid−rel:1[THEN ≡df E] &E by blast
      AOT-hence ‹□□∀ x1...∀ xn ([R]x1...xn → □[R]x1...xn)›
        by (metis S5Basic:6 ≡E(1))
      AOT-hence ‹□Rigid(R)›
        apply (AOT-subst-def df−rigid−rel:1)
        using 2 KBasic:3 ≡S(2) ≡E(2) by blast
      AOT-thus ‹□Rigid1−1(R)›
        apply (AOT-subst-def df−1−1:2)
        using 4 KBasic:3 ≡S(2) ≡E(2) by blast
}
qed

AOT-define InDomainOf :: ‹τ ⇒ τ ⇒ φ› (‹InDomainOf′(-,-′)›)
  df−1−1:5: ‹InDomainOf(x, R) ≡df ∃ y [R]xy›

AOT-register-rigid-restricted-type
  RigidOneToOneRelation: ‹Rigid1−1(Π)›
proof
  AOT-modally-strict {
    AOT-show ‹∃ α Rigid1−1(α)›
    proof (rule ∃ I(1)[where τ=‹«(=E)»›])
      AOT-show ‹Rigid1−1((=E))›
      proof (safe intro!: df−1−1:2[THEN ≡df I] &I df−1−1:1[THEN ≡df I]
                      GEN →I df−rigid−rel:1[THEN ≡df I] =E[denotes])
        fix x y z
        AOT-assume ‹x =E z & y =E z›
        AOT-thus ‹x = y›
          by (metis rule=E &E(1) Conjunction Simplification(2)
                  =E−simple:2 id-sym →E)
      next
        AOT-have ‹∀ x∀ y □(x =E y → □x =E y)›
        proof(rule GEN; rule GEN)
          AOT-show ‹□(x =E y → □x =E y)› for x y
            by (meson RN deduction−theorem id−nec3:1 ≡E(1))
        qed
```

    AOT-hence ‹∀ x_1...∀ x_n □([(=$_E$)]x_1...x_n → □[(=$_E$)]x_1...x_n)›
      by (*rule tuple-forall*[*THEN* ≡$_{df}$*I*])
    AOT-thus ‹□∀ x_1...∀ x_n ([(=$_E$)]x_1...x_n → □[(=$_E$)]x_1...x_n)›
      using *BF*[*THEN* →*E*] **by** *fast*
   qed
  qed(*fact* =*E*[*denotes*])
 }
next
 AOT-modally-strict {
  AOT-show ‹*Rigid*$_{1-1}$(Π) → Π↓› **for** Π
  proof(*rule* →*I*)
   AOT-assume ‹*Rigid*$_{1-1}$(Π)›
   AOT-hence ‹*1*−*1*(Π)›
    using *df*−*1*−*1*:*2*[*THEN* ≡$_{df}$*E*] &*E* **by** *blast*
   AOT-thus ‹Π↓›
    using *df*−*1*−*1*:*1*[*THEN* ≡$_{df}$*E*] &*E* **by** *blast*
  qed
 }
next
 AOT-modally-strict {
  AOT-show ‹∀ *F*(*Rigid*$_{1-1}$(*F*) → □*Rigid*$_{1-1}$(*F*))›
   by (*safe intro*!: *GEN df*−*1*−*1*:*4*[*THEN* ∀ *E*(*2*)])
 }
qed
AOT-register-variable-names
 RigidOneToOneRelation: $\mathcal{R}$ $\mathcal{S}$

AOT-define *IdentityRestrictedToDomain* :: ‹τ ⇒ Π› (‹′(=-′)›)
 id−*d*−*R*: ‹(=$_\mathcal{R}$) =$_{df}$ [λ*xy* ∃ *z* ([$\mathcal{R}$]*xz* & [$\mathcal{R}$]*yz*)]›

syntax -*AOT-id-d-R-infix* :: ‹τ ⇒ τ ⇒ τ ⇒ φ› (‹(- =-/ -)› [*50*, *51*, *51*] *50*)
translations
 -*AOT-id-d-R-infix* κ Π κ′ ==
 CONST AOT-exe (*CONST IdentityRestrictedToDomain* Π) (κ,κ′)

AOT-theorem *id*−*R*−*thm*:*1*: ‹*x* =$_\mathcal{R}$ *y* ≡ ∃ *z* ([$\mathcal{R}$]*xz* & [$\mathcal{R}$]*yz*)›
proof −
 AOT-have *0*: ‹[λ*xy* ∃ *z* ([$\mathcal{R}$]*xz* & [$\mathcal{R}$]*yz*)]↓› **by** *cqt*:*2*
 show *?thesis*
  apply (*rule* =$_{df}$*I*(*1*)[*OF id*−*d*−*R*])
  apply (*fact 0*)
  apply (*rule beta*−*C*−*meta*[*THEN* →*E*, *OF 0*, *unvarify* $\nu_1\nu_n$,
               where τ=‹(-,-)›, *simplified*])
  by (*simp add*: &*I ex*:*1*:*a prod-denotesI rule*−*ui*:*3*)
qed

AOT-theorem *id*−*R*−*thm*:*2*:
 ‹*x* =$_\mathcal{R}$ *y* → (*InDomainOf*(*x*, $\mathcal{R}$) & *InDomainOf*(*y*, $\mathcal{R}$))›
proof(*rule* →*I*)
 AOT-assume ‹*x* =$_\mathcal{R}$ *y*›
 AOT-hence ‹∃ *z* ([$\mathcal{R}$]*xz* & [$\mathcal{R}$]*yz*)›
  using *id*−*R*−*thm*:*1*[*THEN* ≡*E*(*1*)] **by** *simp*
 then AOT-obtain *z* **where** *z-prop*: ‹[$\mathcal{R}$]*xz* & [$\mathcal{R}$]*yz*›
  using ∃ *E*[*rotated*] **by** *blast*
 AOT-show ‹*InDomainOf*(*x*, $\mathcal{R}$) & *InDomainOf*(*y*, $\mathcal{R}$)›
 proof (*safe intro*!: &*I df*−*1*−*1*:*5*[*THEN* ≡$_{df}$*I*])
  AOT-show ‹∃ *y* [$\mathcal{R}$]*xy*›
   using *z-prop*[*THEN* &*E*(*1*)] ∃ *I* **by** *fast*
 next
  AOT-show ‹∃ *z* [$\mathcal{R}$]*yz*›
   using *z-prop*[*THEN* &*E*(*2*)] ∃ *I* **by** *fast*
 qed
qed

AOT-theorem *id−R−thm:3*: ‹$x =_{\mathcal{R}} y \rightarrow x = y$›
proof(*rule* →*I*)
  AOT-assume ‹$x =_{\mathcal{R}} y$›
  AOT-hence ‹$\exists z\ ([\mathcal{R}]xz\ \&\ [\mathcal{R}]yz)$›
    using *id−R−thm:1*[*THEN* ≡*E*(*1*)] **by** *simp*
  then AOT-obtain *z* **where** *z-prop*: ‹$[\mathcal{R}]xz\ \&\ [\mathcal{R}]yz$›
    using ∃*E*[*rotated*] **by** *blast*
  AOT-thus ‹$x = y$›
    using *df−1−1:3*[*THEN* →*E*, *OF RigidOneToOneRelation.ψ*,
                    THEN qml:2[*axiom-inst*, *THEN* →*E*],
                    THEN ≡$_{df}$*E*[*OF df−1−1:1*], *THEN* &*E*(*2*),
                    THEN ∀*E*(*2*), *THEN* ∀*E*(*2*),
                    THEN ∀*E*(*2*), *THEN* →*E*]
      by *blast*
qed

AOT-theorem *id−R−thm:4*:
  ‹$(InDomainOf(x, \mathcal{R}) \vee InDomainOf(y, \mathcal{R})) \rightarrow (x =_{\mathcal{R}} y \equiv x = y)$›
proof (*rule* →*I*)
  AOT-assume ‹$InDomainOf(x, \mathcal{R}) \vee InDomainOf(y, \mathcal{R})$›
  moreover {
    AOT-assume ‹$InDomainOf(x, \mathcal{R})$›
    AOT-hence ‹$\exists z\ [\mathcal{R}]xz$›
      by (*metis* ≡$_{df}$*E df−1−1:5*)
    then AOT-obtain *z* **where** *z-prop*: ‹$[\mathcal{R}]xz$›
      using ∃*E*[*rotated*] **by** *blast*
    AOT-have ‹$x =_{\mathcal{R}} y \equiv x = y$›
    proof(*safe intro*!: ≡*I* →*I id−R−thm:3*[*THEN* →*E*])
      AOT-assume ‹$x = y$›
      AOT-hence ‹$[\mathcal{R}]yz$›
        using *z-prop rule*=*E* **by** *fast*
      AOT-hence ‹$[\mathcal{R}]xz\ \&\ [\mathcal{R}]yz$›
        using *z-prop* &*I* **by** *blast*
      AOT-hence ‹$\exists z\ ([\mathcal{R}]xz\ \&\ [\mathcal{R}]yz)$›
        by (*rule* ∃*I*)
      AOT-thus ‹$x =_{\mathcal{R}} y$›
        using *id−R−thm:1* ≡*E*(*2*) **by** *blast*
    qed
  }
  moreover {
    AOT-assume ‹$InDomainOf(y, \mathcal{R})$›
    AOT-hence ‹$\exists z\ [\mathcal{R}]yz$›
      by (*metis* ≡$_{df}$*E df−1−1:5*)
    then AOT-obtain *z* **where** *z-prop*: ‹$[\mathcal{R}]yz$›
      using ∃*E*[*rotated*] **by** *blast*
    AOT-have ‹$x =_{\mathcal{R}} y \equiv x = y$›
    proof(*safe intro*!: ≡*I* →*I id−R−thm:3*[*THEN* →*E*])
      AOT-assume ‹$x = y$›
      AOT-hence ‹$[\mathcal{R}]xz$›
        using *z-prop rule*=*E id-sym* **by** *fast*
      AOT-hence ‹$[\mathcal{R}]xz\ \&\ [\mathcal{R}]yz$›
        using *z-prop* &*I* **by** *blast*
      AOT-hence ‹$\exists z\ ([\mathcal{R}]xz\ \&\ [\mathcal{R}]yz)$›
        by (*rule* ∃*I*)
      AOT-thus ‹$x =_{\mathcal{R}} y$›
        using *id−R−thm:1* ≡*E*(*2*) **by** *blast*
    qed
  }
  ultimately AOT-show ‹$x =_{\mathcal{R}} y \equiv x = y$›
    by (*metis* ∨*E*(*2*) *raa−cor:1*)
qed

AOT-theorem *id−R−thm:5*: ‹*InDomainOf*(*x*, $\mathcal{R}$) → $x =_{\mathcal{R}} x$›
proof (*rule* →*I*)
  AOT-assume ‹*InDomainOf*(*x*, $\mathcal{R}$)›
  AOT-hence ‹∃ *z* [$\mathcal{R}$]*xz*›
    by (*metis* $\equiv_{df}E$ *df−1−1:5*)
  then AOT-obtain *z* **where** *z-prop*: ‹[$\mathcal{R}$]*xz*›
    using ∃ *E*[*rotated*] **by** *blast*
  AOT-hence ‹[$\mathcal{R}$]*xz* & [$\mathcal{R}$]*xz*›
    using &*I* **by** *blast*
  AOT-hence ‹∃ *z* ([$\mathcal{R}$]*xz* & [$\mathcal{R}$]*xz*)›
    using ∃ *I* **by** *fast*
  AOT-thus ‹$x =_{\mathcal{R}} x$›
    using *id−R−thm:1* ≡*E*(*2*) **by** *blast*
qed

AOT-theorem *id−R−thm:6*: ‹$x =_{\mathcal{R}} y \rightarrow y =_{\mathcal{R}} x$›
proof(*rule* →*I*)
  AOT-assume *0*: ‹$x =_{\mathcal{R}} y$›
  AOT-hence *1*: ‹*InDomainOf*(*x*,$\mathcal{R}$) & *InDomainOf*(*y*,$\mathcal{R}$)›
    using *id−R−thm:2*[*THEN* →*E*] **by** *blast*
  AOT-hence ‹$x =_{\mathcal{R}} y \equiv x = y$›
    using *id−R−thm:4*[*THEN* →*E*, *OF* ∨*I*(*1*)] &*E* **by** *blast*
  AOT-hence ‹*x* = *y*›
    using *0* **by** (*metis* ≡*E*(*1*))
  AOT-hence ‹*y* = *x*›
    using *id-sym* **by** *blast*
  moreover AOT-have ‹$y =_{\mathcal{R}} x \equiv y = x$›
    using *id−R−thm:4*[*THEN* →*E*, *OF* ∨*I*(*2*)] *1* &*E* **by** *blast*
  ultimately AOT-show ‹$y =_{\mathcal{R}} x$›
    by (*metis* ≡*E*(*2*))
qed

AOT-theorem *id−R−thm:7*: ‹$x =_{\mathcal{R}} y$ & $y =_{\mathcal{R}} z \rightarrow x =_{\mathcal{R}} z$›
proof (*rule* →*I*; *frule* &*E*(*1*); *drule* &*E*(*2*))
  AOT-assume *0*: ‹$x =_{\mathcal{R}} y$›
  AOT-hence *1*: ‹*InDomainOf*(*x*,$\mathcal{R}$) & *InDomainOf*(*y*,$\mathcal{R}$)›
    using *id−R−thm:2*[*THEN* →*E*] **by** *blast*
  AOT-hence ‹$x =_{\mathcal{R}} y \equiv x = y$›
    using *id−R−thm:4*[*THEN* →*E*, *OF* ∨*I*(*1*)] &*E* **by** *blast*
  AOT-hence *x-eq-y*: ‹*x* = *y*›
    using *0* **by** (*metis* ≡*E*(*1*))
  AOT-assume *2*: ‹$y =_{\mathcal{R}} z$›
  AOT-hence *3*: ‹*InDomainOf*(*y*,$\mathcal{R}$) & *InDomainOf*(*z*,$\mathcal{R}$)›
    using *id−R−thm:2*[*THEN* →*E*] **by** *blast*
  AOT-hence ‹$y =_{\mathcal{R}} z \equiv y = z$›
    using *id−R−thm:4*[*THEN* →*E*, *OF* ∨*I*(*1*)] &*E* **by** *blast*
  AOT-hence ‹*y* = *z*›
    using *2* **by** (*metis* ≡*E*(*1*))
  AOT-hence *x-eq-z*: ‹*x* = *z*›
    using *x-eq-y id-trans* **by** *blast*
  AOT-have ‹*InDomainOf*(*x*,$\mathcal{R}$) & *InDomainOf*(*z*,$\mathcal{R}$)›
    using *1 3* &*I* &*E* **by** *meson*
  AOT-hence ‹$x =_{\mathcal{R}} z \equiv x = z$›
    using *id−R−thm:4*[*THEN* →*E*, *OF* ∨*I*(*1*)] &*E* **by** *blast*
  AOT-thus ‹$x =_{\mathcal{R}} z$›
    using *x-eq-z* ≡*E*(*2*) **by** *blast*
qed

AOT-define *WeakAncestral* :: ‹Π ⇒ Π› (‹-$^{+}$›)
  w−ances−df: ‹$[\mathcal{R}]^{+} =_{df} [\lambda xy\ [\mathcal{R}]^{*}xy \vee x =_{\mathcal{R}} y]$›

AOT-theorem *w−ances−df*[*den1*]: ‹$[\lambda xy\ [\Pi]^{*}xy \vee x =_{\Pi} y]\downarrow$›
  by *cqt:2*

AOT-theorem *w−ances−df*[*den2*]: ‹$[\Pi]^+\downarrow$›
  using *w−ances−df*[*den1*] $=_{df}I(1)$[*OF w−ances−df*] **by** *blast*

AOT-theorem *w−ances*: ‹$[\mathcal{R}]^+xy \equiv ([\mathcal{R}]^*xy \vee x =_{\mathcal{R}} y)$›
proof −
  AOT-have *0*: ‹$[\lambda xy\ [\mathcal{R}^*]xy \vee x =_{\mathcal{R}} y]\downarrow$›
    by *cqt*:*2*
  AOT-have *1*: ‹«(*AOT-term-of-var x*,*AOT-term-of-var y*)»↓›
    by (*simp add*: &*I ex*:*1*:*a prod-denotesI rule−ui*:*3*)
  have *2*: ‹«$[\lambda\mu_1...\mu_n\ [\mathcal{R}^*]\mu_1...\mu_n \vee [(=_{\mathcal{R}})]\mu_1...\mu_n]xy$» =
      «$[\lambda xy\ [\mathcal{R}^*]xy \vee [(=_{\mathcal{R}})]xy]xy$»›
    by (*simp add*: *cond-case-prod-eta*)
  show *?thesis*
    apply (*rule* $=_{df}I(1)$[*OF w−ances−df*])
     apply (*fact w−ances−df*[*den1*])
    using *beta−C−meta*[*THEN* →*E*, *OF 0*, *unvarify* $\nu_1\nu_n$,
        where τ=‹(-,-)›, *simplified*, *OF 1*] *2* **by** *simp*
qed

AOT-theorem *w−ances−her*:*1*: ‹$[\mathcal{R}]xy \rightarrow [\mathcal{R}]^+xy$›
proof(*rule* →*I*)
  AOT-assume ‹$[\mathcal{R}]xy$›
  AOT-hence ‹$[\mathcal{R}]^*xy$›
    using *anc−her*:*1*[*THEN* →*E*] **by** *blast*
  AOT-thus ‹$[\mathcal{R}]^+xy$›
    using *w−ances*[*THEN* ≡*E*(*2*)] ∨*I* **by** *blast*
qed

AOT-theorem *w−ances−her*:*2*:
  ‹$[F]x$ & $[\mathcal{R}]^+xy$ & *Hereditary*(F, $\mathcal{R}$) → $[F]y$›
proof(*rule* →*I*; (*frule* &*E*(*1*); *drule* &*E*(*2*))+)
  AOT-assume *0*: ‹$[F]x$›
  AOT-assume *1*: ‹*Hereditary*(F, $\mathcal{R}$)›
  AOT-assume ‹$[\mathcal{R}]^+xy$›
  AOT-hence ‹$[\mathcal{R}]^*xy \vee x =_{\mathcal{R}} y$›
    using *w−ances*[*THEN* ≡*E*(*1*)] **by** *simp*
  moreover {
    AOT-assume ‹$[\mathcal{R}]^*xy$›
    AOT-hence ‹$[F]y$›
      using *anc−her*:*3*[*THEN* →*E*, *OF* &*I*, *OF* &*I*] *0 1* **by** *blast*
  }
  moreover {
    AOT-assume ‹$x =_{\mathcal{R}} y$›
    AOT-hence ‹$x = y$›
      using *id−R−thm*:*3*[*THEN* →*E*] **by** *blast*
    AOT-hence ‹$[F]y$›
      using *0 rule*=*E* **by** *blast*
  }
  ultimately AOT-show ‹$[F]y$›
    by (*metis* ∨*E*(*3*) *raa−cor*:*1*)
qed

AOT-theorem *w−ances−her*:*3*: ‹$([\mathcal{R}]^+xy$ & $[\mathcal{R}]yz) \rightarrow [\mathcal{R}]^*xz$›
proof(*rule* →*I*; *frule* &*E*(*1*); *drule* &*E*(*2*))
  AOT-assume ‹$[\mathcal{R}]^+xy$›
  moreover AOT-assume *Ryz*: ‹$[\mathcal{R}]yz$›
  ultimately AOT-have ‹$[\mathcal{R}]^*xy \vee x =_{\mathcal{R}} y$›
    using *w−ances*[*THEN* ≡*E*(*1*)] **by** *metis*
  moreover {
    AOT-assume *R-star-xy*: ‹$[\mathcal{R}]^*xy$›
    AOT-have ‹$[\mathcal{R}]^*xz$›
    proof (*safe intro*!: *ances*[*THEN* ≡*E*(*2*)] →*I GEN*)
      fix *F*

```
      AOT-assume 0: ‹∀z ([ℛ]xz → [F]z) & Hereditary(F,ℛ)›
      AOT-hence ‹[F]y›
        using R-star-xy ances[THEN ≡E(1), OF R-star-xy,
                              THEN ∀E(2), THEN →E] by blast
      AOT-thus ‹[F]z›
        using hered:1[THEN ≡df E, OF 0[THEN &E(2)], THEN &E(2)]
              ∀E(2) →E Ryz by blast
    qed
  }
  moreover {
    AOT-assume ‹x =ℛ y›
    AOT-hence ‹x = y›
      using id−R−thm:3[THEN →E] by blast
    AOT-hence ‹[ℛ]xz›
      using Ryz rule=E id-sym by fast
    AOT-hence ‹[ℛ]*xz›
      by (metis anc−her:1[THEN →E])
  }
  ultimately AOT-show ‹[ℛ]*xz›
    by (metis ∨E(3) raa−cor:1)
qed

AOT-theorem w−ances−her:4: ‹([ℛ]*xy & [ℛ]yz) → [ℛ]+xz›
proof(rule →I; frule &E(1); drule &E(2))
  AOT-assume ‹[ℛ]*xy›
  AOT-hence ‹[ℛ]*xy ∨ x =ℛ y›
    using ∨I by blast
  AOT-hence ‹[ℛ]+xy›
    using w−ances[THEN ≡E(2)] by blast
  moreover AOT-assume ‹[ℛ]yz›
  ultimately AOT-have ‹[ℛ]*xz›
    using w−ances−her:3[THEN →E, OF &I] by simp
  AOT-hence ‹[ℛ]*xz ∨ x =ℛ z›
    using ∨I by blast
  AOT-thus ‹[ℛ]+xz›
    using w−ances[THEN ≡E(2)] by blast
qed

AOT-theorem w−ances−her:5: ‹([ℛ]xy & [ℛ]+yz) → [ℛ]*xz›
proof(rule →I; frule &E(1); drule &E(2))
  AOT-assume 0: ‹[ℛ]xy›
  AOT-assume ‹[ℛ]+yz›
  AOT-hence ‹[ℛ]*yz ∨ y =ℛ z›
    by (metis ≡E(1) w−ances)
  moreover {
    AOT-assume ‹[ℛ]*yz›
    AOT-hence ‹[ℛ]*xz›
      using 0 by (metis anc−her:4 Adjunction →E)
  }
  moreover {
    AOT-assume ‹y =ℛ z›
    AOT-hence ‹y = z›
      by (metis id−R−thm:3 →E)
    AOT-hence ‹[ℛ]xz›
      using 0 rule=E by fast
    AOT-hence ‹[ℛ]*xz›
      by (metis anc−her:1 →E)
  }
  ultimately AOT-show ‹[ℛ]*xz› by (metis ∨E(2) reductio−aa:1)
qed

AOT-theorem w−ances−her:6: ‹([ℛ]+xy & [ℛ]+yz) → [ℛ]+xz›
proof(rule →I; frule &E(1); drule &E(2))
```

```
  AOT-assume 0: ‹[ℛ]^+xy›
  AOT-hence 1: ‹[ℛ]^*xy ∨ x =_ℛ y›
    by (metis ≡E(1) w−ances)
  AOT-assume 2: ‹[ℛ]^+yz›
  {
    AOT-assume ‹x =_ℛ y›
    AOT-hence ‹x = y›
      by (metis id−R−thm:3 →E)
    AOT-hence ‹[ℛ]^+xz›
      using 2 rule=E id-sym by fast
  }
  moreover {
    AOT-assume ‹¬(x =_ℛ y)›
    AOT-hence 3: ‹[ℛ]^*xy›
      using 1 by (metis ∨E(3))
    AOT-have ‹[ℛ]^*yz ∨ y =_ℛ z›
      using 2 by (metis ≡E(1) w−ances)
    moreover {
      AOT-assume ‹[ℛ]^*yz›
      AOT-hence ‹[ℛ]^*xz›
        using 3 by (metis anc−her:6 Adjunction →E)
      AOT-hence ‹[ℛ]^+xz›
        by (metis ∨I(1) ≡E(2) w−ances)
    }
    moreover {
      AOT-assume ‹y =_ℛ z›
      AOT-hence ‹y = z›
        by (metis id−R−thm:3 →E)
      AOT-hence ‹[ℛ]^+xz›
        using 0 rule=E id-sym by fast
    }
    ultimately AOT-have ‹[ℛ]^+xz›
      by (metis ∨E(3) reductio−aa:1)
  }
  ultimately AOT-show ‹[ℛ]^+xz›
    by (metis reductio−aa:1)
qed

AOT-theorem w−ances−her:7: ‹[ℛ]^*xy → ∃z([ℛ]^+xz & [ℛ]zy)›
proof(rule →I)
  AOT-assume 0: ‹[ℛ]^*xy›
  AOT-have 1: ‹∀z ([ℛ]xz → [Π]z) & Hereditary(Π,ℛ) → [Π]y› if ‹Π↓› for Π
    using ances[THEN ≡E(1), THEN ∀E(1), OF 0] that by blast
  AOT-have ‹[λy ∃z([ℛ]^+xz & [ℛ]zy)]y›
  proof (rule 1[THEN →E]; cqt:2[lambda]?;
          safe intro!: &I GEN →I hered:1[THEN ≡_df I] cqt:2)
    fix z
    AOT-assume 0: ‹[ℛ]xz›
    AOT-hence ‹∃z [ℛ]xz› by (rule ∃I)
    AOT-hence ‹InDomainOf(x, ℛ)› by (metis ≡_df I df−1−1:5)
    AOT-hence ‹x =_ℛ x› by (metis id−R−thm:5 →E)
    AOT-hence ‹[ℛ]^+xx› by (metis ∨I(2) ≡E(2) w−ances)
    AOT-hence ‹[ℛ]^+xx & [ℛ]xz› using 0 &I by blast
    AOT-hence ‹∃y ([ℛ]^+xy & [ℛ]yz)› by (rule ∃I)
    AOT-thus ‹[λy ∃z ([ℛ]^+xz & [ℛ]zy)]z›
      by (auto intro!: β←C(1) cqt:2)
  next
    fix x' y
    AOT-assume Rx'y: ‹[ℛ]x'y›
    AOT-assume ‹[λy ∃z ([ℛ]^+xz & [ℛ]zy)]x'›
    AOT-hence ‹∃z ([ℛ]^+xz & [ℛ]zx')›
      using β→C(1) by blast
    then AOT-obtain c where c-prop: ‹[ℛ]^+xc & [ℛ]cx'›
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      using ∃E[rotated] by blast
    AOT-hence ‹[ℛ]*xx'›
      by (meson Rx'y anc−her:1 anc−her:6 Adjunction →E w−ances−her:3)
    AOT-hence ‹[ℛ]*xx' ∨ x =ℛ x'› by (rule ∨I)
    AOT-hence ‹[ℛ]+xx'› by (metis ≡E(2) w−ances)
    AOT-hence ‹[ℛ]+xx' & [ℛ]x'y› using Rx'y by (metis &I)
    AOT-hence ‹∃z ([ℛ]+xz & [ℛ]zy)› by (rule ∃I)
    AOT-thus ‹[λy ∃z ([ℛ]+xz & [ℛ]zy)]y›
      by (auto intro!: β←C(1) cqt:2)
  qed
  AOT-thus ‹∃z([ℛ]+xz & [ℛ]zy)›
    using β→C(1) by fast
qed

AOT-theorem 1−1−R:1: ‹([ℛ]xy & [ℛ]*zy) → [ℛ]+zx›
proof(rule →I; frule &E(1); drule &E(2))
  AOT-assume ‹[ℛ]*zy›
  AOT-hence ‹∃x ([ℛ]+zx & [ℛ]xy)›
    using w−ances−her:7[THEN →E] by simp
  then AOT-obtain a where a-prop: ‹[ℛ]+za & [ℛ]ay›
    using ∃E[rotated] by blast
  moreover AOT-assume ‹[ℛ]xy›
  ultimately AOT-have ‹x = a›
    using df−1−1:2[THEN ≡dfE, OF RigidOneToOneRelation.ψ, THEN &E(1),
                   THEN ≡dfE[OF df−1−1:1], THEN &E(2), THEN ∀E(2),
                   THEN ∀E(2), THEN ∀E(2), THEN →E, OF &I]
    &E by blast
  AOT-thus ‹[ℛ]+zx›
    using a-prop[THEN &E(1)] rule=E id-sym by fast
qed

AOT-theorem 1−1−R:2: ‹[ℛ]xy → (¬[ℛ]*xx → ¬[ℛ]*yy)›
proof(rule →I; rule useful−tautologies:5[THEN →E]; rule →I)
  AOT-assume 0: ‹[ℛ]xy›
  moreover AOT-assume ‹[ℛ]*yy›
  ultimately AOT-have ‹[ℛ]+yx›
    using 1−1−R:1[THEN →E, OF &I] by blast
  AOT-thus ‹[ℛ]*xx›
    using 0 by (metis &I →E w−ances−her:5)
qed

AOT-theorem 1−1−R:3: ‹¬[ℛ]*xx → ([ℛ]+xy → ¬[ℛ]*yy)›
proof(safe intro!: →I)
  AOT-have 0: ‹[λz ¬[ℛ]*zz]↓› by cqt:2
  AOT-assume 1: ‹¬[ℛ]*xx›
  AOT-assume 2: ‹[ℛ]+xy›
  AOT-have ‹[λz ¬[ℛ]*zz]y›
  proof(rule w−ances−her:2[unvarify F, OF 0, THEN →E];
        safe intro!: &I hered:1[THEN ≡dfI] cqt:2 GEN →I)
    AOT-show  ‹[λz ¬[ℛ]*zz]x›
      by (auto intro!: β←C(1) cqt:2 simp: 1)
  next
    AOT-show ‹[ℛ]+xy› by (fact 2)
  next
    fix x y
    AOT-assume ‹[λz ¬[ℛ*]zz]x›
    AOT-hence ‹¬[ℛ]*xx› by (rule β→C(1))
    moreover AOT-assume ‹[ℛ]xy›
    ultimately AOT-have ‹¬[ℛ]*yy›
      using 1−1−R:2[THEN →E, THEN →E] by blast
    AOT-thus ‹[λz ¬[ℛ*]zz]y›
      by (auto intro!: β←C(1) cqt:2)
  qed
```

```
  AOT-thus ‹¬[ℛ]*yy›
    using β→C(1) by blast
qed

AOT-theorem 1−1−R:4: ‹[ℛ]*xy → InDomainOf(x,ℛ)›
proof(rule →I; rule df−1−1:5[THEN ≡df I])
  AOT-assume 1: ‹[ℛ]*xy›
  AOT-have ‹[λz [ℛ*]xz → ∃ y [ℛ]xy]y›
  proof (safe intro!: anc−her:2[unvarify F, THEN →E];
         safe intro!: cqt:2 &I GEN →I hered:1[THEN ≡df I])
    AOT-show ‹[ℛ]*xy› by (fact 1)
  next
    fix z
    AOT-assume ‹[ℛ]xz›
    AOT-thus ‹[λz [ℛ*]xz → ∃ y [ℛ]xy]z›
      by (safe intro!: β←C(1) cqt:2)
         (meson →I existential:2[const-var])
  next
    fix x′ y
    AOT-assume Rx′y: ‹[ℛ]x′y›
    AOT-assume ‹[λz [ℛ*]xz → ∃ y [ℛ]xy]x′›
    AOT-hence 0: ‹[ℛ*]xx′ → ∃ y [ℛ]xy› by (rule β→C(1))
    AOT-have 1: ‹[ℛ*]xy → ∃ y [ℛ]xy›
    proof(rule →I)
      AOT-assume ‹[ℛ]*xy›
      AOT-hence ‹[ℛ]+xx′› by (metis Rx′y &I 1−1−R:1 →E)
      AOT-hence ‹[ℛ]*xx′ ∨ x =ℛ x′› by (metis ≡E(1) w−ances)
      moreover {
        AOT-assume ‹[ℛ]*xx′›
        AOT-hence ‹∃ y [ℛ]xy› using 0 by (metis →E)
      }
      moreover {
        AOT-assume ‹x =ℛ x′›
        AOT-hence ‹x = x′› by (metis id−R−thm:3 →E)
        AOT-hence ‹[ℛ]xy› using Rx′y rule=E id-sym by fast
        AOT-hence ‹∃ y [ℛ]xy› by (rule ∃ I)
      }
      ultimately AOT-show ‹∃ y [ℛ]xy›
        by (metis ∨E(3) reductio−aa:1)
    qed
    AOT-show ‹[λz [ℛ*]xz → ∃ y [ℛ]xy]y›
      by (auto intro!: β←C(1) cqt:2 1)
  qed
  AOT-hence ‹[ℛ*]xy → ∃ y [ℛ]xy› by (rule β→C(1))
  AOT-thus ‹∃ y [ℛ]xy› using 1 →E by blast
qed

AOT-theorem 1−1−R:5: ‹[ℛ]+xy → InDomainOf(x,ℛ)›
proof (rule →I)
  AOT-assume ‹[ℛ]+xy›
  AOT-hence ‹[ℛ]*xy ∨ x =ℛ y›
    by (metis ≡E(1) w−ances)
  moreover {
    AOT-assume ‹[ℛ]*xy›
    AOT-hence ‹InDomainOf(x,ℛ)›
      using 1−1−R:4 →E by blast
  }
  moreover {
    AOT-assume ‹x =ℛ y›
    AOT-hence ‹InDomainOf(x,ℛ)›
      by (metis Conjunction Simplification(1) id−R−thm:2 →E)
  }
  ultimately AOT-show ‹InDomainOf(x,ℛ)›
```

by (*metis* ∨*E*(*3*) *reductio−aa*:*1*)
qed

AOT-theorem *pre−ind*:
‹([*F*]*z* & ∀ *x*∀ *y*(([ℛ]$^+$*zx* & [ℛ]$^+$*zy*) → ([ℛ]*xy* → ([*F*]*x* → [*F*]*y*)))) →
∀ *x* ([ℛ]$^+$*zx* → [*F*]*x*)›
proof(*safe intro*!: →*I GEN*)
AOT-have *den*: ‹[λ*y* [*F*]*y* & [ℛ]$^+$*zy*]↓› **by** *cqt*:*2*
fix *x*
AOT-assume *ϑ*: ‹[*F*]*z* & ∀ *x*∀ *y*(([ℛ]$^+$*zx* & [ℛ]$^+$*zy*) → ([ℛ]*xy* → ([*F*]*x* → [*F*]*y*)))›
AOT-assume *0*: ‹[ℛ]$^+$*zx*›

AOT-have ‹[λ*y* [*F*]*y* & [ℛ]$^+$*zy*]*x*›
proof (*rule w−ances−her*:*2*[*unvarify F*, *OF den*, *THEN* →*E*]; *safe intro*!: &*I*)
AOT-show ‹[λ*y* [*F*]*y* & [ℛ]$^+$*zy*]*z*›
proof (*safe intro*!: *β*←*C*(*1*) *cqt*:*2* &*I*)
AOT-show ‹[*F*]*z*› **using** *ϑ* &*E* **by** *blast*
next
AOT-show ‹[ℛ]$^+$*zz*›
by (*rule w−ances*[*THEN* ≡*E*(*2*), *OF* ∨*I*(*2*)])
(*meson 0 id−R−thm*:*5 1−1−R*:*5* →*E*)
qed
next
AOT-show ‹[ℛ]$^+$*zx*› **by** (*fact 0*)
next
AOT-show ‹*Hereditary*([λ*y* [*F*]*y* & [ℛ]$^+$*zy*],ℛ)›
proof (*safe intro*!: *hered*:*1*[*THEN* $\equiv_{df}$*I*] &*I cqt*:*2 GEN* →*I*)
fix *x*′ *y*
AOT-assume *1*: ‹[ℛ]*x*′*y*›
AOT-assume ‹[λ*y* [*F*]*y* & [ℛ]$^+$*zy*]*x*′›
AOT-hence *2*: ‹[*F*]*x*′ & [ℛ]$^+$*zx*′› **by** (*rule β*→*C*(*1*))
AOT-have ‹[ℛ]**zy*› **using** *1 2*[*THEN* &*E*(*2*)]
by (*metis Adjunction modus−tollens*:*1 reductio−aa*:*1 w−ances−her*:*3*)
AOT-hence *3*: ‹[ℛ]$^+$*zy*› **by** (*metis* ∨*I*(*1*) ≡*E*(*2*) *w−ances*)
AOT-show ‹[λ*y* [*F*]*y* & [ℛ]$^+$*zy*]*y*›
proof (*safe intro*!: *β*←*C*(*1*) *cqt*:*2* &*I 3*)
AOT-show ‹[*F*]*y*›
proof (*rule ϑ*[*THEN* &*E*(*2*), *THEN* ∀ *E*(*2*), *THEN* ∀ *E*(*2*),
THEN →*E*, *THEN* →*E*, *THEN* →*E*])
AOT-show ‹[ℛ]$^+$*zx*′ & [ℛ]$^+$*zy*›
using *2 3* &*E* &*I* **by** *blast*
next
AOT-show ‹[ℛ]*x*′*y*› **by** (*fact 1*)
next
AOT-show ‹[*F*]*x*′› **using** *2* &*E* **by** *blast*
qed
qed
qed
qed
AOT-thus ‹[*F*]*x*› **using** *β*→*C*(*1*) &*E*(*1*) **by** *fast*
qed

The following is not part of PLM, but a theorem of AOT. It states that the predecessor relation coexists with numbering a property. We will use this fact to derive the predecessor axiom, which asserts that the predecessor relation denotes, from the fact that our models validate that numbering a property denotes.

AOT-theorem *pred-coex*:
‹[λ*xy* ∃ *F*∃ *u* ([*F*]*u* & *Numbers*(*y*,*F*) & *Numbers*(*x*,[*F*]$^{-u}$))]↓ ≡ ∀ *F* ([λ*x Numbers*(*x*,*F*)]↓)›
proof(*safe intro*!: ≡*I* →*I GEN*)
fix *F*
let *?P* = ‹«[λ*xy* ∃ *F*∃ *u* ([*F*]*u* & *Numbers*(*y*,*F*) & *Numbers*(*x*,[*F*]$^{-u}$))]»›
AOT-assume ‹[«*?P*»]↓›
AOT-hence ‹□[«*?P*»]↓›
using *exist−nec* →*E* **by** *blast*

```
moreover AOT-have
 ‹□[«?P»]↓ → □(∀x∀y(∀F([F]x ≡ [F]y) → (Numbers(x,F) ≡ Numbers(y,F))))›
proof(rule RM; safe intro!: →I GEN)
  AOT-modally-strict {
    fix x y
    AOT-assume pred-den: ‹[«?P»]↓›
    AOT-hence pred-equiv:
      ‹[«?P»]xy ≡ ∃F∃u ([F]u & Numbers(y,F) & Numbers(x,[F]^-u))› for x y
      by (safe intro!: beta−C−meta[unvarify ν1νn, where τ=‹(-,-)›, THEN →E,
                                   rotated, OF pred-den, simplified]
                  tuple-denotes[THEN ≡df I] &I cqt:2)
```

We show as a subproof that any natural cardinal that is not zero has a predecessor.

```
    AOT-have CardinalPredecessor:
      ‹∃y [«?P»]yx› if card-x: ‹NaturalCardinal(x)› and x-nonzero: ‹x ≠ 0› for x
    proof −
      AOT-have ‹∃G x = #G›
        using card[THEN ≡df E, OF card-x].
      AOT-hence ‹∃G Numbers(x,G)›
        using eq−df−num[THEN ≡E(1)] by blast
      then AOT-obtain G' where numxG': ‹Numbers(x,G')›
        using ∃E[rotated] by blast
      AOT-obtain G where ‹Rigidifies(G,G')›
        using rigid−der:3 ∃E[rotated] by blast

      AOT-hence H: ‹Rigid(G) & ∀x ([G]x ≡ [G']x)›
        using df−rigid−rel:2[THEN ≡df E] by blast
      AOT-have H-rigid: ‹□∀x ([G]x → □[G]x)›
        using H[THEN &E(1), THEN df−rigid−rel:1[THEN ≡df E], THEN &E(2)].
      AOT-hence ‹∀x □([G]x → □[G]x)›
        using CBF →E by blast
      AOT-hence R: ‹□([G]x → □[G]x)› for x using ∀E(2) by blast
      AOT-hence rigid: ‹[G]x ≡ 𝒜[G]x› for x
         by (metis ≡E(6) oth−class−taut:3:a sc−eq−fur:2 →E)
      AOT-have ‹G ≡E G'›
      proof (safe intro!: eqE[THEN ≡df I] &I cqt:2 GEN →I)
        AOT-show ‹[G]x ≡ [G']x› for x using H[THEN &E(2)] ∀E(2) by fast
      qed
      AOT-hence ‹G ≈E G'›
        by (rule apE−eqE:2[THEN →E, OF &I, rotated])
           (simp add: eq−part:1)
      AOT-hence numxG: ‹Numbers(x,G)›
        using num−tran:1[THEN →E, THEN ≡E(2)] numxG' by blast

      {
        AOT-assume ‹¬∃y(y ≠ x & [«?P»]yx)›
        AOT-hence ‹∀y ¬(y ≠ x & [«?P»]yx)›
          using cqt−further:4 →E by blast
        AOT-hence ‹¬(y ≠ x & [«?P»]yx)› for y
          using ∀E(2) by blast
        AOT-hence 0: ‹¬y ≠ x ∨ ¬[«?P»]yx› for y
          using ¬¬E intro−elim:3:c oth−class−taut:5:a by blast
        {
          fix y
          AOT-assume ‹[«?P»]yx›
          AOT-hence ‹¬y ≠ x›
            using 0 ¬¬I con−dis−i−e:4:c by blast
          AOT-hence ‹y = x›
            using =−infix ≡df I raa−cor:4 by blast
        } note Pxy-imp-eq = this
        AOT-have ‹[«?P»]xx›
        proof(rule raa−cor:1)
          AOT-assume notPxx: ‹¬[«?P»]xx›
```

```
    AOT-hence ‹¬∃ F∃ u([F]u & Numbers(x,F) & Numbers(x,[F]^-u))›
      using pred-equiv intro−elim:3:c by blast
    AOT-hence ‹∀ F ¬∃ u([F]u & Numbers(x,F) & Numbers(x,[F]^-u))›
      using cqt−further:4[THEN →E] by blast
    AOT-hence ‹¬∃ u([F]u & Numbers(x,F) & Numbers(x,[F]^-u))› for F
      using ∀ E(2) by blast
    AOT-hence ‹∀ y ¬(O!y & ([F]y & Numbers(x,F) & Numbers(x,[F]^-y)))› for F
      using cqt−further:4[THEN →E] by blast
    AOT-hence 0: ‹¬(O!u & ([F]u & Numbers(x,F) & Numbers(x,[F]^-u)))› for F u
      using ∀ E(2) by blast
    AOT-have ‹□¬∃ u [G]u›
    proof(rule raa−cor:1)
      AOT-assume ‹¬□¬∃ u [G]u›
      AOT-hence ‹◇∃ u [G]u›
        using ≡df I conventions:5 by blast
      AOT-hence ‹∃ u ◇[G]u›
        by (metis Ordinary.res−var−bound−reas[BF◇][THEN →E])
      then AOT-obtain u where posGu: ‹◇[G]u›
        using Ordinary.∃ E[rotated] by meson
      AOT-hence Gu: ‹[G]u›
        by (meson B◇ K◇ →E R)
      AOT-have ‹¬([G]u & Numbers(x,G) & Numbers(x,[G]^-u))›
        using 0 Ordinary.ψ
        by (metis con−dis−i−e:1 raa−cor:1)
      AOT-hence notnumx: ‹¬Numbers(x,[G]^-u)›
        using Gu numxG con−dis−i−e:1 raa−cor:5 by metis
      AOT-obtain y where numy: ‹Numbers(y,[G]^-u)›
        using num:1[unvarify G, OF F−u[den]] ∃ E[rotated] by blast
      AOT-hence ‹[G]u & Numbers(x,G) & Numbers(y,[G]^-u)›
        using Gu numxG &I by blast
      AOT-hence ‹∃ u ([G]u & Numbers(x,G) & Numbers(y,[G]^-u))›
        by (rule Ordinary.∃ I)
      AOT-hence ‹∃ G∃ u ([G]u & Numbers(x,G) & Numbers(y,[G]^-u))›
        by (rule ∃ I)
      AOT-hence ‹[«?P»]yx›
        using pred-equiv[THEN ≡E(2)] by blast
      AOT-hence ‹y = x› using Pxy-imp-eq by blast
      AOT-hence ‹Numbers(x,[G]^-u)›
        using numy rule=E by fast
      AOT-thus ‹p & ¬p› for p using notnumx reductio−aa:1 by blast
    qed
    AOT-hence ‹¬∃ u [G]u›
      using qml:2[axiom-inst, THEN →E] by blast
    AOT-hence num0G: ‹Numbers(0, G)›
      using 0F:1[THEN ≡E(1)] by blast
    AOT-hence ‹x = 0›
      using pre−Hume[unvarify x, THEN →E, OF zero:2, OF &I,
                    THEN ≡E(2), OF num0G, OF numxG, OF eq−part:1]
        id-sym by blast
    moreover AOT-have ‹¬x = 0›
      using x-nonzero
      using =−infix ≡df E by blast
    ultimately AOT-show ‹p & ¬p› for p using reductio−aa:1 by blast
  qed
}
AOT-hence ‹[«?P»]xx ∨ ∃ y (y ≠ x & [«?P»]yx)›
  using con−dis−i−e:3:a con−dis−i−e:3:b raa−cor:1 by blast
moreover {
  AOT-assume ‹[«?P»]xx›
  AOT-hence ‹∃ y [«?P»]yx›
    by (rule ∃ I)
}
moreover {
```

AOT-assume ‹∃ *y* (*y* ≠ *x* & [«*?P*»]*yx*)›
then AOT-obtain *y* **where** ‹*y* ≠ *x* & [«*?P*»]*yx*›
using ∃ *E*[*rotated*] **by** *blast*
AOT-hence ‹[«*?P*»]*yx*›
using &*E* **by** *blast*
AOT-hence ‹∃ *y* [«*?P*»]*yx*›
by (*rule* ∃ *I*)
}
ultimately AOT-show ‹∃ *y* [«*?P*»]*yx*›
using ∨*E*(*1*) →*I* **by** *blast*
qed

Given above lemma, we can show that if one of two indistinguishable objects numbers a property, the other one numbers this property as well.

AOT-assume *indist*: ‹∀ *F*([*F*]*x* ≡ [*F*]*y*)›
AOT-assume *numxF*: ‹*Numbers*(*x*,*F*)›
AOT-hence *0*: ‹*NaturalCardinal*(*x*)›
by (*metis eq−num*:*6 vdash−properties*:*10*)

We show by case distinction that x equals y. As first case we consider x to be non-zero.

{
AOT-assume ‹¬(*x* = *0*)›
AOT-hence ‹*x* ≠ *0*›
by (*metis* =−*infix* $\equiv_{df}$*I*)
AOT-hence ‹∃ *y* [«*?P*»]*yx*›
using *CardinalPredecessor 0* **by** *blast*
then AOT-obtain *z* **where** *Pxz*: ‹[«*?P*»]*zx*›
using ∃ *E*[*rotated*] **by** *blast*
AOT-hence ‹[λ*y* [«*?P*»]*zy*]*x*›
by (*safe intro*!: *β*←*C cqt*:*2*)
AOT-hence ‹[λ*y* [«*?P*»]*zy*]*y*›
by (*safe intro*!: *indist*[*THEN* ∀ *E*(*1*), *THEN* ≡*E*(*1*)] *cqt*:*2*)
AOT-hence *Pyz*: ‹[«*?P*»]*zy*›
using *β*→*C*(*1*) **by** *blast*
AOT-hence ‹∃ *F*∃ *u* ([*F*]*u* & *Numbers*(*y*,*F*) & *Numbers*(*z*,$[F]^{-u}$))›
using *Pyz pred-equiv*[*THEN* ≡*E*(*1*)] **by** *blast*
then AOT-obtain F_1 **where** ‹∃ *u* ([F_1]*u* & *Numbers*(*y*,F_1) & *Numbers*(*z*,$[F_1]^{-u}$))›
using ∃ *E*[*rotated*] **by** *blast*
then AOT-obtain *u* **where** *u-prop*: ‹[F_1]*u* & *Numbers*(*y*,F_1) & *Numbers*(*z*,$[F_1]^{-u}$)›
using *Ordinary*.∃ *E*[*rotated*] **by** *meson*
AOT-have ‹∃ *F*∃ *u* ([*F*]*u* & *Numbers*(*x*,*F*) & *Numbers*(*z*,$[F]^{-u}$))›
using *Pxz pred-equiv*[*THEN* ≡*E*(*1*)] **by** *blast*
then AOT-obtain F_2 **where** ‹∃ *u* ([F_2]*u* & *Numbers*(*x*,F_2) & *Numbers*(*z*,$[F_2]^{-u}$))›
using ∃ *E*[*rotated*] **by** *blast*
then AOT-obtain *v* **where** *v-prop*: ‹[F_2]*v* & *Numbers*(*x*,F_2) & *Numbers*(*z*,$[F_2]^{-v}$)›
using *Ordinary*.∃ *E*[*rotated*] **by** *meson*
AOT-have ‹$[F_2]^{-v} \approx_E [F_1]^{-u}$›
using *hume*−*strict*:*1*[*unvarify F G*, *THEN* ≡*E*(*1*), *OF F*−*u*[*den*],
OF F−*u*[*den*], *OF* ∃ *I*(*2*)[**where** *β*=*z*], *OF* &*I*]
v-prop u-prop &*E* **by** *blast*
AOT-hence ‹$F_2 \approx_E F_1$›
using *P*′−*eq*[*THEN* →*E*, *OF* &*I*, *OF* &*I*]
u-prop v-prop &*E* **by** *meson*
AOT-hence ‹*x* = *y*›
using *pre*−*Hume*[*THEN* →*E*, *THEN* ≡*E*(*2*), *OF* &*I*]
v-prop u-prop &*E* **by** *blast*
}

The second case handles x being equal to zero.

moreover {
fix *u*
AOT-assume *x-is-zero*: ‹*x* = *0*›
moreover AOT-have ‹*Numbers*(*0*,$[\lambda z\ z =_E u]^{-u}$)›

```
    proof (safe intro!: 0F:1[unvarify F, THEN ≡E(1)] cqt:2 raa−cor:2
                  F−u[den][unvarify F])
      AOT-assume ‹∃ v [[λz z =E u]⁻ᵘ]v›
      then AOT-obtain v where ‹[[λz z =E u]⁻ᵘ]v›
        using Ordinary.∃ E[rotated] by meson
      AOT-hence ‹[λz z =E u]v & v ≠E u›
        by (auto intro: F−u[THEN =df E(1), where τ1τn=(-,-), simplified]
                 intro!: cqt:2 F−u[equiv][unvarify F, THEN ≡E(1)]
                   F−u[den][unvarify F])
      AOT-thus ‹p & ¬p› for p
        using β→C thm−neg=E[THEN ≡E(1)] &E &I
            raa−cor:3 by fast
    qed
    ultimately AOT-have 0: ‹Numbers(x,[λz z =E u]⁻ᵘ)›
      using rule=E id-sym by fast
    AOT-have ‹∃ y Numbers(y,[λz z =E u])›
      by (safe intro!: num:1[unvarify G] cqt:2)
    then AOT-obtain z where ‹Numbers(z,[λz z =E u])›
      using ∃ E by metis
    moreover AOT-have ‹[λz z=E u]u›
      by (safe intro!: β←C cqt:2 ord=Eequiv:1[THEN →E] Ordinary.ψ)
    ultimately AOT-have
      1: ‹[λz z=E u]u & Numbers(z,[λz z=E u]) & Numbers(x,[λz z=E u]⁻ᵘ)›
      using 0 &I by auto
    AOT-hence ‹∃ v([λz z=E u]v & Numbers(z,[λz z =E u]) & Numbers(x,[λz z=E u]⁻ᵛ))›
      by (rule Ordinary.∃ I)
    AOT-hence ‹∃ F∃ u([F]u & Numbers(z,[F]) & Numbers(x,[F]⁻ᵘ))›
      by (rule ∃ I; cqt:2)
    AOT-hence Px1: ‹[«?P»]xz›
      using beta−C−cor:2[THEN →E, OF pred-den,
            THEN tuple-forall[THEN ≡df E], THEN ∀ E(2),
            THEN ∀ E(2), THEN ≡E(2)] by simp
    AOT-hence ‹[λy [«?P»]yz]x›
      by (safe intro!: β←C cqt:2)
    AOT-hence ‹[λy [«?P»]yz]y›
      by (safe intro!: indist[THEN ∀ E(1), THEN ≡E(1)] cqt:2)
    AOT-hence Py1: ‹[«?P»]yz›
      using β→C by blast
    AOT-hence ‹∃ F∃ u([F]u & Numbers(z,[F]) & Numbers(y,[F]⁻ᵘ))›
      using β→C by fast
    then AOT-obtain G where ‹∃ u([G]u & Numbers(z,[G]) & Numbers(y,[G]⁻ᵘ))›
      using ∃ E[rotated] by blast
    then AOT-obtain v where 2: ‹[G]v & Numbers(z,[G]) & Numbers(y,[G]⁻ᵛ)›
      using Ordinary.∃ E[rotated] by meson
    with 1 2 AOT-have ‹[λz z =E u] ≈E G›
      by (auto intro!: hume−strict:1[unvarify F, THEN ≡E(1), rotated,
                      OF ∃ I(2)[where β=z], OF &I] cqt:2
            dest: &E)
    AOT-hence 3: ‹[λz z =E u]⁻ᵘ ≈E [G]⁻ᵛ›
      using 1 2
      by (safe-step intro!: eqP′[unvarify F, THEN →E])
        (auto dest: &E intro!: cqt:2 &I)
    with 1 2 AOT-have ‹x = y›
      by (auto intro!: pre−Hume[unvarify G H, THEN →E,
                          THEN ≡E(2), rotated 3, OF 3]
                 F−u[den][unvarify F] cqt:2 &I
            dest: &E)
  }
  ultimately AOT-have ‹x = y›
    using ∨E(1) →I reductio−aa:1 by blast
```

Now since x numbers F, so does y.

```
  AOT-hence ‹Numbers(y,F)›
```

        using *numxF rule=E* **by** *fast*
    } **note** *0 = this*

The only thing left is to generalize this result to a biconditional.

```
    AOT-modally-strict {
      fix x y
      AOT-assume ‹[«?P»]↓›
      moreover AOT-assume ‹∀F([F]x ≡ [F]y)›
      moreover AOT-have ‹∀F([F]y ≡ [F]x)›
        by (metis cqt−basic:11 intro−elim:3:a calculation(2))
      ultimately AOT-show ‹Numbers(x,F) ≡ Numbers(y,F)›
        using 0 ≡I →I by auto
    }
  qed
  ultimately AOT-show ‹[λx Numbers(x,F)]↓›
    using kirchner−thm:1[THEN ≡E(2)] →E by fast
next
```

The converse can be shown by coexistence.

```
  AOT-assume ‹∀F [λx Numbers(x,F)]↓›
  AOT-hence ‹[λx Numbers(x,F)]↓› for F
    using ∀E(2) by blast
  AOT-hence ‹□[λx Numbers(x,F)]↓› for F
    using exist−nec[THEN →E] by blast
  AOT-hence ‹∀F □[λx Numbers(x,F)]↓›
    by (rule GEN)
  AOT-hence ‹□∀F [λx Numbers(x,F)]↓›
    using BF[THEN →E] by fast
  moreover AOT-have
   ‹□∀F [λx Numbers(x,F)]↓ →
    □∀x ∀y (∃F ∃u ([F]u & [λz Numbers(z,F)]y & [λz Numbers(z,[F]^−u)]x) ≡
           ∃F ∃u ([F]u & Numbers(y,F) & Numbers(x,[F]^−u)))›
  proof(rule RM; safe intro!: →I GEN)
    AOT-modally-strict {
      fix x y
      AOT-assume 0: ‹∀F [λx Numbers(x,F)]↓›
      AOT-show ‹∃F ∃u ([F]u & [λz Numbers(z,F)]y & [λz Numbers(z,[F]^−u)]x) ≡
             ∃F ∃u ([F]u & Numbers(y,F) & Numbers(x,[F]^−u))›
      proof(safe intro!: ≡I →I)
        AOT-assume ‹∃F ∃u ([F]u & [λz Numbers(z,F)]y & [λz Numbers(z,[F]^−u)]x)›
        then AOT-obtain F where
          ‹∃u ([F]u & [λz Numbers(z,F)]y & [λz Numbers(z,[F]^−u)]x)›
          using ∃E[rotated] by blast
        then AOT-obtain u where ‹[F]u & [λz Numbers(z,F)]y & [λz Numbers(z,[F]^−u)]x›
          using Ordinary.∃E[rotated] by meson
        AOT-hence ‹[F]u & Numbers(y,F) & Numbers(x,[F]^−u)›
          by (auto intro!: &I dest: &E β→C)
        AOT-thus ‹∃F ∃u ([F]u & Numbers(y,F) & Numbers(x,[F]^−u))›
          using ∃I Ordinary.∃I by fast
      next
        AOT-assume ‹∃F ∃u ([F]u & Numbers(y,F) & Numbers(x,[F]^−u))›
        then AOT-obtain F where ‹∃u ([F]u & Numbers(y,F) & Numbers(x,[F]^−u))›
          using ∃E[rotated] by blast
        then AOT-obtain u where ‹[F]u & Numbers(y,F) & Numbers(x,[F]^−u)›
          using Ordinary.∃E[rotated] by meson
        AOT-hence ‹[F]u & [λz Numbers(z,F)]y & [λz Numbers(z,[F]^−u)]x›
          by (auto intro!: &I β←C 0[THEN ∀E(1)] F−u[den]
                  dest: &E intro: cqt:2)
        AOT-hence ‹∃u([F]u & [λz Numbers(z,F)]y & [λz Numbers(z,[F]^−u)]x)›
          by (rule Ordinary.∃I)
        AOT-thus ‹∃F∃u([F]u & [λz Numbers(z,F)]y & [λz Numbers(z,[F]^−u)]x)›
          by (rule ∃I)
      qed
```

```
    }
  qed
  ultimately AOT-have
    ‹□∀ x ∀ y (∃ F ∃ u ([F]u & [λz Numbers(z,F)]y & [λz Numbers(z,[F]⁻ᵘ)]x) ≡
            ∃ F ∃ u ([F]u & Numbers(y,F) & Numbers(x,[F]⁻ᵘ)))›
    using →E by blast
  AOT-thus ‹[λxy ∃ F ∃ u ([F]u & Numbers(y,F) & Numbers(x,[F]⁻ᵘ))]↓›
    by (rule safe−ext[2][axiom-inst, THEN →E, OF &I, rotated]) cqt:2
qed
```

The following is not part of PLM, but a consequence of extended relation comprehension and can be used to *derive* the predecessor axiom.

```
AOT-theorem numbers-prop-den: ‹[λx Numbers(x,G)]↓›
proof (rule safe−ext[axiom-inst, THEN →E, OF &I])
  AOT-show ‹[λx A!x & [λx ∀ F (x[F] ≡ [λz 𝓐[F]z] ≈E G)]x]↓›
    by cqt:2
next
  AOT-have 0: ‹⊢□ [λx ∀ F (x[F] ≡ [λz 𝓐[F]z] ≈E G)]↓›
  proof(safe intro!: Comprehension-3[THEN →E] →I RN GEN)
      AOT-modally-strict {
        fix F H
        AOT-assume ‹□H ≡E F›
        AOT-hence ‹□∀ u ([H]u ≡ [F]u)›
          by (AOT-subst (reverse) ‹∀ u ([H]u ≡ [F]u)› ‹H ≡E F›)
              (safe intro!: eqE[THEN ≡Df, THEN ≡S(1), OF &I] cqt:2)
        AOT-hence ‹∀ u □([H]u ≡ [F]u)›
          by (metis Ordinary.res−var−bound−reas[CBF] →E)
        AOT-hence ‹□([H]u ≡ [F]u)› for u
          using Ordinary.∀ E by fast
        AOT-hence ‹𝓐([H]u ≡ [F]u)› for u
          by (metis nec−imp−act →E)
        AOT-hence ‹𝓐([F]u ≡ [H]u)› for u
          by (metis Act−Basic:5 Commutativity of ≡ intro−elim:3:b)
        AOT-hence ‹[λz 𝓐[F]z] ≡E [λz 𝓐[H]z]›
          by (safe intro!: eqE[THEN ≡df I] &I cqt:2 Ordinary.GEN;
              AOT-subst ‹[λz 𝓐[F]z]u› ‹𝓐[F]u› for: u F)
             (auto intro!: beta−C−meta[THEN →E] cqt:2
                          Act−Basic:5[THEN ≡E(1)])
        AOT-hence ‹[λz 𝓐[F]z] ≈E [λz 𝓐[H]z]›
          by (safe intro!: apE−eqE:1[unvarify F G, THEN →E] cqt:2)
        AOT-thus ‹[λz 𝓐[F]z] ≈E G ≡ [λz 𝓐[H]z] ≈E G›
          using ≡I eq−part:2[terms] eq−part:3[terms] →E →I
          by metis
      }
  qed
  AOT-show ‹□∀ x (A!x & [λx ∀ F (x[F] ≡ [λz 𝓐[F]z] ≈E G)]x ≡ Numbers(x,G))›
  proof (safe intro!: RN GEN)
    AOT-modally-strict {
      fix x
      AOT-show ‹A!x & [λx ∀ F (x[F] ≡ [λz 𝓐[F]z] ≈E G)]x ≡ Numbers(x,G)›
        by (AOT-subst-def numbers; AOT-subst-thm beta−C−meta[THEN →E, OF 0])
           (auto intro!: beta−C−meta[THEN →E, OF 0] ≡I →I &I cqt:2
                   dest: &E)
    }
  qed
qed
```

The two theorems above allow us to derive the predecessor axiom of PLM as theorem.

```
AOT-theorem pred: ‹[λxy ∃ F∃ u ([F]u & Numbers(y,F) & Numbers(x,[F]⁻ᵘ))]↓›
  using pred-coex numbers-prop-den[∀ I G] ≡E by blast

AOT-define Predecessor :: ‹Π› (‹ℙ›)
  pred−thm:1:
```

```
  ‹ℙ =df [λxy ∃F∃u ([F]u & Numbers(y,F) & Numbers(x,[F]^-u))]›

AOT-theorem pred−thm:2: ‹ℙ↓›
  using pred pred−thm:1 rule−id−df:2:b[zero] by blast

AOT-theorem pred−thm:3:
  ‹[ℙ]xy ≡ ∃F∃u ([F]u & Numbers(y,F) & Numbers(x,[F]^-u))›
    by (auto intro!: beta−C−meta[unvarify ν1νn, where τ=‹(-,-)›, THEN →E,
                                  rotated, OF pred, simplified]
                   tuple-denotes[THEN ≡df I] &I cqt:2 pred
             intro: =df I(2)[OF pred−thm:1])

AOT-theorem pred−1−1:1: ‹[ℙ]xy → □[ℙ]xy›
proof(rule →I)
  AOT-assume ‹[ℙ]xy›
  AOT-hence ‹∃F∃u ([F]u & Numbers(y,F) & Numbers(x,[F]^-u))›
    using ≡E(1) pred−thm:3 by fast
  then AOT-obtain F where ‹∃u ([F]u & Numbers(y,F) & Numbers(x,[F]^-u))›
    using ∃E[rotated] by blast
  then AOT-obtain u where props: ‹[F]u & Numbers(y,F) & Numbers(x,[F]^-u)›
    using Ordinary.∃E[rotated] by meson
  AOT-obtain G where Ridigifies-G-F: ‹Rigidifies(G, F)›
    by (metis instantiation rigid−der:3)
  AOT-hence ξ: ‹□∀x([G]x → □[G]x)› and ζ: ‹∀x([G]x ≡ [F]x)›
    using df−rigid−rel:2[THEN ≡df E, THEN &E(1),
                         THEN ≡df E[OF df−rigid−rel:1], THEN &E(2)]
          df−rigid−rel:2[THEN ≡df E, THEN &E(2)] by blast+

  AOT-have rigid-num-nec: ‹Numbers(x,F) & Rigidifies(G,F) → □Numbers(x,G)›
    for x G F
  proof(rule →I; frule &E(1); drule &E(2))
    fix G F x
    AOT-assume Numbers-xF: ‹Numbers(x,F)›
    AOT-assume ‹Rigidifies(G,F)›
    AOT-hence ξ: ‹Rigid(G)› and ζ: ‹∀x([G]x ≡ [F]x)›
      using df−rigid−rel:2[THEN ≡df E] &E by blast+
    AOT-thus ‹□Numbers(x,G)›
    proof (safe intro!:
          num−cont:2[THEN →E, OF ξ, THEN qml:2[axiom-inst, THEN →E],
                     THEN ∀E(2), THEN →E]
          num−tran:3[THEN →E, THEN ≡E(1), rotated, OF Numbers-xF]
          eqE[THEN ≡df I]
            &I cqt:2[const-var][axiom-inst] Ordinary.GEN →I)
      AOT-show ‹[F]u ≡ [G]u› for u
        using ζ[THEN ∀E(2)] by (metis ≡E(6) oth−class−taut:3:a)
    qed
  qed
  AOT-have ‹□Numbers(y,G)›
    using rigid-num-nec[THEN →E, OF &I, OF props[THEN &E(1), THEN &E(2)],
                        OF Ridigifies-G-F].
  moreover {
    AOT-have ‹Rigidifies([G]^-u, [F]^-u)›
    proof (safe intro!: df−rigid−rel:1[THEN ≡df I] df−rigid−rel:2[THEN ≡df I]
                        &I F−u[den] GEN ≡I →I)
      AOT-have ‹□∀x([G]x → □[G]x) → □∀x([[G]^-u]x → □[[G]^-u]x)›
      proof (rule RM; safe intro!: →I GEN)
        AOT-modally-strict {
          fix x
          AOT-assume 0: ‹∀x([G]x → □[G]x)›
          AOT-assume 1: ‹[[G]^-u]x›
          AOT-have ‹[λx [G]x & x ≠E u]x›
            apply (rule F−u[THEN =df E(1), where τ1τn=(-,-), simplified])
             apply cqt:2[lambda]
```

```
          by (fact 1)
        AOT-hence ‹[G]x & x ≠E u›
          by (rule β→C(1))
        AOT-hence 2: ‹□[G]x› and 3: ‹□x ≠E u›
          using &E 0[THEN ∀E(2), THEN →E] id−nec4:1 ≡E(1) by blast+
        AOT-show ‹□[[G]−u]x›
          apply (AOT-subst ‹[[G]−u]x› ‹[G]x & x ≠E u›)
           apply (rule F−u[THEN =df I(1), where τ1τn=(-,-), simplified])
            apply cqt:2[lambda]
           apply (rule beta−C−meta[THEN →E])
          apply cqt:2[lambda]
          using 2 3 KBasic:3 ≡S(2) ≡E(2) by blast
      }
    qed
    AOT-thus ‹□∀x([[G]−u]x → □[[G]−u]x)› using ξ →E by blast
  next
    fix x
    AOT-assume ‹[[G]−u]x›
    AOT-hence ‹[λx [G]x & x ≠E u]x›
      by (auto intro: F−u[THEN =df E(1), where τ1τn=(-,-), simplified]
              intro!: cqt:2)
    AOT-hence ‹[G]x & x ≠E u›
      by (rule β→C(1))
    AOT-hence ‹[F]x & x ≠E u›
      using ζ &I &E(1) &E(2) ≡E(1) rule−ui:3 by blast
    AOT-hence ‹[λx [F]x & x ≠E u]x›
      by (auto intro!: β←C(1) cqt:2)
    AOT-thus ‹[[F]−u]x›
      by (auto intro: F−u[THEN =df I(1), where τ1τn=(-,-), simplified]
              intro!: cqt:2)
  next
    fix x
    AOT-assume ‹[[F]−u]x›
    AOT-hence ‹[λx [F]x & x ≠E u]x›
      by (auto intro: F−u[THEN =df E(1), where τ1τn=(-,-), simplified]
              intro!: cqt:2)
    AOT-hence ‹[F]x & x ≠E u›
      by (rule β→C(1))
    AOT-hence ‹[G]x & x ≠E u›
      using ζ &I &E(1) &E(2) ≡E(2) rule−ui:3 by blast
    AOT-hence ‹[λx [G]x & x ≠E u]x›
      by (auto intro!: β←C(1) cqt:2)
    AOT-thus ‹[[G]−u]x›
      by (auto intro: F−u[THEN =df I(1), where τ1τn=(-,-), simplified]
              intro!: cqt:2)
  qed
  AOT-hence ‹□Numbers(x,[G]−u)›
    using rigid-num-nec[unvarify F G, OF F−u[den], OF F−u[den], THEN →E,
                        OF &I, OF props[THEN &E(2)]] by blast
}
moreover AOT-have ‹□[G]u›
  using props[THEN &E(1), THEN &E(1), THEN ζ[THEN ∀E(2), THEN ≡E(2)]]
        ξ[THEN qml:2[axiom-inst, THEN →E], THEN ∀E(2), THEN →E]
  by blast
ultimately AOT-have ‹□([G]u & Numbers(y,G) & Numbers(x,[G]−u))›
  by (metis KBasic:3 &I ≡E(2))
AOT-hence ‹∃u (□([G]u & Numbers(y,G) & Numbers(x,[G]−u)))›
  by (rule Ordinary.∃I)
AOT-hence ‹□∃u ([G]u & Numbers(y,G) & Numbers(x,[G]−u))›
  using Ordinary.res−var−bound−reas[Buridan] →E by fast
AOT-hence ‹∃F □∃u ([F]u & Numbers(y,F) & Numbers(x,[F]−u))›
  by (rule ∃I)
AOT-hence 0: ‹□∃F∃u ([F]u & Numbers(y,F) & Numbers(x,[F]−u))›
```

```
    using Buridan vdash-properties:10 by fast
  AOT-show ‹□[ℙ]xy›
    by (AOT-subst ‹[ℙ]xy› ‹∃F∃u ([F]u & Numbers(y,F) & Numbers(x,[F]⁻ᵘ))›;
        simp add: pred-thm:3 0)
qed

AOT-theorem pred-1-1:2: ‹Rigid(ℙ)›
  by (safe intro!: df-rigid-rel:1[THEN ≡df I] pred-thm:2 &I
                  RN tuple-forall[THEN ≡df I];
      safe intro!: GEN pred-1-1:1)

AOT-theorem pred-1-1:3: ‹1-1(ℙ)›
proof (safe intro!: df-1-1:1[THEN ≡df I] pred-thm:2 &I GEN →I;
       frule &E(1); drule &E(2))
  fix x y z
  AOT-assume ‹[ℙ]xz›
  AOT-hence ‹∃F∃u ([F]u & Numbers(z,F) & Numbers(x,[F]⁻ᵘ))›
    using pred-thm:3[THEN ≡E(1)] by blast
  then AOT-obtain F where ‹∃u ([F]u & Numbers(z,F) & Numbers(x,[F]⁻ᵘ))›
    using ∃E[rotated] by blast
  then AOT-obtain u where u-prop: ‹[F]u & Numbers(z,F) & Numbers(x,[F]⁻ᵘ)›
    using Ordinary.∃E[rotated] by meson
  AOT-assume ‹[ℙ]yz›
  AOT-hence ‹∃F∃u ([F]u & Numbers(z,F) & Numbers(y,[F]⁻ᵘ))›
    using pred-thm:3[THEN ≡E(1)] by blast
  then AOT-obtain G where ‹∃u ([G]u & Numbers(z,G) & Numbers(y,[G]⁻ᵘ))›
    using ∃E[rotated] by blast
  then AOT-obtain v where v-prop: ‹[G]v & Numbers(z,G) & Numbers(y,[G]⁻ᵛ)›
    using Ordinary.∃E[rotated] by meson
  AOT-show ‹x = y›
  proof (rule pre-Hume[unvarify G H, OF F-u[den], OF F-u[den],
                       THEN →E, OF &I, THEN ≡E(2)])
    AOT-show ‹Numbers(x, [F]⁻ᵘ)›
      using u-prop &E by blast
  next
    AOT-show ‹Numbers(y, [G]⁻ᵛ)›
      using v-prop &E by blast
  next
    AOT-have ‹F ≈E G›
      using u-prop[THEN &E(1), THEN &E(2)]
      using v-prop[THEN &E(1), THEN &E(2)]
      using num-tran:2[THEN →E, OF &I] by blast
    AOT-thus ‹[F]⁻ᵘ ≈E [G]⁻ᵛ›
      using u-prop[THEN &E(1), THEN &E(1)]
      using v-prop[THEN &E(1), THEN &E(1)]
      using eqP'[THEN →E, OF &I, OF &I]
      by blast
  qed
qed

AOT-theorem pred-1-1:4: ‹Rigid₁₋₁(ℙ)›
  by (meson ≡df I &I df-1-1:2 pred-1-1:2 pred-1-1:3)

AOT-theorem assume-anc:1:
  ‹[ℙ]* = [λxy ∀F((∀z([ℙ]xz → [F]z) & Hereditary(F,ℙ)) → [F]y)]›
  apply (rule =df I(1)[OF ances-df])
   apply cqt:2[lambda]
  apply (rule =I(1))
  by cqt:2[lambda]

AOT-theorem assume-anc:2: ‹ℙ*↓›
  using t=t-proper:1 assume-anc:1 vdash-properties:10 by blast
```

```
AOT-theorem assume−anc:3:
  ‹[ℙ*]xy ≡ ∀ F((∀ z([ℙ]xz → [F]z) & ∀ x'∀ y'([ℙ]x'y' → ([F]x' → [F]y'))) → [F]y)›
proof −
  AOT-have prod-den: ‹⊢□ «(AOT-term-of-var x1,AOT-term-of-var x2)»↓›
    for x1 x2 :: ‹κ AOT-var›
    by (simp add: &I ex:1:a prod-denotesI rule−ui:3)
  AOT-have den: ‹[λxy ∀ F((∀ z([ℙ]xz → [F]z) & Hereditary(F,ℙ)) → [F]y)]↓›
    by cqt:2[lambda]
  AOT-have 1: ‹[ℙ*]xy ≡ ∀ F((∀ z([ℙ]xz → [F]z) & Hereditary(F,ℙ)) → [F]y)›
    apply (rule rule=E[rotated, OF assume−anc:1[symmetric]])
    by (rule beta−C−meta[unvarify ν1νn, OF prod-den, THEN →E,
                        simplified, OF den, simplified])
  show ?thesis
    apply (AOT-subst (reverse) ‹∀ x'∀ y' ([ℙ]x'y' → ([F]x' → [F]y'))›
                             ‹Hereditary(F,ℙ)› for: F :: ‹<κ>›)
    using hered:1[THEN ≡Df, THEN ≡S(1), OF &I, OF pred−thm:2,
                  OF cqt:2[const-var][axiom-inst]] apply blast
    by (fact 1)
qed

AOT-theorem no−pred−0:1: ‹¬∃ x [ℙ]x 0›
proof(rule raa−cor:2)
  AOT-assume ‹∃ x [ℙ]x 0›
  then AOT-obtain a where ‹[ℙ]a 0›
    using ∃ E[rotated] by blast
  AOT-hence ‹∃ F∃ u ([F]u & Numbers(0, F) & Numbers(a, [F]^-u))›
    using pred−thm:3[unvarify y, OF zero:2, THEN ≡E(1)] by blast
  then AOT-obtain F where ‹∃ u ([F]u & Numbers(0, F) & Numbers(a, [F]^-u))›
    using ∃ E[rotated] by blast
  then AOT-obtain u where ‹[F]u & Numbers(0, F) & Numbers(a, [F]^-u)›
    using Ordinary.∃ E[rotated] by meson
  AOT-hence ‹[F]u› and num0-F: ‹Numbers(0, F)›
    using &E &I by blast+
  AOT-hence ‹∃ u [F]u›
    using Ordinary.∃ I by fast
  moreover AOT-have ‹¬∃ u [F]u›
    using num0-F ≡E(2) 0F:1 by blast
  ultimately AOT-show ‹p & ¬p› for p
    by (metis raa−cor:3)
qed

AOT-theorem no−pred−0:2: ‹¬∃ x [ℙ*]x 0›
proof(rule raa−cor:2)
  AOT-assume ‹∃ x [ℙ*]x 0›
  then AOT-obtain a where ‹[ℙ*]a 0›
    using ∃ E[rotated] by blast
  AOT-hence ‹∃ z [ℙ]z 0›
    using anc−her:5[unvarify R y, OF zero:2,
                    OF pred−thm:2, THEN →E] by auto
  AOT-thus ‹∃ z [ℙ]z 0 & ¬∃ z [ℙ]z 0›
    by (metis no−pred−0:1 raa−cor:3)
qed

AOT-theorem no−pred−0:3: ‹¬[ℙ*]0 0›
  by (metis existential:1 no−pred−0:2 reductio−aa:1 zero:2)

AOT-theorem assume1:1: ‹(=ℙ) = [λxy ∃ z ([ℙ]xz & [ℙ]yz)]›
  apply (rule =dfI(1)[OF id−d−R])
   apply cqt:2[lambda]
  apply (rule =I(1))
  by cqt:2[lambda]

AOT-theorem assume1:2: ‹x =ℙ y ≡ ∃ z ([ℙ]xz & [ℙ]yz)›
```

```
proof (rule rule=E[rotated, OF assume1:1[symmetric]])
  AOT-have prod-den: ‹⊢□ «(AOT-term-of-var x₁,AOT-term-of-var x₂)»↓›
    for x₁ x₂ :: ‹κ AOT-var›
    by (simp add: &I ex:1:a prod-denotesI rule−ui:3)
  AOT-have 1: ‹[λxy ∃z ([ℙ]xz & [ℙ]yz)]↓›
    by cqt:2
  AOT-show ‹[λxy ∃z ([ℙ]xz & [ℙ]yz)]xy ≡ ∃z ([ℙ]xz & [ℙ]yz)›
    using beta−C−meta[THEN →E, OF 1, unvarify ν₁νₙ,
                       OF prod-den, simplified] by blast
qed

AOT-theorem assume1:3: ‹[ℙ]⁺ = [λxy [ℙ]*xy ∨ x =ℙ y]›
  apply (rule =dfI(1)[OF w−ances−df])
   apply (simp add: w−ances−df[den1])
  apply (rule rule=E[rotated, OF assume1:1[symmetric]])
  apply (rule =dfI(1)[OF id−d−R])
   apply cqt:2[lambda]
  apply (rule =I(1))
  by cqt:2[lambda]

AOT-theorem assume1:4: ‹[ℙ]⁺↓›
  using w−ances−df[den2].

AOT-theorem assume1:5: ‹[ℙ]⁺xy ≡ [ℙ]*xy ∨ x =ℙ y›
proof −
  AOT-have 0: ‹[λxy [ℙ]*xy ∨ x =ℙ y]↓› by cqt:2
  AOT-have prod-den: ‹⊢□ «(AOT-term-of-var x₁, AOT-term-of-var x₂)»↓›
    for x₁ x₂ :: ‹κ AOT-var›
    by (simp add: &I ex:1:a prod-denotesI rule−ui:3)
  show ?thesis
    apply (rule rule=E[rotated, OF assume1:3[symmetric]])
    using beta−C−meta[THEN →E, OF 0, unvarify ν₁νₙ, OF prod-den, simplified]
    by (simp add: cond-case-prod-eta)
qed

AOT-define NaturalNumber :: ‹τ› (‹ℕ›)
  nnumber:1: ‹ℕ =df [λx [ℙ]⁺0x]›

AOT-theorem nnumber:2: ‹ℕ↓›
  by (rule =dfI(2)[OF nnumber:1]; cqt:2[lambda])

AOT-theorem nnumber:3: ‹[ℕ]x ≡ [ℙ]⁺0x›
  apply (rule =dfI(2)[OF nnumber:1])
   apply cqt:2[lambda]
  apply (rule beta−C−meta[THEN →E])
  by cqt:2[lambda]

AOT-theorem 0−n: ‹[ℕ]0›
proof (safe intro!: nnumber:3[unvarify x, OF zero:2, THEN ≡E(2)]
    assume1:5[unvarify x y, OF zero:2, OF zero:2, THEN ≡E(2)]
    ∨I(2) assume1:2[unvarify x y, OF zero:2, OF zero:2, THEN ≡E(2)])
  fix u
  AOT-have den: ‹[λx O!x & x =E u]↓› by cqt:2[lambda]
  AOT-obtain a where a-prop: ‹Numbers(a, [λx O!x & x =E u])›
    using num:1[unvarify G, OF den] ∃E[rotated] by blast
  AOT-have ‹[ℙ]0a›
  proof (safe intro!: pred−thm:3[unvarify x, OF zero:2, THEN ≡E(2)]
                      ∃I(1)[where τ=‹«[λx O!x & x =E u]»›]
                      Ordinary.∃I[where β=u] &I den
                      0F:1[unvarify F, OF F−u[den], unvarify F,
                           OF den, THEN ≡E(1)])
    AOT-show ‹[λx [O!]x & x =E u]u›
      by (auto intro!: β←C(1) cqt:2 &I ord=Eequiv:1[THEN →E]
```

```
                Ordinary.ψ)
  next
    AOT-show ‹Numbers(a,[λx [O!]x & x =E u])›
      using a-prop.
  next
    AOT-show ‹¬∃ v [[λx [O!]x & x =E u]⁻ᵘ]v›
    proof(rule raa−cor:2)
      AOT-assume ‹∃ v [[λx [O!]x & x =E u]⁻ᵘ]v›
      then AOT-obtain v where ‹[[λx [O!]x & x =E u]⁻ᵘ]v›
        using Ordinary.∃ E[rotated] &E by blast
      AOT-hence ‹[λz [λx [O!]x & x =E u]z & z ≠E u]v›
        apply (rule F−u[THEN =df E(1), where τ1τn=(-,-), simplified, rotated])
        by cqt:2[lambda]
      AOT-hence ‹[λx [O!]x & x =E u]v & v ≠E u›
        by (rule β→C(1))
      AOT-hence ‹v =E u› and ‹v ≠E u›
        using β→C(1) &E by blast+
      AOT-hence ‹v =E u & ¬(v =E u)›
        by (metis ≡E(4) reductio−aa:1 thm−neg=E)
      AOT-thus ‹p & ¬p› for p
        by (metis raa−cor:1)
    qed
  qed
  AOT-thus ‹∃ z ([ℙ]0z & [ℙ]0z)›
    by (safe intro!: &I ∃ I(2)[where β=a])
qed

AOT-theorem mod−col−num:1: ‹[ℕ]x → □[ℕ]x›
proof(rule →I)
  AOT-have nec0N: ‹[λx □[ℕ]x]0›
    by (auto intro!: β←C(1) cqt:2 simp: zero:2 RN 0−n)
  AOT-have 1: ‹[λx □[ℕ]x]0 &
    ∀ x∀ y ([[ℙ]⁺]0x & [[ℙ]⁺]0y → ([ℙ]xy → ([λx □[ℕ]x]x → [λx □[ℕ]x]y))) →
    ∀ x ([[ℙ]⁺]0x → [λx □[ℕ]x]x)›
    by (auto intro!: cqt:2
              intro: pre−ind[unconstrain ℛ, unvarify β, OF pred−thm:2,
                            THEN →E, OF pred−1−1:4, unvarify z, OF zero:2,
                            unvarify F])
  AOT-have ‹∀ x ([[ℙ]⁺]0x → [λx □[ℕ]x]x)›
  proof (rule 1[THEN →E]; safe intro!: &I GEN →I nec0N;
         frule &E(1); drule &E(2))
    fix x y
    AOT-assume ‹[ℙ]xy›
    AOT-hence 0: ‹□[ℙ]xy›
      by (metis pred−1−1:1 →E)
    AOT-assume ‹[λx □[ℕ]x]x›
    AOT-hence ‹□[ℕ]x›
      by (rule β→C(1))
    AOT-hence ‹□([ℙ]xy & [ℕ]x)›
      by (metis 0 KBasic:3 Adjunction ≡E(2) →E)
    moreover AOT-have ‹□([ℙ]xy & [ℕ]x) → □[ℕ]y›
    proof (rule RM; rule →I; frule &E(1); drule &E(2))
      AOT-modally-strict {
        AOT-assume 0: ‹[ℙ]xy›
        AOT-assume ‹[ℕ]x›
        AOT-hence 1: ‹[[ℙ]⁺]0x›
          by (metis ≡E(1) nnumber:3)
        AOT-show ‹[ℕ]y›
          apply (rule nnumber:3[THEN ≡E(2)])
          apply (rule assume1:5[unvarify x, OF zero:2, THEN ≡E(2)])
          apply (rule ∨I(1))
          apply (rule w−ances−her:3[unconstrain ℛ, unvarify β, OF pred−thm:2,
                                 THEN →E, OF pred−1−1:4, unvarify x,
```

OF zero:*2*, *THEN* →*E*])
apply (*rule* &*I*)
apply (*fact 1*)
by (*fact 0*)
}
qed
ultimately AOT-have ‹□[ℕ]*y*›
by (*metis* →*E*)
AOT-thus ‹[λ*x* □[ℕ]*x*]*y*›
by (*auto intro*!: *β*←*C*(*1*) *cqt*:*2*)
qed
AOT-hence *0*: ‹[[ℙ]$^+$]*0x* → [λ*x* □[ℕ]*x*]*x*›
using ∀ *E*(*2*) **by** *blast*
AOT-assume ‹[ℕ]*x*›
AOT-hence ‹[[ℙ]$^+$]*0x*›
by (*metis* ≡*E*(*1*) *nnumber*:*3*)
AOT-hence ‹[λ*x* □[ℕ]*x*]*x*›
using *0*[*THEN* →*E*] **by** *blast*
AOT-thus ‹□[ℕ]*x*›
by (*rule β*→*C*(*1*))
qed

AOT-theorem *mod*−*col*−*num*:*2*: ‹*Rigid*(ℕ)›
by (*safe intro*!: *df*−*rigid*−*rel*:*1*[*THEN* $\equiv_{df}$*I*] &*I RN GEN*
mod−*col*−*num*:*1 nnumber*:*2*)

AOT-register-rigid-restricted-type
Number: ‹[ℕ]*κ*›
proof
AOT-modally-strict {
AOT-show ‹∃ *x* [ℕ]*x*›
by (*rule* ∃ *I*(*1*)[**where** *τ*=‹«*0*»›]; *simp add*: *0*−*n zero*:*2*)
}
next
AOT-modally-strict {
AOT-show ‹[ℕ]*κ* → *κ*↓› **for** *κ*
by (*simp add*: →*I cqt*:*5*:*a*[*1*][*axiom-inst*, *THEN* →*E*, *THEN* &*E*(*2*)])
}
next
AOT-modally-strict {
AOT-show ‹∀ *x*([ℕ]*x* → □[ℕ]*x*)›
by (*simp add*: *GEN mod*−*col*−*num*:*1*)
}
qed
AOT-register-variable-names
Number: *m n k i j*

AOT-theorem *0*−*pred*: ‹¬∃ *n* [ℙ]*n 0*›
proof (*rule raa*−*cor*:*2*)
AOT-assume ‹∃ *n* [ℙ]*n 0*›
then AOT-obtain *n* **where** ‹[ℙ]*n 0*›
using *Number*.∃ *E*[*rotated*] **by** *meson*
AOT-hence ‹∃ *x* [ℙ]*x 0*›
using &*E* ∃ *I* **by** *fast*
AOT-thus ‹∃ *x* [ℙ]*x 0* & ¬∃ *x* [ℙ]*x 0*›
using *no*−*pred*−*0*:*1* &*I* **by** *auto*
qed

AOT-theorem *no*−*same*−*succ*:
‹∀ *n*∀ *m*∀ *k*([ℙ]*nk* & [ℙ]*mk* → *n* = *m*)›
proof(*safe intro*!: *Number*.*GEN* →*I*)
fix *n m k*
AOT-assume ‹[ℙ]*nk* & [ℙ]*mk*›

```
  AOT-thus ‹n = m›
    by (safe intro!: cqt:2[const-var][axiom-inst] df−1−1:3[
          unvarify R, OF pred−thm:2,
          THEN →E, OF pred−1−1:4, THEN qml:2[axiom-inst, THEN →E],
          THEN ≡df E[OF df−1−1:1], THEN &E(2), THEN ∀ E(1), THEN ∀ E(1),
          THEN ∀ E(1)[where τ=‹AOT-term-of-var (Number.Rep k)›], THEN →E])
qed

AOT-theorem induction:
  ‹∀ F([F]0 & ∀ n∀ m([ℙ]nm → ([F]n → [F]m)) → ∀ n[F]n)›
proof (safe intro!: GEN[where 'a=‹<κ>›] Number.GEN &I →I;
       frule &E(1); drule &E(2))
  fix F n
  AOT-assume F0: ‹[F]0›
  AOT-assume 0: ‹∀ n∀ m([ℙ]nm → ([F]n → [F]m))›
  {
    fix x y
    AOT-assume ‹[[ℙ]+]0x & [[ℙ]+]0y›
    AOT-hence ‹[ℕ]x› and ‹[ℕ]y›
      using &E ≡E(2) nnumber:3 by blast+
    moreover AOT-assume ‹[ℙ]xy›
    moreover AOT-assume ‹[F]x›
    ultimately AOT-have ‹[F]y›
      using 0[THEN ∀ E(2), THEN →E, THEN ∀ E(2), THEN →E,
            THEN →E, THEN →E] by blast
  } note 1 = this
  AOT-have 0: ‹[[ℙ]+]0n›
    by (metis ≡E(1) nnumber:3 Number.ψ)
  AOT-show ‹[F]n›
    apply (rule pre−ind[unconstrain ℛ, unvarify β, THEN →E, OF pred−thm:2,
                          OF pred−1−1:4, unvarify z, OF zero:2, THEN →E,
                          THEN ∀ E(2), THEN →E];
          safe intro!: 0 &I GEN →I F0)
    using 1 by blast
qed

AOT-theorem suc−num:1: ‹[ℙ]nx → [ℕ]x›
proof(rule →I)
  AOT-have ‹[[ℙ]+]0 n›
    by (meson Number.ψ ≡E(1) nnumber:3)
  moreover AOT-assume ‹[ℙ]nx›
  ultimately AOT-have ‹[[ℙ]*]0 x›
    using w−ances−her:3[unconstrain ℛ, unvarify β, OF pred−thm:2, THEN →E,
                          OF pred−1−1:4, unvarify x, OF zero:2,
                          THEN →E, OF &I]
    by blast
  AOT-hence ‹[[ℙ]+]0 x›
    using assume1:5[unvarify x, OF zero:2, THEN ≡E(2), OF ∨I(1)]
    by blast
  AOT-thus ‹[ℕ]x›
    by (metis ≡E(2) nnumber:3)
qed

AOT-theorem suc−num:2: ‹[[ℙ]*]nx → [ℕ]x›
proof(rule →I)
  AOT-have ‹[[ℙ]+]0 n›
    using Number.ψ ≡E(1) nnumber:3 by blast
  AOT-assume ‹[[ℙ]*]n x›
  AOT-hence ‹∀ F (∀ z ([ℙ]nz → [F]z) & ∀ x'∀ y' ([ℙ]x'y' → ([F]x' → [F]y')) → [F]x)›
    using assume−anc:3[THEN ≡E(1)] by blast
  AOT-hence ϑ: ‹∀ z ([ℙ]nz → [ℕ]z) & ∀ x'∀ y' ([ℙ]x'y' → ([ℕ]x' → [ℕ]y')) → [ℕ]x›
    using ∀ E(1) nnumber:2 by blast
  AOT-show ‹[ℕ]x›
```

```
  proof (safe intro!: ϑ[THEN →E] GEN →I &I)
    AOT-show ‹[ℕ]z› if ‹[ℙ]nz› for z
      using Number.ψ suc−num:1 that →E by blast
  next
    AOT-show ‹[ℕ]y› if ‹[ℙ]xy› and ‹[ℕ]x› for x y
      using suc−num:1[unconstrain n, THEN →E] that →E by blast
  qed
qed

AOT-theorem suc−num:3: ‹[ℙ]+nx → [ℕ]x›
proof (rule →I)
  AOT-assume ‹[ℙ]+nx›
  AOT-hence ‹[ℙ]*nx ∨ n =ℙ x›
    by (metis assume1:5 ≡E(1))
  moreover {
    AOT-assume ‹[ℙ]*nx›
    AOT-hence ‹[ℕ]x›
      by (metis suc−num:2 →E)
  }
  moreover {
    AOT-assume ‹n =ℙ x›
    AOT-hence ‹n = x›
      using id−R−thm:3[unconstrain ℛ, unvarify β, OF pred−thm:2,
                       THEN →E, OF pred−1−1:4, THEN →E] by blast
    AOT-hence ‹[ℕ]x›
      by (metis rule=E Number.ψ)
  }
  ultimately AOT-show ‹[ℕ]x›
    by (metis ∨E(3) reductio−aa:1)
qed

AOT-theorem pred−num: ‹[ℙ]xn → [ℕ]x›
proof (rule →I)
  AOT-assume 0: ‹[ℙ]xn›
  AOT-have ‹[[ℙ]+]0 n›
    using Number.ψ ≡E(1) nnumber:3 by blast
  AOT-hence ‹[[ℙ]*]0 n ∨ 0 =ℙ n›
    using assume1:5[unvarify x, OF zero:2] by (metis ≡E(1))
  moreover {
    AOT-assume ‹0 =ℙ n›
    AOT-hence ‹∃z ([ℙ]0z & [ℙ]nz)›
      using assume1:2[unvarify x, OF zero:2, THEN ≡E(1)] by blast
    then AOT-obtain a where ‹[ℙ]0a & [ℙ]na› using ∃E[rotated] by blast
    AOT-hence ‹0 = n›
      using pred−1−1:3[THEN df−1−1:1[THEN ≡df E], THEN &E(2),
                       THEN ∀E(1), OF zero:2, THEN ∀E(2),
                       THEN ∀E(2), THEN →E] by blast
    AOT-hence ‹[ℙ]x 0›
      using 0 rule=E id-sym by fast
    AOT-hence ‹∃x [ℙ]x 0›
      by (rule ∃I)
    AOT-hence ‹∃x [ℙ]x 0 & ¬∃x [ℙ]x 0›
      by (metis no−pred−0:1 raa−cor:3)
  }
  ultimately AOT-have ‹[[ℙ]*]0n›
    by (metis ∨E(3) raa−cor:1)
  AOT-hence ‹∃z ([[ℙ]+]0z & [ℙ]zn)›
    using w−ances−her:7[unconstrain ℛ, unvarify β, OF pred−thm:2,
                        THEN →E, OF pred−1−1:4, unvarify x,
                        OF zero:2, THEN →E] by blast
  then AOT-obtain b where b-prop: ‹[[ℙ]+]0b & [ℙ]bn›
    using ∃E[rotated] by blast
  AOT-hence ‹[ℕ]b›
```

```
    by (metis &E(1) ≡E(2) nnumber:3)
  moreover AOT-have ‹x = b›
    using pred−1−1:3[THEN df−1−1:1[THEN ≡df E], THEN &E(2),
                  THEN ∀E(2), THEN ∀E(2), THEN ∀E(2), THEN →E,
                  OF &I, OF 0, OF b-prop[THEN &E(2)]].
  ultimately AOT-show ‹[ℕ]x›
    using rule=E id-sym by fast
qed

AOT-theorem nat−card: ‹[ℕ]x → NaturalCardinal(x)›
proof(rule →I)
  AOT-assume ‹[ℕ]x›
  AOT-hence ‹[[ℙ]+]0x›
    by (metis ≡E(1) nnumber:3)
  AOT-hence ‹[[ℙ]*]0x ∨ 0 =ℙ x›
    using assume1:5[unvarify x, OF zero:2, THEN ≡E(1)] by blast
  moreover {
    AOT-assume ‹[[ℙ]*]0x›
    then AOT-obtain a where ‹[ℙ]ax›
      using anc−her:5[unvarify R x, OF zero:2, OF pred−thm:2, THEN →E]
          ∃E[rotated] by blast
    AOT-hence ‹∃F∃u ([F]u & Numbers(x,F) & Numbers(a,[F]−u))›
      using pred−thm:3[THEN ≡E(1)] by blast
    then AOT-obtain F where ‹∃u ([F]u & Numbers(x,F) & Numbers(a,[F]−u))›
      using ∃E[rotated] by blast
    then AOT-obtain u where ‹[F]u & Numbers(x,F) & Numbers(a,[F]−u)›
      using Ordinary.∃E[rotated] by meson
    AOT-hence ‹NaturalCardinal(x)›
      using eq−num:6[THEN →E] &E by blast
  }
  moreover {
    AOT-assume ‹0 =ℙ x›
    AOT-hence ‹0 = x›
      using id−R−thm:3[unconstrain ℛ, unvarify β, OF pred−thm:2,
                    THEN →E, OF pred−1−1:4, unvarify x,
                    OF zero:2, THEN →E] by blast
    AOT-hence ‹NaturalCardinal(x)›
      by (metis rule=E zero−card)
  }
  ultimately AOT-show ‹NaturalCardinal(x)›
    by (metis ∨E(2) raa−cor:1)
qed

AOT-theorem pred−func:1: ‹[ℙ]xy & [ℙ]xz → y = z›
proof (rule →I; frule &E(1); drule &E(2))
  AOT-assume ‹[ℙ]xy›
  AOT-hence ‹∃F∃u ([F]u & Numbers(y,F) & Numbers(x,[F]−u))›
    using pred−thm:3[THEN ≡E(1)] by blast
  then AOT-obtain F where ‹∃u ([F]u & Numbers(y,F) & Numbers(x,[F]−u))›
    using ∃E[rotated] by blast
  then AOT-obtain a where
        Oa: ‹O!a›
    and a-prop: ‹[F]a & Numbers(y,F) & Numbers(x,[F]−a)›
    using ∃E[rotated] &E by blast
  AOT-assume ‹[ℙ]xz›
  AOT-hence ‹∃F∃u ([F]u & Numbers(z,F) & Numbers(x,[F]−u))›
    using pred−thm:3[THEN ≡E(1)] by blast
  then AOT-obtain G where ‹∃u ([G]u & Numbers(z,G) & Numbers(x,[G]−u))›
    using ∃E[rotated] by blast
  then AOT-obtain b where Ob: ‹O!b›
                and b-prop: ‹[G]b & Numbers(z,G) & Numbers(x,[G]−b)›
    using ∃E[rotated] &E by blast
  AOT-have ‹[F]−a ≈E [G]−b›
```

```
    using num−tran:2[unvarify G H, OF F−u[den], OF F−u[den],
                     THEN →E, OF &I, OF a-prop[THEN &E(2)],
                     OF b-prop[THEN &E(2)]].
  AOT-hence ‹F ≈E G›
    using P′−eq[unconstrain u, THEN →E, OF Oa, unconstrain v, THEN →E,
                OF Ob, THEN →E, OF &I, OF &I]
          a-prop[THEN &E(1), THEN &E(1)]
          b-prop[THEN &E(1), THEN &E(1)] by blast
  AOT-thus ‹y = z›
    using pre−Hume[THEN →E, THEN ≡E(2), OF &I,
                   OF a-prop[THEN &E(1), THEN &E(2)],
                   OF b-prop[THEN &E(1), THEN &E(2)]]
    by blast
qed

AOT-theorem pred−func:2: ‹[ℙ]nm & [ℙ]nk → m = k›
  using pred−func:1.

AOT-theorem being-number-of-den: ‹[λx x = #G]↓›
proof (rule safe−ext[axiom-inst, THEN →E]; safe intro!: &I GEN RN)
  AOT-show ‹[λx Numbers(x,[λz 𝓐[G]z])]↓›
    by (rule numbers-prop-den[unvarify G]) cqt:2[lambda]
next
  AOT-modally-strict {
    AOT-show ‹Numbers(x,[λz 𝓐[G]z]) ≡ x = #G› for x
      using eq−num:2.
  }
qed

axiomatization ω-nat :: ‹ω ⇒ nat› where ω-nat: ‹surj ω-nat›
```

Unfortunately, since the axiom requires the type ω to have an infinite domain, **nitpick** can only find a potential model and no genuine model. However, since we could trivially choose ω as a copy of *nat*, we can still be assured that above axiom is consistent.

```
lemma ‹True› nitpick[satisfy, user-axioms, card nat=1, expect = potential] ..

AOT-axiom modal−axiom:
  ‹∃x([ℕ]x & x = #G) → ◇∃y([E!]y & ∀u (𝓐[G]u → u ≠E y))›
proof(rule AOT-model-axiomI) AOT-modally-strict {
```

The actual extension on the ordinary objects of a property is the set of ordinary urelements that exemplifies the property in the designated actual world.

```
  define act-ωext :: ‹<κ> ⇒ ω set› where
    ‹act-ωext ≡ λ Π . {x :: ω . [w0 ⊨ [Π]«ωκ x»]}›
```

Encoding a property with infinite actual extension on the ordinary objects denotes a property by extended relation comprehension.

```
  AOT-have enc-finite-act-ωext-den:
    ‹⊢□ [λx ∃F(¬«εo w. finite (act-ωext F)» & x[F])]↓›
  proof(safe intro!: Comprehension-1[THEN →E] RN GEN →I)
    AOT-modally-strict {
      fix F G
      AOT-assume ‹□G ≡E F›
      AOT-hence ‹𝓐G ≡E F›
        using nec−imp−act[THEN →E] by blast
      AOT-hence ‹𝓐(G↓ & F↓ & ∀u([G]u ≡ [F]u))›
        by (AOT-subst-def (reverse) eqE)
      hence ‹[w0 ⊨ [G]«ωκ x»] = [w0 ⊨ [F]«ωκ x»]› for x
        by (auto dest!: ∀E(1) →E
                simp: AOT-model-denotes-κ-def AOT-sem-denotes AOT-sem-conj
                      AOT-model-ωκ-ordinary AOT-sem-act AOT-sem-equiv)
      AOT-thus ‹¬«εo w. finite (act-ωext (AOT-term-of-var F))» ≡
```

```
        ¬«ε_o w. finite (act-ωext (AOT-term-of-var G))»›
      by (simp add: AOT-sem-not AOT-sem-equiv act-ωext-def
              AOT-model-proposition-choice-simp)
  }
qed
```

By coexistence, encoding only properties with finite actual extension on the ordinary objects denotes.

```
AOT-have ‹[λx ∀F(x[F] → «ε_o w. finite (act-ωext F)»)]↓›
proof(rule safe−ext[axiom-inst, THEN →E]; safe intro!: &I RN GEN)
  AOT-show ‹[λx ¬[λx ∃F(¬«ε_o w. finite (act-ωext F)» & x[F])]x]↓›
    by cqt:2
next
  AOT-modally-strict {
    fix x
    AOT-show ‹¬[λx ∃F (¬«ε_o w. finite (act-ωext F)» & x[F])]x ≡
          ∀F(x[F] → «ε_o w. finite (act-ωext F)»)›
      by (AOT-subst ‹[λx ∃F (¬«ε_o w. finite (act-ωext F)» & x[F])]x›
                 ‹∃F (¬«ε_o w. finite (act-ωext F)» & x[F])›;
          (rule beta−C−meta[THEN →E])?)
         (auto simp: enc-finite-act-ωext-den AOT-sem-equiv AOT-sem-not
                 AOT-sem-forall AOT-sem-imp AOT-sem-conj AOT-sem-exists)
  }
qed
```

We show by induction that any property encoded by a natural number has a finite actual extension on the ordinary objects.

```
AOT-hence ‹[λx ∀F(x[F] → «ε_o w. finite (act-ωext F)»)]n› for n
proof(rule induction[THEN ∀E(1), THEN →E, THEN Number.∀E];
     safe intro!: &I Number.GEN β←C zero:2 →I cqt:2
         dest!: β→C)
  AOT-show ‹∀F(0[F] → «ε_o w. finite (act-ωext F)»)›
  proof(safe intro!: GEN →I)
    fix F
    AOT-assume ‹0[F]›
    AOT-actually {
      AOT-hence ‹¬∃u [F]u›
        using zero=:2 intro−elim:3:a AOT-sem-enc-nec by blast
      AOT-hence ‹∀x ¬(O!x & [F]x)›
        using cqt−further:4 vdash−properties:10 by blast
      hence ‹¬([w_0 ⊨ [F]«ωκ x»])› for x
        by (auto dest!: ∀E(1)[where τ=‹ωκ x›]
              simp: AOT-sem-not AOT-sem-conj AOT-model-ωκ-ordinary
                 russell−axiom[exe,1].ψ-denotes-asm)
    }
    AOT-thus ‹«ε_o w. finite (act-ωext (AOT-term-of-var F))»›
      by (auto simp: AOT-model-proposition-choice-simp act-ωext-def)
  qed
next
  fix n m
  AOT-assume ‹[ℙ]nm›
  AOT-hence ‹∃F∃u ([F]u & Numbers(m,F) & Numbers(n,[F]^-u))›
    using pred−thm:3[THEN ≡E(1)] by blast
  then AOT-obtain G where ‹∃u ([G]u & Numbers(m,G) & Numbers(n,[G]^-u))›
    using ∃E[rotated] by blast
  then AOT-obtain u where 0: ‹[G]u & Numbers(m,G) & Numbers(n,[G]^-u)›
    using Ordinary.∃E[rotated] by meson

  AOT-assume n-prop: ‹∀F(n[F] → «ε_o w. finite (act-ωext F)»)›
  AOT-show ‹∀F(m[F] → «ε_o w. finite (act-ωext F)»)›
  proof(safe intro!: GEN →I)
    fix F
    AOT-assume ‹m[F]›
    AOT-hence 1: ‹[λx 𝓐[F]x] ≈_E G›
```

```
  using 0[THEN &E(1), THEN &E(2), THEN numbers[THEN ≡df E],
        THEN &E(2), THEN ∀E(2), THEN ≡E(1)] by auto
AOT-show ‹«εo w. finite (act-ωext (AOT-term-of-var F))»›
proof(rule raa−cor:1)
  AOT-assume ‹¬«εo w. finite (act-ωext (AOT-term-of-var F))»›
  hence inf: ‹infinite (act-ωext (AOT-term-of-var F))›
    by (auto simp: AOT-sem-not AOT-model-proposition-choice-simp)
  then AOT-obtain v where act-F-v: ‹𝓐[F]v›
    unfolding AOT-sem-act act-ωext-def
    by (metis AOT-term-of-var-cases AOT-model-ωκ-ordinary
            AOT-model-denotes-κ-def Ordinary.Rep-cases κ.disc(7)
            mem-Collect-eq not-finite-existsD)
  AOT-hence ‹[λx 𝓐[F]x]v›
    by (safe intro!: β←C cqt:2)
  AOT-hence ‹[λx 𝓐[F]x]⁻ᵛ ≈E [G]⁻ᵘ›
    by (safe intro!: eqP′[unvarify F, THEN →E] &I cqt:2 1
                  0[THEN &E(1), THEN &E(1)])
  moreover AOT-have ‹[λx 𝓐[F]x]⁻ᵛ ≈E [λx 𝓐[λy [F]y & y ≠E v]x]›
  proof(safe intro!: apE−eqE:1[unvarify F G, THEN →E] cqt:2
                  F−u[den][unvarify F] eqE[THEN ≡df I] &I
                  Ordinary.GEN)
    fix u
    AOT-have ‹[λx [λx 𝓐[F]x]x & x ≠E v]u ≡ [λx 𝓐[F]x]u & u ≠E v›
      by (safe intro!: beta−C−meta[THEN →E] cqt:2)
    also AOT-have ‹[λx 𝓐[F]x]u & u ≠E v ≡ 𝓐[F]u & u ≠E v›
      by (AOT-subst ‹[λx 𝓐[F]x]u› ‹𝓐[F]u›)
         (safe intro!: beta−C−meta[THEN →E] cqt:2
                   oth−class−taut:3:a)
    also AOT-have ‹𝓐[F]u & u ≠E v ≡ 𝓐([F]u & u ≠E v)›
      using id−act2:2 AOT-sem-conj AOT-sem-equiv AOT-sem-act by auto
    also AOT-have ‹𝓐([F]u & u ≠E v) ≡ 𝓐[λy [F]y & y ≠E v]u›
      by (AOT-subst ‹[λy [F]y & y ≠E v]u› ‹[F]u & u ≠E v›)
         (safe intro!: beta−C−meta[THEN →E] cqt:2
                   oth−class−taut:3:a)
    also AOT-have ‹𝓐[λy [F]y & y ≠E v]u ≡ [λx 𝓐[λy [F]y & y ≠E v]x]u›
      by (safe intro!: beta−C−meta[THEN →E, symmetric] cqt:2)
    finally AOT-show ‹[[λx 𝓐[F]x]⁻ᵛ]u ≡ [λx 𝓐[λy [F]y & y ≠E v]x]u›
      by (auto intro!: cqt:2
             intro: rule−id−df:2:b[OF F−u, where τ1τn=‹(-,-)›, simplified])
  qed
  ultimately AOT-have ‹[λx 𝓐[λy [F]y & y ≠E v]x] ≈E [G]⁻ᵘ›
    using eq−part:2[terms] eq−part:3[terms] →E by blast
  AOT-hence ‹n[λy [F]y & y ≠E v]›
    by (safe intro!: 0[THEN &E(2), THEN numbers[THEN ≡df E],
        THEN &E(2), THEN ∀E(1), THEN ≡E(2)] cqt:2)
  hence finite: ‹finite (act-ωext «[λy [F]y & y ≠E v]»)›
    by (safe intro!: n-prop[THEN ∀E(1), THEN →E,
                      simplified AOT-model-proposition-choice-simp]
                  cqt:2)
  obtain y where y-def: ‹ωκ y = AOT-term-of-var (Ordinary.Rep v)›
    by (metis AOT-model-ordinary-ωκ Ordinary.restricted-var-condition)
  AOT-actually {
    fix x
    AOT-assume ‹[λy [F]y & y ≠E v]«ωκ x»›
    AOT-hence ‹[F]«ωκ x»›
      by (auto dest!: β→C &E(1))
  }
  moreover AOT-actually {
    AOT-have ‹[F]«ωκ y»›
      unfolding y-def using act-F-v AOT-sem-act by blast
  }
  moreover AOT-actually {
    fix x
```

```
        assume noteq: ‹x ≠ y›
        AOT-assume ‹[F]«ωκ x»›
        moreover AOT-have ωκ-x-den: ‹«ωκ x»↓›
          using AOT-sem-exe calculation by blast
        moreover {
          AOT-have ‹¬(«ωκ x» =E v)›
          proof(rule raa−cor:2)
            AOT-assume ‹«ωκ x» =E v›
            AOT-hence ‹«ωκ x» = v›
              using =E−simple:2[unvarify x, THEN →E, OF ωκ-x-den]
              by blast
            hence ‹ωκ x = ωκ y›
              unfolding y-def AOT-sem-eq
              by meson
            hence ‹x = y›
              by blast
            AOT-thus ‹p & ¬p› for p using noteq by blast
          qed
          AOT-hence ‹«ωκ x» ≠E v›
            by (safe intro!: thm−neg=E[unvarify x, THEN ≡E(2)] ωκ-x-den)
        }
        ultimately AOT-have ‹[λy [F]y & y ≠E v]«ωκ x»›
          by (auto intro!: β←C cqt:2 &I)
      }
      ultimately have ‹(insert y (act-ωext «[λy [F]y & y ≠E v]»)) =
                   (act-ωext (AOT-term-of-var F))›
        unfolding act-ωext-def
        by auto
      hence ‹finite (act-ωext (AOT-term-of-var F))›
        using finite finite.insertI by metis
      AOT-thus ‹p & ¬p› for p
        using inf by blast
    qed
  qed
qed
AOT-hence nat-enc-finite: ‹∀ F(n[F] → «εo w. finite (act-ωext F)»)› for n
  using β→C(1) by blast
```

The main proof can now generate a witness, since we required the domain of ordinary objects to be infinite.

```
AOT-show ‹∃ x ([ℕ]x & x = #G) → ◇∃ y (E!y & ∀ u (𝓐[G]u → u ≠E y))›
proof(safe intro!: →I)
  AOT-assume ‹∃ x ([ℕ]x & x = #G)›
  then AOT-obtain n where ‹n = #G›
    using Number.∃ E[rotated] by meson
  AOT-hence ‹Numbers(n,[λx 𝓐[G]x])›
    using eq−num:3 rule=E id-sym by fast
  AOT-hence ‹n[G]›
    by (auto intro!: numbers[THEN ≡df E, THEN &E(2),
                        THEN ∀ E(2), THEN ≡E(2)]
                  eq−part:1[unvarify F] cqt:2)
  AOT-hence ‹«εo w. finite (act-ωext (AOT-term-of-var G))»›
    using nat-enc-finite[THEN ∀ E(2), THEN →E] by blast
  hence finite: ‹finite (act-ωext (AOT-term-of-var G))›
    by (auto simp: AOT-model-proposition-choice-simp)
  AOT-have ‹∃ u ¬𝓐[G]u›
  proof(rule raa−cor:1)
    AOT-assume ‹¬∃ u ¬𝓐[G]u›
    AOT-hence ‹∀ x ¬(O!x & ¬𝓐[G]x)›
      by (metis cqt−further:4 →E)
    AOT-hence ‹𝓐[G]x› if ‹O!x› for x
      using ∀ E(2) AOT-sem-conj AOT-sem-not that by blast
    hence ‹[w0 ⊨ [G]«ωκ x»]› for x
```

```
        by (metis AOT-term-of-var-cases AOT-model-ωκ-ordinary
                  AOT-model-denotes-κ-def AOT-sem-act κ.disc(7))
      hence ‹(act-ωext (AOT-term-of-var G)) = UNIV›
        unfolding act-ωext-def by auto
      moreover have ‹infinite (UNIV::ω set)›
        by (metis ω-nat finite-imageI infinite-UNIV-char-0)
      ultimately have ‹infinite (act-ωext (AOT-term-of-var G))›
        by simp
      AOT-thus ‹p & ¬p› for p using finite by blast
    qed
    then AOT-obtain x where x-prop: ‹O!x & ¬𝓐[G]x›
      using ∃ E[rotated] by blast
    AOT-hence ‹◇E!x›
      by (metis betaC:1:a con−dis−i−e:2:a AOT-sem-ordinary)
    moreover AOT-have ‹□∀ u (𝓐[G]u → u ≠E x)›
    proof(safe intro!: RN GEN →I)
      AOT-modally-strict {
        fix y
        AOT-assume ‹O!y›
        AOT-assume 0: ‹𝓐[G]y›
        AOT-show ‹y ≠E x›
        proof (safe intro!: thm−neg=E[THEN ≡E(2)] raa−cor:2)
          AOT-assume ‹y =E x›
          AOT-hence ‹y = x›
            by (metis =E−simple:2 vdash−properties:10)
          hence ‹y = x›
            by (simp add: AOT-sem-eq AOT-term-of-var-inject)
          AOT-hence ‹¬𝓐[G]y›
            using x-prop &E AOT-sem-not AOT-sem-act by metis
          AOT-thus ‹𝓐[G]y & ¬𝓐[G]y›
            using 0 &I by blast
        qed
      }
    qed
    ultimately AOT-have ‹◇(∀ u (𝓐[G]u → u ≠E x) & E!x)›
      using KBasic:16[THEN →E, OF &I] by blast
    AOT-hence ‹◇(E!x & ∀ u (𝓐[G]u → u ≠E x))›
      by (AOT-subst ‹E!x & ∀ u (𝓐[G]u → u ≠E x)› ‹∀ u (𝓐[G]u → u ≠E x) & E!x›)
         (auto simp: oth−class−taut:2:a)
    AOT-hence ‹∃ y ◇(E!y & ∀ u (𝓐[G]u → u ≠E y))›
      using ∃ I by fast
    AOT-thus ‹◇∃ y (E!y & ∀ u (𝓐[G]u → u ≠E y))›
      using CBF◇[THEN →E] by fast
  qed
} qed

AOT-theorem modal−lemma:
  ‹◇∀ u(𝓐[G]u → u ≠E v) → ∀ u(𝓐[G]u → u ≠E v)›
proof(safe intro!: →I Ordinary.GEN)
  AOT-modally-strict {
    fix u
    AOT-assume act-Gu: ‹𝓐[G]u›
    AOT-have ‹∀ u (𝓐[G]u → u ≠E v) → u ≠E v›
    proof(rule →I)
      AOT-assume ‹∀ u (𝓐[G]u → u ≠E v)›
      AOT-hence ‹𝓐[G]u → u ≠E v›
        using Ordinary.∀ E by fast
      AOT-thus ‹u ≠E v›
        using act-Gu →E by blast
    qed
  } note 0 = this
  AOT-have ϑ: ‹□(∀ u (𝓐[G]u → u ≠E v) → u ≠E v)› if ‹□𝓐[G]u› for u
  proof −
```

AOT-have ‹$\Box\mathcal{A}[G]u \to \Box(\forall u\ (\mathcal{A}[G]u \to u \neq_E v) \to u \neq_E v)$›
apply (*rule RM*) **using** *0* &*E* →*I* **by** *blast*
thus *?thesis* **using** *that* →*E* **by** *blast*
qed
fix *u*
AOT-assume *1*: ‹$\Diamond\forall u(\mathcal{A}[G]u \to u \neq_E v)$›
AOT-assume ‹$\mathcal{A}[G]u$›
AOT-hence ‹$\Box\mathcal{A}[G]u$›
by (*metis Act−Basic*:*6* ≡*E*(*1*))
AOT-hence ‹$\Box(\forall u\ (\mathcal{A}[G]u \to u \neq_E v) \to u \neq_E v)$›
using *Ordinary.ψ ϑ* **by** *blast*
AOT-hence ‹$\Diamond u \neq_E v$›
using *1 K*◊[*THEN* →*E*, *THEN* →*E*] **by** *blast*
AOT-thus ‹$u \neq_E v$›
by (*metis id−nec4*:*2* ≡*E*(*1*))
qed

AOT-theorem *th−succ*: ‹$\forall n \exists! m\ [\mathbb{P}]nm$›
proof(*safe intro*!: *Number.GEN* →*I uniqueness*:*1*[*THEN* $\equiv_{df}$*I*])
fix *n*
AOT-have ‹*NaturalCardinal*(*n*)›
by (*metis nat−card Number.ψ* →*E*)
AOT-hence ‹$\exists G(n = \#G)$›
by (*metis* $\equiv_{df}$*E card*)
then AOT-obtain *G* **where** *n-num-G*: ‹$n = \#G$›
using ∃*E*[*rotated*] **by** *blast*
AOT-hence ‹$\exists n\ (n = \#G)$›
by (*rule Number*.∃*I*)
AOT-hence ‹$\Diamond\exists y\ ([E!]y\ \&\ \forall u(\mathcal{A}[G]u \to u \neq_E y))$›
using *modal−axiom*[*axiom-inst*, *THEN* →*E*] **by** *blast*
AOT-hence ‹$\exists y\ \Diamond([E!]y\ \&\ \forall u(\mathcal{A}[G]u \to u \neq_E y))$›
using *BF*◊[*THEN* →*E*] **by** *auto*
then AOT-obtain *y* **where** ‹$\Diamond([E!]y\ \&\ \forall u(\mathcal{A}[G]u \to u \neq_E y))$›
using ∃*E*[*rotated*] **by** *blast*
AOT-hence ‹$\Diamond E!y$› **and** *2*: ‹$\Diamond\forall u(\mathcal{A}[G]u \to u \neq_E y)$›
using *KBasic2*:*3* &*E* →*E* **by** *blast+*
AOT-hence *Oy*: ‹$O!y$›
by (*auto intro*!: *β*←*C*(*1*) *cqt*:*2 intro*: *AOT-ordinary*[*THEN* $=_{df}$*I*(*2*)])
AOT-have *0*: ‹$\forall u(\mathcal{A}[G]u \to u \neq_E y)$›
using *2 modal−lemma*[*unconstrain v*, *THEN* →*E*, *OF Oy*, *THEN* →*E*] **by** *simp*
AOT-have *1*: ‹$[\lambda x\ \mathcal{A}[G]x \lor x =_E y]\downarrow$›
by *cqt*:*2*
AOT-obtain *b* **where** *b-prop*: ‹*Numbers*(*b*, $[\lambda x\ \mathcal{A}[G]x \lor x =_E y]$)›
using *num*:*1*[*unvarify G*, *OF 1*] ∃*E*[*rotated*] **by** *blast*
AOT-have *Pnb*: ‹$[\mathbb{P}]nb$›
proof(*safe intro*!: *pred−thm*:*3*[*THEN* ≡*E*(*2*)]
∃*I*(*1*)[**where** *τ*=‹«$[\lambda x\ \mathcal{A}[G]x \lor x =_E y]$»›]
1 ∃*I*(*2*)[**where** *β*=*y*] &*I Oy b-prop*)
AOT-show ‹$[\lambda x\ \mathcal{A}[G]x \lor x =_E y]y$›
by (*auto intro*!: *β*←*C*(*1*) *cqt*:*2* ∨*I*(*2*)
ord=*Eequiv*:*1*[*THEN* →*E*, *OF Oy*])
next
AOT-have *equinum*: ‹$[\lambda x\ \mathcal{A}[G]x \lor x =_E y]^{-y} \approx_E [\lambda x\ \mathcal{A}[G]x]$›
proof(*rule apE−eqE*:*1*[*unvarify F G*, *THEN* →*E*];
(*cqt*:*2*[*lambda*] | *rule F−u*[*den*][*unvarify F*]; *cqt*:*2*[*lambda*])*?*)
AOT-show ‹$[\lambda x\ \mathcal{A}[G]x \lor x =_E y]^{-y} \equiv_E [\lambda x\ \mathcal{A}[G]x]$›
proof (*safe intro*!: *eqE*[*THEN* $\equiv_{df}$*I*] &*I F−u*[*den*][*unvarify F*]
Ordinary.GEN →*I*; *cqt*:*2?*)
fix *u*
AOT-have ‹$[[\lambda x\ \mathcal{A}[G]x \lor [(=_E)]xy]^{-y}]u \equiv ([\lambda x\ \mathcal{A}[G]x \lor x =_E y]u)\ \&\ u \neq_E y$›
apply (*rule F−u*[*THEN* $=_{df}$*I*(*1*)][**where** $\tau_1\tau_n$=‹(-,-)›], *simplified*]; *cqt*:*2?*)
by (*rule beta−C−cor*:*2*[*THEN* →*E*, *THEN* ∀*E*(*2*)]; *cqt*:*2*)
also AOT-have ‹$\ldots \equiv (\mathcal{A}[G]u \lor u =_E y)\ \&\ u \neq_E y$›

apply (*AOT-subst* ‹[λx 𝓐[G]x ∨ [(=$_E$)]xy]u› ‹𝓐[G]u ∨ u =$_E$ y›)
apply (*rule beta−C−cor*:*2*[*THEN* →*E*, *THEN* ∀*E*(*2*)]; *cqt*:*2*)
using *oth−class−taut*:*3*:*a* **by** *blast*
also AOT-have ‹. . . ≡ 𝓐[G]u›
proof(*safe intro*!: ≡*I* →*I*)
AOT-assume ‹(𝓐[G]u ∨ u =$_E$ y) & u ≠$_E$ y›
AOT-thus ‹𝓐[G]u›
by (*metis* &*E*(*1*) &*E*(*2*) ∨*E*(*3*) ≡*E*(*1*) *thm−neg*=*E*)
next
AOT-assume ‹𝓐[G]u›
AOT-hence ‹u ≠$_E$ y› **and** ‹𝓐[G]u ∨ u =$_E$ y›
using *0*[*THEN* ∀*E*(*2*), *THEN* →*E*, *OF Ordinary*.ψ, *THEN* →*E*]
∨*I* **by** *blast*+
AOT-thus ‹(𝓐[G]u ∨ u =$_E$ y) & u ≠$_E$ y›
using &*I* **by** *simp*
qed
also AOT-have ‹. . . ≡ [λx 𝓐[G]x]u›
by (*rule beta−C−cor*:*2*[*THEN* →*E*, *THEN* ∀*E*(*2*), *symmetric*]; *cqt*:*2*)
finally AOT-show ‹[[λx 𝓐[G]x ∨ [(=$_E$)]xy]$^{-y}$]u ≡ [λx 𝓐[G]x]u›**.**
qed
qed
AOT-have *2*: ‹[λx 𝓐[G]x]↓› **by** *cqt*:*2*[*lambda*]
AOT-show ‹*Numbers*(n,[λx 𝓐[G]x ∨ x =$_E$ y]$^{-y}$)›
using *num−tran*:*1*[*unvarify G H*, *OF 2*, *OF F−u*[*den*][*unvarify F*, *OF 1*],
THEN →*E*, *OF equinum*, *THEN* ≡*E*(*2*),
OF eq−num:*2*[*THEN* ≡*E*(*2*), *OF n-num-G*]]**.**
qed
AOT-show ‹∃α ([ℕ]α & [ℙ]nα & ∀β ([ℕ]β & [ℙ]nβ → β = α))›
proof(*safe intro*!: ∃*I*(*2*)[**where** *β*=*b*] &*I Pnb* →*I GEN*)
AOT-show ‹[ℕ]b› **using** *suc−num*:*1*[*THEN* →*E*, *OF Pnb*]**.**
next
fix *y*
AOT-assume *0*: ‹[ℕ]y & [ℙ]ny›
AOT-show ‹y = b›
apply (*rule pred−func*:*1*[*THEN* →*E*])
using *0*[*THEN* &*E*(*2*)] *Pnb* &*I* **by** *blast*
qed
qed

AOT-define *Successor* :: ‹τ ⇒ κ$_s$› (‹-′′› [*100*] *100*)
def−suc: ‹n′ =$_{df}$ ιm([ℙ]nm)›

Note: not explicitly in PLM

AOT-theorem *def−suc*[*den1*]: ‹ιm([ℙ]nm)↓›
using *A−Exists*:*2 RA*[*2*] ≡*E*(*2*) *th−succ*[*THEN Number*.∀*E*] **by** *blast*

Note: not explicitly in PLM

AOT-theorem *def−suc*[*den2*]: **shows** ‹n′↓›
by (*rule def−suc*[*THEN* =$_{df}$*I*(*1*)])
(*auto simp*: *def−suc*[*den1*])

AOT-theorem *suc-eq-desc*: ‹n′ = ιm([ℙ]nm)›
by (*rule def−suc*[*THEN* =$_{df}$*I*(*1*)])
(*auto simp*: *def−suc*[*den1*] *rule*=*I*:*1*)

AOT-theorem *suc−fact*: ‹n = m → n′ = m′›
proof (*rule* →*I*)
AOT-assume *0*: ‹n = m›
AOT-show ‹n′ = m′›
apply (*rule rule*=*E*[*rotated*, *OF 0*])
by (*rule* =*I*(*1*)[*OF def−suc*[*den2*]])

```
qed

AOT-theorem ind−gnd: ‹m = 0 ∨ ∃ n(m = n')›
proof −
  AOT-have ‹[[ℙ]+]0m›
    using Number.ψ ≡E(1) nnumber:3 by blast
  AOT-hence ‹[[ℙ]*]0m ∨ 0 =ℙ m›
    using assume1:5[unvarify x, OF zero:2, THEN ≡E(1)] by blast
  moreover {
    AOT-assume ‹[[ℙ]*]0m›
    AOT-hence ‹∃ z ([[ℙ]+]0z & [ℙ]zm)›
      using w−ances−her:7[unconstrain ℛ, unvarify β x, OF zero:2,
                          OF pred−thm:2, THEN →E, OF pred−1−1:4,
                          THEN →E]
      by blast
    then AOT-obtain z where ϑ: ‹[[ℙ]+]0z› and ξ: ‹[ℙ]zm›
      using &E ∃ E[rotated] by blast
    AOT-have Nz: ‹[ℕ]z›
      using ϑ ≡E(2) nnumber:3 by blast
    moreover AOT-have ‹m = z'›
    proof (rule def−suc[THEN =df I(1)];
          safe intro!: def−suc[den1][unconstrain n, THEN →E, OF Nz]
                        nec−hintikka−scheme[THEN ≡E(2)] &I
                        GEN →I Act−Basic:2[THEN ≡E(2)])
      AOT-show ‹𝓐[ℕ]m› using Number.ψ
        by (meson mod−col−num:1 nec−imp−act →E)
    next
      AOT-show ‹𝓐[ℙ]zm› using ξ
        by (meson nec−imp−act pred−1−1:1 →E)
    next
      fix y
      AOT-assume ‹𝓐([ℕ]y & [ℙ]zy)›
      AOT-hence ‹𝓐[ℕ]y› and ‹𝓐[ℙ]zy›
        using Act−Basic:2 &E ≡E(1) by blast+
      AOT-hence 0: ‹[ℙ]zy›
        by (metis RN ≡E(1) pred−1−1:1 sc−eq−fur:2 →E)
      AOT-thus ‹y = m›
        using pred−func:1[THEN →E, OF &I] ξ by metis
    qed
    ultimately AOT-have ‹[ℕ]z & m = z'›
      by (rule &I)
    AOT-hence ‹∃ n m = n'›
      by (rule ∃ I)
    hence ?thesis
      by (rule ∨I)
  }
  moreover {
    AOT-assume ‹0 =ℙ m›
    AOT-hence ‹0 = m›
      using id−R−thm:3[unconstrain ℛ, unvarify β x, OF zero:2, OF pred−thm:2,
                       THEN →E, OF pred−1−1:4, THEN →E]
      by auto
    hence ?thesis using id-sym ∨I by blast
  }
  ultimately show ?thesis
    by (metis ∨E(2) raa−cor:1)
qed

AOT-theorem suc−thm: ‹[ℙ]n n'›
proof −
  AOT-obtain x where m-is-n: ‹x = n'›
    using free−thms:1[THEN ≡E(1), OF def−suc[den2]]
    using ∃ E by metis
```

```
  AOT-have ‹𝓐([ℕ]n' & [ℙ]n n')›
    apply (rule rule=E[rotated, OF suc-eq-desc[symmetric]])
    apply (rule actual−desc:4[THEN →E])
    by (simp add: def−suc[den1])
  AOT-hence ‹𝓐[ℕ]n'› and ‹𝓐[ℙ]n n'›
    using Act−Basic:2 ≡E(1) &E by blast+
  AOT-hence ‹𝓐[ℙ]nx›
    using m-is-n[symmetric] rule=E by fast+
  AOT-hence ‹[ℙ]nx›
    by (metis RN ≡E(1) pred−1−1:1 sc−eq−fur:2 →E)
  thus ?thesis
    using m-is-n rule=E by fast
qed

AOT-define Numeral1 :: ‹κs› (‹1›)
  numerals:1: ‹1 =df 0'›

AOT-theorem prec−facts:1: ‹[ℙ]0 1›
  by (auto intro: numerals:1[THEN rule−id−df:2:b[zero],
                              OF def−suc[den2][unconstrain n, unvarify β,
                                               OF zero:2, THEN →E, OF 0−n]]
                  suc−thm[unconstrain n, unvarify β, OF zero:2,
                          THEN →E, OF 0−n])




AOT-define Finite :: ‹τ ⇒ φ› (‹Finite'(-')›)
  inf−card:1: ‹Finite(x) ≡df NaturalCardinal(x) & [ℕ]x›
AOT-define Infinite :: ‹τ ⇒ φ› (‹Infinite'(-')›)
  inf−card:2: ‹Infinite(x) ≡df NaturalCardinal(x) & ¬Finite(x)›

AOT-theorem inf−card−exist:1: ‹NaturalCardinal(#O!)›
  by (safe intro!: card[THEN ≡dfI] ∃I(1)[where τ=‹«O!»›] =I
                   num−def:2[unvarify G] oa−exist:1)

AOT-theorem inf−card−exist:2: ‹Infinite(#O!)›
proof (safe intro!: inf−card:2[THEN ≡dfI] &I inf−card−exist:1)
  AOT-show ‹¬Finite(#O!)›
  proof(rule raa−cor:2)
    AOT-assume ‹Finite(#O!)›
    AOT-hence 0: ‹[ℕ]#O!›
      using inf−card:1[THEN ≡dfE] &E(2) by blast
    AOT-have ‹Numbers(#O!, [λz 𝓐O!z])›
      using eq−num:3[unvarify G, OF oa−exist:1].
    AOT-hence ‹#O! = #O!›
      using eq−num:2[unvarify x G, THEN ≡E(1), OF oa−exist:1,
                    OF num−def:2[unvarify G], OF oa−exist:1]
      by blast
    AOT-hence ‹[ℕ]#O! & #O! = #O!›
      using 0 &I by blast
    AOT-hence ‹∃x ([ℕ]x & x = #O!)›
      using num−def:2[unvarify G, OF oa−exist:1] ∃I(1) by fast
    AOT-hence ‹◇∃y ([E!]y & ∀u (𝓐[O!]u → u ≠E y))›
      using modal−axiom[axiom-inst, unvarify G, THEN →E, OF oa−exist:1] by blast
    AOT-hence ‹∃y ◇([E!]y & ∀u (𝓐[O!]u → u ≠E y))›
      using BF◇[THEN →E] by blast
    then AOT-obtain b where ‹◇([E!]b & ∀u (𝓐[O!]u → u ≠E b))›
      using ∃E[rotated] by blast
    AOT-hence ‹◇[E!]b› and 2: ‹◇∀u (𝓐[O!]u → u ≠E b)›
      using KBasic2:3[THEN →E] &E by blast+
    AOT-hence ‹[λx ◇[E!]x]b›
      by (auto intro!: β←C(1) cqt:2)
```

moreover AOT-have ‹$O! = [\lambda x\ \Diamond[E!]x]$›
by (*rule rule−id−df:1[zero][OF oa:1]*) *cqt:2*
ultimately AOT-have *b-ord*: ‹$O!b$›
using *rule=E id-sym* **by** *fast*
AOT-hence ‹$\mathcal{A}O!b$›
by (*meson ≡E(1) oa−facts:7*)
moreover AOT-have *2*: ‹$\forall u\ (\mathcal{A}[O!]u \to u \neq_E b)$›
using *modal−lemma[unvarify G, unconstrain v, OF oa−exist:1, THEN →E, OF b-ord, THEN →E, OF 2]*.
ultimately AOT-have ‹$b \neq_E b$›
using *Ordinary.∀ E[OF 2, unconstrain α, THEN →E, OF b-ord, THEN →E]* **by** *blast*
AOT-hence ‹$\neg(b =_E b)$›
by (*metis ≡E(1) thm−neg=E*)
moreover AOT-have ‹$b =_E b$›
using *ord=Eequiv:1[THEN →E, OF b-ord]*.
ultimately AOT-show ‹$p\ \&\ \neg p$› **for** *p*
by (*metis raa−cor:3*)
qed
qed

theory *AOT-misc*
imports *AOT-NaturalNumbers*
begin

14 Miscellaneous Theorems

AOT-theorem *PossiblyNumbersEmptyPropertyImpliesZero*:
‹$\Diamond Numbers(x,[\lambda z\ O!z\ \&\ z \neq_E z]) \to x = 0$›
proof(*rule →I*)
AOT-have ‹$Rigid([\lambda z\ O!z\ \&\ z \neq_E z])$›
proof (*safe intro!: df−rigid−rel:1[THEN ≡$_{df}$I] &I cqt:2; rule RN; safe intro!: GEN →I*)
AOT-modally-strict {
fix *x*
AOT-assume ‹$[\lambda z\ O!z\ \&\ z \neq_E z]x$›
AOT-hence ‹$O!x\ \&\ x \neq_E x$› **by** (*rule β→C*)
moreover AOT-have ‹$x =_E x$› **using** *calculation[THEN &E(1)]*
by (*metis ord=Eequiv:1 vdash−properties:10*)
ultimately AOT-have ‹$x =_E x\ \&\ \neg x =_E x$›
by (*metis con−dis−i−e:1 con−dis−i−e:2:b intro−elim:3:a thm−neg=E*)
AOT-thus ‹$\Box[\lambda z\ O!z\ \&\ z \neq_E z]x$› **using** *raa−cor:1* **by** *blast*
}
qed
AOT-hence ‹$\Box\forall x\ (Numbers(x,[\lambda z\ O!z\ \&\ z \neq_E z]) \to \Box Numbers(x,[\lambda z\ O!z\ \&\ z \neq_E z]))$›
by (*safe intro!: num−cont:2[unvarify G, THEN →E] cqt:2*)
AOT-hence ‹$\forall x\ \Box(Numbers(x,[\lambda z\ O!z\ \&\ z \neq_E z]) \to \Box Numbers(x,[\lambda z\ O!z\ \&\ z \neq_E z]))$›
using *BFs:2[THEN →E]* **by** *blast*
AOT-hence ‹$\Box(Numbers(x,[\lambda z\ O!z\ \&\ z \neq_E z]) \to \Box Numbers(x,[\lambda z\ O!z\ \&\ z \neq_E z]))$›
using *∀ E(2)* **by** *auto*
moreover AOT-assume ‹$\Diamond Numbers(x,[\lambda z\ O!z\ \&\ z \neq_E z])$›
ultimately AOT-have ‹$\mathcal{A}Numbers(x,[\lambda z\ O!z\ \&\ z \neq_E z])$›
using *sc−eq−box−box:1[THEN ≡E(1), THEN →E, THEN nec−imp−act[THEN →E]]*
by *blast*
AOT-hence ‹$Numbers(x,[\lambda z\ \mathcal{A}[\lambda z\ O!z\ \&\ z \neq_E z]z])$›
by (*safe intro!: eq−num:1[unvarify G, THEN ≡E(1)] cqt:2*)
AOT-hence ‹$x = \#[\lambda z\ O!z\ \&\ z \neq_E z]$›
by (*safe intro!: eq−num:2[unvarify G, THEN ≡E(1)] cqt:2*)
AOT-thus ‹$x = 0$›
using *cqt:2(1) rule−id−df:2:b[zero] rule=E zero:1* **by** *blast*
qed

```
AOT-define Numbers' :: ‹τ ⇒ τ ⇒ φ› (‹Numbers'''(-,-')›)
  ‹Numbers'(x, G) ≡df A!x & G↓ & ∀F (x[F] ≡ F ≈E G)›
AOT-theorem Numbers'equiv: ‹Numbers'(x,G) ≡ A!x & ∀F (x[F] ≡ F ≈E G)›
  by (AOT-subst-def Numbers')
     (auto intro!: ≡I →I &I cqt:2 dest: &E)

AOT-theorem Numbers'DistinctZeroes:
  ‹∃x∃y (◇Numbers'(x,[λz O!z & z ≠E z]) & ◇Numbers'(y,[λz O!z & z ≠E z]) & x ≠ y)›
proof −
  AOT-obtain w1 where ‹∃w w1 ≠ w›
    using two−worlds−exist:4 PossibleWorld.∃E[rotated] by fast
  then AOT-obtain w2 where distinct-worlds: ‹w1 ≠ w2›
    using PossibleWorld.∃E[rotated] by blast
  AOT-obtain x where x-prop:
    ‹A!x & ∀F (x[F] ≡ w1 ⊨ F ≈E [λz O!z & z ≠E z])›
    using A−objects[axiom-inst] ∃E[rotated] by fast
  moreover AOT-obtain y where y-prop:
    ‹A!y & ∀F (y[F] ≡ w2 ⊨ F ≈E [λz O!z & z ≠E z])›
    using A−objects[axiom-inst] ∃E[rotated] by fast
  moreover {
    fix x w
    AOT-assume x-prop: ‹A!x & ∀F (x[F] ≡ w ⊨ F ≈E [λz O!z & z ≠E z])›
    AOT-have ‹∀F w ⊨ (x[F] ≡ F ≈E [λz O!z & z ≠E z])›
    proof(safe intro!: GEN conj−dist−w:4[unvarify p q, OF log−prop−prop:2,
                             OF log−prop−prop:2,THEN ≡E(2)] ≡I →I)
      fix F
      AOT-assume ‹w ⊨ x[F]›
      AOT-hence ‹◇x[F]›
        using fund:1[unvarify p, OF log−prop−prop:2, THEN ≡E(2),
                      OF PossibleWorld.∃I] by blast
      AOT-hence ‹x[F]›
        by (metis en−eq:3[1] intro−elim:3:a)
      AOT-thus ‹w ⊨ (F ≈E [λz O!z & z ≠E z])›
        using x-prop[THEN &E(2), THEN ∀E(2), THEN ≡E(1)] by blast
    next
      fix F
      AOT-assume ‹w ⊨ (F ≈E [λz O!z & z ≠E z])›
      AOT-hence ‹x[F]›
        using x-prop[THEN &E(2), THEN ∀E(2), THEN ≡E(2)] by blast
      AOT-hence ‹□x[F]›
        using pre−en−eq:1[1][THEN →E] by blast
      AOT-thus ‹w ⊨ x[F]›
        using fund:2[unvarify p, OF log−prop−prop:2, THEN ≡E(1)]
              PossibleWorld.∀E by fast
    qed
    AOT-hence ‹w ⊨ ∀F (x[F] ≡ F ≈E [λz O!z & z ≠E z])›
      using conj−dist−w:5[THEN ≡E(2)] by fast
    moreover {
      AOT-have ‹□[λz O!z & z ≠E z]↓›
        by (safe intro!: RN cqt:2)
      AOT-hence ‹w ⊨ [λz O!z & z ≠E z]↓›
        using fund:2[unvarify p, OF log−prop−prop:2, THEN ≡E(1),
                      THEN PossibleWorld.∀E] by blast
    }
    moreover {
      AOT-have ‹□A!x›
        using x-prop[THEN &E(1)] by (metis oa−facts:2 →E)
      AOT-hence ‹w ⊨ A!x›
        using fund:2[unvarify p, OF log−prop−prop:2,
                      THEN ≡E(1), THEN PossibleWorld.∀E] by blast
    }
    ultimately AOT-have ‹w ⊨ (A!x & [λz O!z & z ≠E z]↓ &
```

```
                        ∀F (x[F] ≡ F ≈E [λz O!z & z ≠E z]))›
    using conj−dist−w:1[unvarify p q, OF log−prop−prop:2,
          OF log−prop−prop:2, THEN ≡E(2), OF &I] by auto
  AOT-hence ‹∃w w ⊨ (A!x & [λz O!z & z ≠E z]↓ &
                    ∀F (x[F] ≡ F ≈E [λz O!z & z ≠E z]))›
    using PossibleWorld.∃I by auto
  AOT-hence ‹◇(A!x & [λz O!z & z ≠E z]↓ & ∀F (x[F] ≡ F ≈E [λz O!z & z ≠E z]))›
    using fund:1[unvarify p, OF log−prop−prop:2, THEN ≡E(2)] by blast
  AOT-hence ‹◇Numbers′(x,[λz O!z & z ≠E z])›
    by (AOT-subst-def Numbers′)
}
ultimately AOT-have ‹◇Numbers′(x,[λz O!z & z ≠E z])›
              and ‹◇Numbers′(y,[λz O!z & z ≠E z])›
  by auto
moreover AOT-have ‹x ≠ y›
proof (rule ab−obey:2[THEN →E])
  AOT-have ‹□¬∃u [λz O!z & z ≠E z]u›
  proof (safe intro!: RN raa−cor:2)
    AOT-modally-strict {
      AOT-assume ‹∃u [λz O!z & z ≠E z]u›
      then AOT-obtain u where ‹[λz O!z & z ≠E z]u›
        using Ordinary.∃E[rotated] by blast
      AOT-hence ‹O!u & u ≠E u›
        by (rule β→C)
      AOT-hence ‹¬(u =E u)›
        by (metis con−dis−taut:2 intro−elim:3:d modus−tollens:1
                raa−cor:3 thm−neg=E)
      AOT-hence ‹u =E u & ¬u =E u›
        by (metis modus−tollens:1 ord=Eequiv:1 raa−cor:3 Ordinary.ψ)
      AOT-thus ‹p & ¬p› for p
        by (metis raa−cor:1)
    }
  qed
  AOT-hence nec-not-ex: ‹∀w w ⊨ ¬∃u [λz O!z & z ≠E z]u›
    using fund:2[unvarify p, OF log−prop−prop:2, THEN ≡E(1)] by blast
  AOT-have ‹□([λy p]x ≡ p)› for x p
    by (safe intro!: RN beta−C−meta[THEN →E] cqt:2)
  AOT-hence ‹∀w w ⊨ ([λy p]x ≡ p)› for x p
    using fund:2[unvarify p, OF log−prop−prop:2, THEN ≡E(1)] by blast
  AOT-hence world-prop-beta: ‹∀w (w ⊨ [λy p]x ≡ w ⊨ p)› for x p
    using conj−dist−w:4[unvarify p, OF log−prop−prop:2, THEN ≡E(1)]
        PossibleWorld.∀E PossibleWorld.∀I by meson

  AOT-have ‹∃p (w1 ⊨ p & ¬w2 ⊨ p)›
  proof(rule raa−cor:1)
    AOT-assume 0: ‹¬∃p (w1 ⊨ p & ¬w2 ⊨ p)›
    AOT-have 1: ‹w1 ⊨ p → w2 ⊨ p› for p
    proof(safe intro!: GEN →I)
      AOT-assume ‹w1 ⊨ p›
      AOT-thus ‹w2 ⊨ p›
        using 0 con−dis−i−e:1 ∃I(2) raa−cor:4 by fast
    qed
    moreover AOT-have ‹w2 ⊨ p → w1 ⊨ p› for p
    proof(safe intro!: GEN →I)
      AOT-assume ‹w2 ⊨ p›
      AOT-hence ‹¬w2 ⊨ ¬p›
        using coherent:2 intro−elim:3:a by blast
      AOT-hence ‹¬w1 ⊨ ¬p›
        using 1[∀I p, THEN ∀E(1), OF log−prop−prop:2]
        by (metis modus−tollens:1)
      AOT-thus ‹w1 ⊨ p›
        using coherent:1 intro−elim:3:b reductio−aa:1 by blast
    qed
```

```
  ultimately AOT-have ‹w₁ ⊨ p ≡ w₂ ⊨ p› for p
    by (metis intro−elim:2)
  AOT-hence ‹w₁ = w₂›
    using sit−identity[unconstrain s, THEN →E,
        OF PossibleWorld.ψ[THEN world:1[THEN ≡df E], THEN &E(1)],
        unconstrain s', THEN →E,
        OF PossibleWorld.ψ[THEN world:1[THEN ≡df E], THEN &E(1)],
        THEN ≡E(2)] GEN by fast
  AOT-thus ‹w₁ = w₂ & ¬w₁ = w₂›
    using =−infix ≡df E con−dis−i−e:1 distinct-worlds by blast
qed
then AOT-obtain p where 0: ‹w₁ ⊨ p & ¬w₂ ⊨ p›
  using ∃E[rotated] by blast
AOT-have ‹y[λy p]›
proof (safe intro!: y-prop[THEN &E(2), THEN ∀E(1), THEN ≡E(2)] cqt:2)
  AOT-show ‹w₂ ⊨ [λy p] ≈E [λz O!z & z ≠E z]›
  proof (safe intro!: cqt:2 empty−approx:1[unvarify F H, THEN RN,
            THEN fund:2[unvarify p, OF log−prop−prop:2, THEN ≡E(1)],
            THEN PossibleWorld.∀E,
            THEN conj−dist−w:2[unvarify p q, OF log−prop−prop:2,
                         OF log−prop−prop:2, THEN ≡E(1)],
                         THEN →E]
            conj−dist−w:1[unvarify p q, OF log−prop−prop:2,
                         OF log−prop−prop:2, THEN ≡E(2)] &I)
    AOT-have ‹¬w₂ ⊨ ∃u [λy p]u›
    proof (rule raa−cor:2)
      AOT-assume ‹w₂ ⊨ ∃u [λy p]u›
      AOT-hence ‹∃x w₂ ⊨ (O!x & [λy p]x)›
        by (metis conj−dist−w:6 intro−elim:3:a)
      then AOT-obtain x where ‹w₂ ⊨ (O!x & [λy p]x)›
        using ∃E[rotated] by blast
      AOT-hence ‹w₂ ⊨ [λy p]x›
        using conj−dist−w:1[unvarify p q, OF log−prop−prop:2,
            OF log−prop−prop:2, THEN ≡E(1), THEN &E(2)] by blast
      AOT-hence ‹w₂ ⊨ p›
        using world-prop-beta[THEN PossibleWorld.∀E, THEN ≡E(1)] by blast
      AOT-thus ‹w₂ ⊨ p & ¬w₂ ⊨ p›
        using 0[THEN &E(2)] &I by blast
    qed
    AOT-thus ‹w₂ ⊨ ¬∃u [λy p]u›
      by (safe intro!: coherent:1[unvarify p, OF log−prop−prop:2,
                         THEN ≡E(2)])
  next
    AOT-show ‹w₂ ⊨ ¬∃v [λz O!z & z ≠E z]v›
      using nec-not-ex[THEN PossibleWorld.∀E] by blast
  qed
qed
moreover AOT-have ‹¬x[λy p]›
proof(rule raa−cor:2)
  AOT-assume ‹x[λy p]›
  AOT-hence w₁ ⊨ [λy p] ≈E [λz O!z & z ≠E z]
    using x-prop[THEN &E(2), THEN ∀E(1), THEN ≡E(1)]
        prop−prop2:2 by blast
  AOT-hence ¬w₁ ⊨ ¬[λy p] ≈E [λz O!z & z ≠E z]
    using coherent:2[unvarify p, OF log−prop−prop:2, THEN ≡E(1)] by blast
  moreover AOT-have w₁ ⊨ ¬([λy p] ≈E [λz O!z & z ≠E z])
  proof (safe intro!: cqt:2 empty−approx:2[unvarify F H, THEN RN,
              THEN fund:2[unvarify p, OF log−prop−prop:2, THEN ≡E(1)],
              THEN PossibleWorld.∀E,
              THEN conj−dist−w:2[unvarify p q, OF log−prop−prop:2,
                  OF log−prop−prop:2, THEN ≡E(1)], THEN →E]
              conj−dist−w:1[unvarify p q, OF log−prop−prop:2,
                         OF log−prop−prop:2, THEN ≡E(2)] &I)
```

fix u
AOT-have ‹$w_1 \models O!u$›
using *Ordinary.ψ[THEN RN,*
THEN fund:2[unvarify p, OF log−prop−prop:2, THEN ≡E(1)],
THEN PossibleWorld.∀E] **by** *simp*
moreover AOT-have ‹$w_1 \models [\lambda y\ p]u$›
by (*safe intro!: world-prop-beta[THEN PossibleWorld.∀E, THEN ≡E(2)]*
0[THEN &E(1)])
ultimately AOT-have ‹$w_1 \models (O!u\ \&\ [\lambda y\ p]u)$›
using *conj−dist−w:1[unvarify p q, OF log−prop−prop:2,*
OF log−prop−prop:2, THEN ≡E(2),
OF &I] **by** *blast*
AOT-hence ‹$\exists x\ w_1 \models (O!x\ \&\ [\lambda y\ p]x)$›
by (*rule ∃I*)
AOT-thus ‹$w_1 \models \exists u\ [\lambda y\ p]u$›
by (*metis conj−dist−w:6 intro−elim:3:b*)
next
AOT-show ‹$w_1 \models \neg\exists v\ [\lambda z\ O!z\ \&\ z \neq_E z]v$›
using *PossibleWorld.∀E nec-not-ex* **by** *fastforce*
qed
ultimately AOT-show ‹$p\ \&\ \neg p$› **for** p
using *raa−cor:3* **by** *blast*
qed
ultimately AOT-have ‹$y[\lambda y\ p]\ \&\ \neg x[\lambda y\ p]$›
using *&I* **by** *blast*
AOT-hence ‹$\exists F\ (y[F]\ \&\ \neg x[F])$›
by (*metis existential:1 prop−prop2:2*)
AOT-thus ‹$\exists F\ (x[F]\ \&\ \neg y[F]) \vee \exists F\ (y[F]\ \&\ \neg x[F])$›
by (*rule ∨I*)
qed
ultimately AOT-have ‹$\Diamond Numbers'(x,[\lambda z\ O!z\ \&\ z \neq_E z])\ \&$
$\Diamond Numbers'(y,[\lambda z\ O!z\ \&\ z \neq_E z])\ \&\ x \neq y$›
using *&I* **by** *blast*
AOT-thus ‹$\exists x \exists y\ (\Diamond Numbers'(x,[\lambda z\ O!z\ \&\ z \neq_E z])\ \&$
$\Diamond Numbers'(y,[\lambda z\ O!z\ \&\ z \neq_E z])\ \&\ x \neq y)$›
using *∃I(2)*[**where** *β=x*] *∃I(2)*[**where** *β=y*] **by** *auto*
qed

AOT-theorem *restricted-identity*:
‹$x =_{\mathcal{R}} y \equiv (InDomainOf(x,\mathcal{R})\ \&\ InDomainOf(y,\mathcal{R})\ \&\ x = y)$›
by (*auto intro!: ≡I →I &I*
dest: id−R−thm:2[THEN →E] &E
id−R−thm:3[THEN →E]
id−R−thm:4[THEN →E, OF ∨I(1), THEN ≡E(2)])

AOT-theorem *induction′*: ‹$\forall F\ ([F]0\ \&\ \forall n([F]n \to [F]n') \to \forall n\ [F]n)$›
proof(*rule GEN; rule →I*)
fix $F\ n$
AOT-assume A: ‹$[F]0\ \&\ \forall n([F]n \to [F]n')$›
AOT-have ‹$\forall n \forall m([\mathbb{P}]nm \to ([F]n \to [F]m))$›
proof(*safe intro!: Number.GEN →I*)
fix $n\ m$
AOT-assume ‹$[\mathbb{P}]nm$›
moreover AOT-have ‹$[\mathbb{P}]n\ n'$›
using *suc−thm*.
ultimately AOT-have *m-eq-suc-n*: ‹$m = n'$›
using *pred−func:1[unvarify z, OF def−suc[den2], THEN →E, OF &I]*
by *blast*
AOT-assume ‹$[F]n$›
AOT-hence ‹$[F]n'$›
using *A[THEN &E(2), THEN Number.∀E, THEN →E]* **by** *blast*
AOT-thus ‹$[F]m$›
using *m-eq-suc-n[symmetric] rule=E* **by** *fast*

```
  qed
  AOT-thus ‹∀ n[F]n›
    using induction[THEN ∀ E(2), THEN →E, OF &I, OF A[THEN &E(1)]]
    by simp
qed

AOT-define ExtensionOf :: ‹τ ⇒ Π ⇒ φ› (‹ExtensionOf′(-,-′)›)
  exten−property:1: ‹ExtensionOf(x,[G]) ≡df A!x & G↓ & ∀ F(x[F] ≡ ∀ z([F]z ≡ [G]z))›

AOT-define OrdinaryExtensionOf :: ‹τ ⇒ Π ⇒ φ› (‹OrdinaryExtensionOf′(-,-′)›)
  ‹OrdinaryExtensionOf(x,[G]) ≡df A!x & G↓ & ∀ F(x[F] ≡ ∀ z(O!z → ([F]z ≡ [G]z)))›

AOT-theorem BeingOrdinaryExtensionOfDenotes:
  ‹[λx OrdinaryExtensionOf(x,[G])]↓›
proof(rule safe−ext[axiom-inst, THEN →E, OF &I])
  AOT-show ‹[λx A!x & G↓ & [λx ∀ F(x[F] ≡ ∀ z(O!z → ([F]z ≡ [G]z)))]x]↓›
    by cqt:2
next
  AOT-show ‹□∀ x (A!x & G↓ & [λx ∀ F (x[F] ≡ ∀ z (O!z → ([F]z ≡ [G]z)))]x ≡
          OrdinaryExtensionOf(x,[G]))›
  proof(safe intro!: RN GEN)
    AOT-modally-strict {
      fix x
      AOT-modally-strict {
        AOT-have ‹[λx ∀ F (x[F] ≡ ∀ z (O!z → ([F]z ≡ [G]z)))]↓›
        proof (safe intro!: Comprehension-3[THEN →E] RN GEN
                          →I ≡I Ordinary.GEN)
          AOT-modally-strict {
            fix F H u
            AOT-assume ‹□H ≡E F›
            AOT-hence ‹∀ u([H]u ≡ [F]u)›
              using eqE[THEN ≡df E, THEN &E(2)] qml:2[axiom-inst, THEN →E]
              by blast
            AOT-hence 0: ‹[H]u ≡ [F]u› using Ordinary.∀ E by fast
            {
              AOT-assume ‹∀ u([F]u ≡ [G]u)›
              AOT-hence 1: ‹[F]u ≡ [G]u› using Ordinary.∀ E by fast
              AOT-show ‹[G]u› if ‹[H]u› using 0 1 ≡E(1) that by blast
              AOT-show ‹[H]u› if ‹[G]u› using 0 1 ≡E(2) that by blast
            }
            {
              AOT-assume ‹∀ u([H]u ≡ [G]u)›
              AOT-hence 1: ‹[H]u ≡ [G]u› using Ordinary.∀ E by fast
              AOT-show ‹[G]u› if ‹[F]u› using 0 1 ≡E(1,2) that by blast
              AOT-show ‹[F]u› if ‹[G]u› using 0 1 ≡E(1,2) that by blast
            }
          }
        qed
      }
      AOT-thus ‹(A!x & G↓ & [λx ∀ F (x[F] ≡ ∀ z (O!z → ([F]z ≡ [G]z)))]x) ≡
              OrdinaryExtensionOf(x,[G])›
        apply (AOT-subst-def OrdinaryExtensionOf)
        apply (AOT-subst ‹[λx ∀ F (x[F] ≡ ∀ z (O!z → ([F]z ≡ [G]z)))]x›
                      ‹∀ F (x[F] ≡ ∀ z (O!z → ([F]z ≡ [G]z)))›)
        by (auto intro!: beta−C−meta[THEN →E] simp: oth−class−taut:3:a)
    }
  qed
qed
```

Fragments of PLM's theory of Concepts.

```
AOT-define FimpG :: ‹Π ⇒ Π ⇒ φ› (infixl ‹⇒› 50)
  F−imp−G: ‹[G] ⇒ [F] ≡df F↓ & G↓ & □∀ x ([G]x → [F]x)›
```

AOT-define *concept* :: ‹Π› (‹*C*!›)
concepts: ‹$C! =_{df}\ A!$›

AOT-register-rigid-restricted-type
Concept: ‹$C!\kappa$›
proof
AOT-modally-strict {
AOT-have ‹$\exists x\ A!x$›
using *o−objects−exist:2 qml:2[axiom-inst]* →*E* **by** *blast*
AOT-thus ‹$\exists x\ C!x$›
using *rule−id−df:1[zero][OF concepts, OF oa−exist:2] rule=E id-sym*
by *fast*
}
next
AOT-modally-strict {
AOT-show ‹$C!\kappa \rightarrow \kappa\downarrow$› **for** κ
using *cqt:5:a[axiom-inst, THEN →E, THEN &E(2)] →I*
by *blast*
}
next
AOT-modally-strict {
AOT-have ‹$\forall x(A!x \rightarrow \Box A!x)$›
by (*simp add: oa−facts:2 GEN*)
AOT-thus ‹$\forall x(C!x \rightarrow \Box C!x)$›
using *rule−id−df:1[zero][OF concepts, OF oa−exist:2] rule=E id-sym*
by *fast*
}
qed

AOT-register-variable-names
Concept: *c d e*

AOT-theorem *concept−comp:1*: ‹$\exists x(C!x\ \&\ \forall F(x[F] \equiv \varphi\{F\}))$›
using *concepts[THEN rule−id−df:1[zero], OF oa−exist:2, symmetric]*
A−objects[axiom-inst]
rule=E **by** *fast*

AOT-theorem *concept−comp:2*: ‹$\exists !x(C!x\ \&\ \forall F(x[F] \equiv \varphi\{F\}))$›
using *concepts[THEN rule−id−df:1[zero], OF oa−exist:2, symmetric]*
A−objects!
rule=E **by** *fast*

AOT-theorem *concept−comp:3*: ‹$\iota x(C!x\ \&\ \forall F(x[F] \equiv \varphi\{F\}))\downarrow$›
using *concept−comp:2 A−Exists:2[THEN ≡E(2)] RA[2]* **by** *blast*

AOT-theorem *concept−comp:4*:
‹$\iota x(C!x\ \&\ \forall F(x[F] \equiv \varphi\{F\})) = \iota x(A!x\ \&\ \forall F(x[F] \equiv \varphi\{F\}))$›
using *=I(1)[OF concept−comp:3]*
rule=E[rotated]
concepts[THEN rule−id−df:1[zero], OF oa−exist:2]
by *fast*

AOT-define *conceptInclusion* :: ‹$\tau \Rightarrow \tau \Rightarrow \varphi$› (**infixl** ‹$\preceq$› *100*)
con:1: ‹$c \preceq d \equiv_{df} \forall F(c[F] \rightarrow d[F])$›

AOT-define *conceptOf* :: ‹$\tau \Rightarrow \tau \Rightarrow \varphi$› (‹*ConceptOf*′(-,-′)›)
concept−of−G: ‹$ConceptOf(c,G) \equiv_{df} G\downarrow\ \&\ \forall F\ (c[F] \equiv [G] \Rightarrow [F])$›

AOT-theorem *ConceptOfOrdinaryProperty*: ‹$([H] \Rightarrow O!) \rightarrow [\lambda x\ ConceptOf(x,H)]\downarrow$›
proof(*rule* →*I*)
AOT-assume ‹$[H] \Rightarrow O!$›
AOT-hence ‹$\Box\forall x([H]x \rightarrow O!x)$›

```
  using F−imp−G[THEN ≡df E] &E by blast
AOT-hence ‹□□∀x([H]x → O!x)›
  using S5Basic:6[THEN ≡E(1)] by blast
moreover AOT-have ‹□□∀x([H]x → O!x) →
                  □∀F∀G(□(G ≡E F) → ([H] ⇒ [F] ≡ [H] ⇒ [G]))›
proof(rule RM; safe intro!: →I GEN ≡I)
  AOT-modally-strict {
    fix F G
    AOT-assume 0: ‹□∀x([H]x → O!x)›
    AOT-assume ‹□G ≡E F›
    AOT-hence 1: ‹□∀u([G]u ≡ [F]u)›
      by (AOT-subst-thm eqE[THEN ≡Df, THEN ≡S(1), OF &I,
            OF cqt:2[const-var][axiom-inst],
            OF cqt:2[const-var][axiom-inst], symmetric])
    {
      AOT-assume ‹[H] ⇒ [F]›
      AOT-hence ‹□∀x([H]x → [F]x)›
        using F−imp−G[THEN ≡df E] &E by blast
      moreover AOT-modally-strict {
        AOT-assume ‹∀x([H]x → O!x)›
        moreover AOT-assume ‹∀u([G]u ≡ [F]u)›
        moreover AOT-assume ‹∀x([H]x → [F]x)›
        ultimately AOT-have ‹[H]x → [G]x› for x
          by (auto intro!: →I dest!: ∀E(2) dest: →E ≡E)
        AOT-hence ‹∀x([H]x → [G]x)›
          by (rule GEN)
      }
      ultimately AOT-have ‹□∀x([H]x → [G]x)›
        using RN[prem][where
            Γ={«∀x([H]x → O!x)», «∀u([G]u ≡ [F]u)», «∀x([H]x → [F]x)»}]
        using 0 1 by fast
      AOT-thus ‹[H] ⇒ [G]›
        by (AOT-subst-def F−imp−G)
           (safe intro!: cqt:2 &I)
    }
    {
      AOT-assume ‹[H] ⇒ [G]›
      AOT-hence ‹□∀x([H]x → [G]x)›
        using F−imp−G[THEN ≡df E] &E by blast
      moreover AOT-modally-strict {
        AOT-assume ‹∀x([H]x → O!x)›
        moreover AOT-assume ‹∀u([G]u ≡ [F]u)›
        moreover AOT-assume ‹∀x([H]x → [G]x)›
        ultimately AOT-have ‹[H]x → [F]x› for x
          by (auto intro!: →I dest!: ∀E(2) dest: →E ≡E)
        AOT-hence ‹∀x([H]x → [F]x)›
          by (rule GEN)
      }
      ultimately AOT-have ‹□∀x([H]x → [F]x)›
        using RN[prem][where
            Γ={«∀x([H]x → O!x)», «∀u([G]u ≡ [F]u)», «∀x([H]x → [G]x)»}]
        using 0 1 by fast
      AOT-thus ‹[H] ⇒ [F]›
        by (AOT-subst-def F−imp−G)
           (safe intro!: cqt:2 &I)
    }
  }
qed
ultimately AOT-have ‹□∀F∀G(□(G ≡E F) → ([H] ⇒ [F] ≡ [H] ⇒ [G]))›
  using →E by blast
AOT-hence 0: ‹[λx ∀F(x[F] ≡ ([H] ⇒ [F]))]↓›
  using Comprehension-3[THEN →E] by blast
AOT-show ‹[λx ConceptOf(x,H)]↓›
```

```
  proof (rule safe-ext[axiom-inst, THEN →E, OF &I])
    AOT-show ‹[λx C!x & [λx ∀F(x[F] ≡ ([H] ⇒ [F]))]x]↓› by cqt:2
  next
    AOT-show ‹□∀x (C!x & [λx ∀F (x[F] ≡ [H] ⇒ [F])]x ≡ ConceptOf(x,H))›
    proof (rule RN[prem][where Γ=‹{«[λx ∀F(x[F] ≡ ([H] ⇒ [F]))]↓»}›, simplified])
      AOT-modally-strict {
        AOT-assume 0: ‹[λx ∀F (x[F] ≡ [H] ⇒ [F])]↓›
        AOT-show ‹∀x (C!x & [λx ∀F (x[F] ≡ [H] ⇒ [F])]x ≡ ConceptOf(x,H))›
        proof(safe intro!: GEN ≡I →I &I)
          fix x
          AOT-assume ‹C!x & [λx ∀F (x[F] ≡ [H] ⇒ [F])]x›
          AOT-thus ‹ConceptOf(x,H)›
            by (AOT-subst-def concept−of−G)
               (auto intro!: &I cqt:2 dest: &E β→C)
        next
          fix x
          AOT-assume ‹ConceptOf(x,H)›
          AOT-hence ‹C!x & (H↓ & ∀F(x[F] ≡ [H] ⇒ [F]))›
            by (AOT-subst-def (reverse) concept−of−G)
          AOT-thus ‹C!x› and ‹[λx ∀F(x[F] ≡ [H] ⇒ [F])]x›
            by (auto intro!: β←C 0 cqt:2 dest: &E)
        qed
      }
    next
      AOT-show ‹□[λx ∀F(x[F] ≡ ([H] ⇒ [F]))]↓›
        using exist−nec[THEN →E] 0 by blast
    qed
  qed
qed

AOT-theorem con−exists:1: ‹∃c ConceptOf(c,G)›
proof −
  AOT-obtain c where ‹∀F (c[F] ≡ [G] ⇒ [F])›
    using concept−comp:1 Concept.∃E[rotated] by meson
  AOT-hence ‹ConceptOf(c,G)›
    by (auto intro!: concept−of−G[THEN ≡dfI] &I cqt:2 Concept.ψ)
  thus ?thesis by (rule Concept.∃I)
qed

AOT-theorem con−exists:2: ‹∃!c ConceptOf(c,G)›
proof −
  AOT-have ‹∃!c ∀F (c[F] ≡ [G] ⇒ [F])›
    using concept−comp:2 by simp
  moreover {
    AOT-modally-strict {
      fix x
      AOT-assume ‹∀F (x[F] ≡ [G] ⇒ [F])›
      moreover AOT-have ‹[G] ⇒ [G]›
        by (safe intro!: F−imp−G[THEN ≡dfI] &I cqt:2 RN GEN →I)
      ultimately AOT-have ‹x[G]›
        using ∀E(2) ≡E by blast
      AOT-hence ‹A!x›
        using encoders−are−abstract[THEN →E, OF ∃I(2)] by simp
      AOT-hence ‹C!x›
        using concepts[THEN rule−id−df:1[zero], OF oa−exist:2, symmetric]
              rule=E[rotated]
        by fast
    }
  }
  ultimately show ?thesis
    by (AOT-subst ‹ConceptOf(c,G)› ‹∀F (c[F] ≡ [G] ⇒ [F])› for: c;
          AOT-subst-def concept−of−G)
       (auto intro!: ≡I →I &I cqt:2 Concept.ψ dest: &E)
```

```
qed

AOT-theorem con−exists:3: ‹ιc ConceptOf(c,G)↓›
  by (safe intro!: A−Exists:2[THEN ≡E(2)] con−exists:2[THEN RA[2]])


AOT-define theConceptOfG :: ‹τ ⇒ κs› (‹c-›)
  concept−G: ‹cG =df ιc ConceptOf(c, G)›

AOT-theorem concept−G[den]: ‹cG↓›
  by (auto intro!: rule−id−df:1[OF concept−G]
                  t=t−proper:1[THEN →E]
                  con−exists:3)


AOT-theorem concept−G[concept]: ‹C!cG›
proof −
  AOT-have ‹𝒜(C!cG & ConceptOf(cG, G))›
    by (auto intro!: actual−desc:2[unvarify x, THEN →E]
                    rule−id−df:1[OF concept−G]
                    concept−G[den]
                    con−exists:3)
  AOT-hence ‹𝒜C!cG›
    by (metis Act−Basic:2 con−dis−i−e:2:a intro−elim:3:a)
  AOT-hence ‹𝒜A!cG›
    using rule−id−df:1[zero][OF concepts, OF oa−exist:2]
          rule=E by fast
  AOT-hence ‹A!cG›
    using oa−facts:8[unvarify x, THEN ≡E(2)] concept−G[den] by blast
  thus ?thesis
    using rule−id−df:1[zero][OF concepts, OF oa−exist:2, symmetric]
          rule=E by fast
qed

AOT-theorem conG−strict: ‹cG = ιc ∀F(c[F] ≡ [G] ⇒ [F])›
proof (rule id−eq:3[unvarify α β γ, THEN →E])
  AOT-have ‹□∀x (C!x & ConceptOf(x,G) ≡ C!x & ∀F (x[F] ≡ [G] ⇒ [F]))›
    by (auto intro!: concept−of−G[THEN ≡df I] RN GEN ≡I →I &I cqt:2
              dest: &E;
        auto dest: ∀E(2) ≡E(1,2) dest!: &E(2) concept−of−G[THEN ≡df E])
  AOT-thus ‹cG = ιc ConceptOf(c, G) & ιc ConceptOf(c, G) = ιc ∀F(c[F] ≡ [G] ⇒ [F])›
    by (auto intro!: &I rule−id−df:1[OF concept−G] con−exists:3
                    equiv−desc−eq:3[THEN →E])
qed(auto simp: concept−G[den] con−exists:3 concept−comp:3)


AOT-theorem conG−lemma:1: ‹∀F(cG[F] ≡ [G] ⇒ [F])›
proof(safe intro!: GEN ≡I →I)
  fix F
  AOT-have ‹𝒜∀F(cG[F] ≡ [G] ⇒ [F])›
    using actual−desc:4[THEN →E, OF concept−comp:3,
                        THEN Act−Basic:2[THEN ≡E(1)],
                        THEN &E(2)]
          conG−strict[symmetric] rule=E by fast
  AOT-hence ‹𝒜(cG[F] ≡ [G] ⇒ [F])›
    using logic−actual−nec:3[axiom-inst, THEN ≡E(1)] ∀E(2)
    by blast
  AOT-hence 0: ‹𝒜cG[F] ≡ 𝒜[G] ⇒ [F]›
    using Act−Basic:5[THEN ≡E(1)] by blast
  {
    AOT-assume ‹cG[F]›
    AOT-hence ‹𝒜cG[F]›
      by(safe intro!: en−eq:10[1][unvarify x1, THEN ≡E(2)]
```

```
                    concept−G[den])
    AOT-hence ‹𝓐[G] ⇒ [F]›
      using 0[THEN ≡E(1)] by blast
    AOT-hence ‹𝓐(F↓ & G↓ & □∀x([G]x → [F]x))›
      by (AOT-subst-def (reverse) F−imp−G)
    AOT-hence ‹𝓐□∀x([G]x → [F]x)›
      using Act−Basic:2[THEN ≡E(1)] &E by blast
    AOT-hence ‹□∀x([G]x → [F]x)›
      using qml−act:2[axiom-inst, THEN ≡E(2)] by simp
    AOT-thus ‹[G] ⇒ [F]›
      by (AOT-subst-def F−imp−G; auto intro!: &I cqt:2)
  }
  {
    AOT-assume ‹[G] ⇒ [F]›
    AOT-hence ‹□∀x([G]x → [F]x)›
      by (safe dest!: F−imp−G[THEN ≡df E] &E(2))
    AOT-hence ‹𝓐□∀x([G]x → [F]x)›
      using qml−act:2[axiom-inst, THEN ≡E(1)] by simp
    AOT-hence ‹𝓐(F↓ & G↓ & □∀x([G]x → [F]x))›
      by (auto intro!: Act−Basic:2[THEN ≡E(2)] &I cqt:2
              intro: RA[2])
    AOT-hence ‹𝓐([G] ⇒ [F])›
      by (AOT-subst-def F−imp−G)
    AOT-hence ‹𝓐c_G[F]›
      using 0[THEN ≡E(2)] by blast
    AOT-thus ‹c_G[F]›
      by(safe intro!: en−eq:10[1][unvarify x₁, THEN ≡E(1)]
                    concept−G[den])
  }
qed

AOT-theorem conH-enc-ord:
  ‹([H] ⇒ O!) → □∀F ∀G (□G ≡_E F → (c_H[F] ≡ c_H[G]))›
proof(rule →I)
  AOT-assume 0: ‹[H] ⇒ O!›
  AOT-have 0: ‹□([H] ⇒ O!)›
    apply (AOT-subst-def F−imp−G)
    using 0[THEN ≡df E[OF F−imp−G]]
    by (auto intro!: KBasic:3[THEN ≡E(2)] &I exist−nec[THEN →E]
              dest: &E 4[THEN →E])
  moreover AOT-have ‹□([H] ⇒ O!) → □∀F ∀G (□G ≡_E F → (c_H[F] ≡ c_H[G]))›
  proof(rule RM; safe intro!: →I GEN)
    AOT-modally-strict {
      fix F G
      AOT-assume ‹[H] ⇒ O!›
      AOT-hence 0: ‹□∀x ([H]x → O!x)›
        by (safe dest!: F−imp−G[THEN ≡df E] &E(2))
      AOT-assume 1: ‹□G ≡_E F›
      AOT-assume ‹c_H[F]›
      AOT-hence ‹[H] ⇒ [F]›
        using conG−lemma:1[THEN ∀E(2), THEN ≡E(1)] by simp
      AOT-hence 2: ‹□∀x ([H]x → [F]x)›
        by (safe dest!: F−imp−G[THEN ≡df E] &E(2))
      AOT-modally-strict {
        AOT-assume 0: ‹∀x ([H]x → O!x)›
        AOT-assume 1: ‹∀x ([H]x → [F]x)›
        AOT-assume 2: ‹G ≡_E F›
        AOT-have ‹∀x ([H]x → [G]x)›
        proof(safe intro!: GEN →I)
          fix x
          AOT-assume ‹[H]x›
          AOT-hence ‹O!x› and ‹[F]x›
            using 0 1 ∀E(2) →E by blast+
```

AOT-thus ‹$[G]x$›
using *2[THEN eqE[THEN* $\equiv_{df}$*E], THEN &E(2)]*
$\forall$*E(2)* $\rightarrow$*E* $\equiv$*E(2) calculation* **by** *blast*
qed
}
AOT-hence ‹$\Box\forall x\ ([H]x \rightarrow [G]x)$›
using *RN[prem]*[**where** Γ=‹{«$\forall x\ ([H]x \rightarrow O!x)$»,
«$\forall x\ ([H]x \rightarrow [F]x)$»,
«$G \equiv_E F$»}›, *simplified] 0 1 2* **by** *fast*
AOT-hence ‹$[H] \Rightarrow [G]$›
by *(safe intro!: F−imp−G[THEN* $\equiv_{df}$*I] &I cqt:2)*
AOT-hence ‹$\mathbf{c}_H[G]$›
using *conG−lemma:1[THEN* $\forall$*E(2), THEN* $\equiv$*E(2)]* **by** *simp*
} note *0 = this*
AOT-modally-strict {
fix *F G*
AOT-assume ‹$[H] \Rightarrow O!$›
moreover AOT-assume ‹$\Box G \equiv_E F$›
moreover AOT-have ‹$\Box F \equiv_E G$›
by *(AOT-subst* ‹$F \equiv_E G$› ‹$G \equiv_E F$›*)*
(auto intro!: calculation(2)
eqE[THEN $\equiv_{df}$*I]*
$\equiv$*I* $\rightarrow$*I &I cqt:2 Ordinary.GEN*
dest!: eqE[THEN $\equiv_{df}$*E] &E(2)*
dest: $\equiv$*E(1,2) Ordinary.*$\forall$*E)*
ultimately AOT-show ‹$(\mathbf{c}_H[F] \equiv \mathbf{c}_H[G])$›
using *0* $\equiv$*I* $\rightarrow$*I* **by** *auto*
}
qed
ultimately AOT-show ‹$\Box\forall F\ \forall G\ (\Box G \equiv_E F \rightarrow (\mathbf{c}_H[F] \equiv \mathbf{c}_H[G]))$›
using $\rightarrow$*E* **by** *blast*
qed

AOT-theorem *concept-inclusion-denotes-1*:
‹$([H] \Rightarrow O!) \rightarrow [\lambda x\ \mathbf{c}_H \preceq x]\downarrow$›
proof(*rule* $\rightarrow$*I)*
AOT-assume *0*: ‹$[H] \Rightarrow O!$›
AOT-show ‹$[\lambda x\ \mathbf{c}_H \preceq x]\downarrow$›
proof(*rule safe−ext[axiom-inst, THEN* $\rightarrow$*E, OF &I])*
AOT-show ‹$[\lambda x\ C!x\ \&\ \forall F(\mathbf{c}_H[F] \rightarrow x[F])]\downarrow$›
by *(safe intro!: conjunction-denotes[THEN* $\rightarrow$*E, OF &I]*
Comprehension-2′[THEN $\rightarrow$*E]*
conH-enc-ord[THEN $\rightarrow$*E, OF 0]) cqt:2*
next
AOT-show ‹$\Box\forall x\ (C!x\ \&\ \forall F\ (\mathbf{c}_H[F] \rightarrow x[F]) \equiv \mathbf{c}_H \preceq x)$›
by *(safe intro!: RN GEN; AOT-subst-def con:1)*
(auto intro!: $\equiv$*I* $\rightarrow$*I &I concept−G[concept] dest: &E)*
qed
qed

AOT-theorem *concept-inclusion-denotes-2*:
‹$([H] \Rightarrow O!) \rightarrow [\lambda x\ x \preceq \mathbf{c}_H]\downarrow$›
proof(*rule* $\rightarrow$*I)*
AOT-assume *0*: ‹$[H] \Rightarrow O!$›
AOT-show ‹$[\lambda x\ x \preceq \mathbf{c}_H]\downarrow$›
proof(*rule safe−ext[axiom-inst, THEN* $\rightarrow$*E, OF &I])*
AOT-show ‹$[\lambda x\ C!x\ \&\ \forall F(x[F] \rightarrow \mathbf{c}_H[F])]\downarrow$›
by *(safe intro!: conjunction-denotes[THEN* $\rightarrow$*E, OF &I]*
Comprehension-1′[THEN $\rightarrow$*E]*
conH-enc-ord[THEN $\rightarrow$*E, OF 0]) cqt:2*
next
AOT-show ‹$\Box\forall x\ (C!x\ \&\ \forall F\ (x[F] \rightarrow \mathbf{c}_H[F]) \equiv x \preceq \mathbf{c}_H)$›
by *(safe intro!: RN GEN; AOT-subst-def con:1)*

(*auto intro*!: ≡*I* →*I* &*I* *concept*−*G*[*concept*] *dest*: &*E*)
qed
qed

AOT-define *ThickForm* :: ‹τ ⇒ τ ⇒ φ› (‹*FormOf*′(-,-′)›)
tform−*of*: ‹$FormOf(x,G) \equiv_{df}$ *A*!*x* & *G*↓ & ∀ *F*(*x*[*F*] ≡ [*G*] ⇒ [*F*])›

AOT-theorem *FormOfOrdinaryProperty*: ‹([*H*] ⇒ *O*!) → [λ*x* *FormOf*(*x*,*H*)]↓›
proof(*rule* →*I*)
AOT-assume *0*: ‹[*H*] ⇒ [*O*!]›
AOT-show ‹[λ*x* *FormOf*(*x*,*H*)]↓›
proof (*rule safe*−*ext*[*axiom-inst*, *THEN* →*E*, *OF* &*I*])
AOT-show ‹[λ*x* *ConceptOf*(*x*,*H*)]↓›
using *0 ConceptOfOrdinaryProperty*[*THEN* →*E*] **by** *blast*
AOT-show ‹□∀ *x* (*ConceptOf*(*x*,*H*) ≡ *FormOf*(*x*,*H*))›
proof(*safe intro*!: *RN GEN*)
AOT-modally-strict {
fix *x*
AOT-modally-strict {
AOT-have ‹*A*!*x* ≡ *A*!*x*›
by (*simp add*: *oth*−*class*−*taut*:*3*:*a*)
AOT-hence ‹*C*!*x* ≡ *A*!*x*›
using *rule*−*id*−*df*:*1*[*zero*][*OF concepts*, *OF oa*−*exist*:*2*]
rule=*E id-sym* **by** *fast*
}
AOT-thus ‹*ConceptOf*(*x*,*H*) ≡ *FormOf*(*x*,*H*)›
by (*AOT-subst-def tform*−*of*;
AOT-subst-def concept−*of*−*G*;
AOT-subst ‹*C*!*x*› ‹*A*!*x*›)
(*auto intro*!: ≡*I* →*I* &*I dest*: &*E*)
}
qed
qed
qed

AOT-theorem *equal-E-rigid-one-to-one*: ‹$Rigid_{1-1}((=_E))$›
proof (*safe intro*!: *df*−*1*−*1*:*2*[*THEN* $\equiv_{df}I$] &*I df*−*1*−*1*:*1*[*THEN* $\equiv_{df}I$]
GEN →*I df*−*rigid*−*rel*:*1*[*THEN* $\equiv_{df}I$] =*E*[*denotes*])
fix *x y z*
AOT-assume ‹$x =_E z$ & $y =_E z$›
AOT-thus ‹*x* = *y*›
by (*metis rule*=*E* &*E*(*1*) *Conjunction Simplification*(*2*)
=*E*−*simple*:*2 id-sym* →*E*)
next
AOT-have ‹∀ *x*∀ *y* □($x =_E y$ → □$x =_E y$)›
proof(*rule GEN*; *rule GEN*)
AOT-show ‹□($x =_E y$ → □$x =_E y$)› **for** *x y*
by (*meson RN deduction*−*theorem id*−*nec3*:*1* ≡*E*(*1*))
qed
AOT-hence ‹∀ x_1...∀ x_n □([$(=_E)$]$x_1...x_n$ → □[$(=_E)$]$x_1...x_n$)›
by (*rule tuple-forall*[*THEN* $\equiv_{df}I$])
AOT-thus ‹□∀ x_1...∀ x_n ([$(=_E)$]$x_1...x_n$ → □[$(=_E)$]$x_1...x_n$)›
using *BF*[*THEN* →*E*] **by** *fast*
qed

AOT-theorem *equal-E-domain*: ‹*InDomainOf*(*x*,$(=_E)$) ≡ *O*!*x*›
proof(*safe intro*!: ≡*I* →*I*)
AOT-assume ‹*InDomainOf*(*x*,$(=_E)$)›
AOT-hence ‹∃ *y* $x =_E y$›
by (*metis* $\equiv_{df}E$ *df*−*1*−*1*:*5*)
then AOT-obtain *y* **where** ‹$x =_E y$›
using ∃ *E*[*rotated*] **by** *blast*
AOT-thus ‹*O*!*x*›

using $=E-simple{:}1[THEN\ \equiv E(1)]\ \&E$ **by** *blast*
next
AOT-assume ‹$O!x$›
AOT-hence ‹$x =_E x$›
by (*metis ord=Eequiv*:*1*[*THEN* →*E*])
AOT-hence ‹$\exists\, y\ x =_E y$›
using ∃ *I*(*2*) **by** *fast*
AOT-thus ‹*InDomainOf*(x,($=_E$))›
by (*metis* $\equiv_{df}I$ *df*−*1*−*1*:*5*)
qed

AOT-theorem *shared-urelement-projection-identity*:
assumes ‹∀ y [λx (y[λz [R]zx])]↓›
shows ‹∀ F([F]a ≡ [F]b) → [λz [R]za] = [λz [R]zb]›
proof(*rule* →*I*)
AOT-assume *0*: ‹∀ F([F]a ≡ [F]b)›
{
fix z
AOT-have ‹[λx (z[λz [R]zx])]↓›
using *assms*[*THEN* ∀ *E*(*2*)]**.**
AOT-hence *1*: ‹∀ x ∀ y (∀ F ([F]x ≡ [F]y) → □(z[λz [R]zx] ≡ z[λz [R]zy]))›
using *kirchner*−*thm*−*cor*:*1*[*THEN* →*E*]
by *blast*
AOT-have ‹□(z[λz [R]za] ≡ z[λz [R]zb])›
using *1*[*THEN* ∀ *E*(*2*), *THEN* ∀ *E*(*2*), *THEN* →*E*, *OF 0*] **by** *blast*
}
AOT-hence ‹∀ z □(z[λz [R]za] ≡ z[λz [R]zb])›
by (*rule GEN*)
AOT-hence ‹□∀ z(z[λz [R]za] ≡ z[λz [R]zb])›
by (*rule BF*[*THEN* →*E*])
AOT-thus ‹[λz [R]za] = [λz [R]zb]›
by (*AOT-subst-def identity*:*2*)
(*auto intro*!: &*I cqt*:*2*)
qed

AOT-theorem *shared-urelement-exemplification-identity*:
assumes ‹∀ y [λx (y[λz [G]x])]↓›
shows ‹∀ F([F]a ≡ [F]b) → ([G]a) = ([G]b)›
proof(*rule* →*I*)
AOT-assume *0*: ‹∀ F([F]a ≡ [F]b)›
{
fix z
AOT-have ‹[λx (z[λz [G]x])]↓›
using *assms*[*THEN* ∀ *E*(*2*)]**.**
AOT-hence *1*: ‹∀ x ∀ y (∀ F ([F]x ≡ [F]y) → □(z[λz [G]x] ≡ z[λz [G]y]))›
using *kirchner*−*thm*−*cor*:*1*[*THEN* →*E*]
by *blast*
AOT-have ‹□(z[λz [G]a] ≡ z[λz [G]b])›
using *1*[*THEN* ∀ *E*(*2*), *THEN* ∀ *E*(*2*), *THEN* →*E*, *OF 0*] **by** *blast*
}
AOT-hence ‹∀ z □(z[λz [G]a] ≡ z[λz [G]b])›
by (*rule GEN*)
AOT-hence ‹□∀ z(z[λz [G]a] ≡ z[λz [G]b])›
by (*rule BF*[*THEN* →*E*])
AOT-hence ‹[λz [G]a] = [λz [G]b]›
by (*AOT-subst-def identity*:*2*)
(*auto intro*!: &*I cqt*:*2*)
AOT-thus ‹([G]a) = ([G]b)›
by (*safe intro*!: *identity*:*4*[*THEN* $\equiv_{df}I$] &*I log*−*prop*−*prop*:*2*)
qed

The assumptions of the theorems above are derivable, if the additional introduction rules for the upcoming extension of *AOT-instance-of-cqt-2* $\varphi \Longrightarrow [\lambda\nu_1...\nu_n\ \varphi\{\nu_1...\nu_n\}]\downarrow \in \Lambda_\Box$ are explicitly allowed (while they are currently not part of the abstraction layer).

notepad
begin
  AOT-modally-strict {
    AOT-have ‹$\forall R \forall y\ [\lambda x\ (y[\lambda z\ [R]zx])]\downarrow$›
      by (*safe intro*!: *GEN cqt*:*2 AOT-instance-of-cqt-2-intro-next*)
    AOT-have ‹$\forall G \forall y\ [\lambda x\ (y[\lambda z\ [G]x])]\downarrow$›
      by (*safe intro*!: *GEN cqt*:*2 AOT-instance-of-cqt-2-intro-next*)
  }
end

end